General Lorentz Transformations, Acting on Some Important Physical Quantities

We are interested in transforming measurements made in a reference frame $\mathcal{O}'$ into measurements of the same quantities as made in a reference frame $\mathcal{O}$, where the reference frame $\mathcal{O}$ measures $\mathcal{O}'$ to be moving with constant velocity $\vec{v}$, so that $\gamma \equiv \gamma_v \equiv 1/\sqrt{1-(v/c)^2}$, where $\vec{v}$ is a 3-vector in a totally arbitrary direction. It is therefore immediately useful to divide all other vector quantities into their parts which are parallel and perpendicular to $\vec{v}$; for instance, for the location vector $\vec{r}$ we write

$$\vec{r} = \vec{r}_{\|} + \vec{r}_{\perp} = (\hat{v}\cdot\vec{r})\hat{v} + \hat{v}\times(\vec{r}\times\hat{v}) \ .$$

The appropriate Lorentz transformation equations for the location vector are then

$$\begin{gathered} \vec{r}_{\|} = \gamma[\vec{r}\,'_{\|} + t'\vec{v}] \ , \qquad \vec{r}_{\perp} = \vec{r}\,'_{\perp} \ , \\ t = \gamma[t' + \vec{v}\cdot\vec{r}\,'/c^2] \ , \\ \text{or} \quad \vec{r} = \vec{r}\,' + (\gamma-1)(\vec{r}\,'\cdot\hat{v})\hat{v} + \gamma t'\vec{v} \ . \end{gathered}$$

Since the transformations mix together $\vec{r}$ and t, it seems profitable to create a 4-dimensional vector, usually referred to simply as a “4-vector,” which then transforms between the two frames via a so-called *“boost matrix”*, denoted by $\Lambda(\vec{v})$. We denote the location-and-time 4-vector by $\tilde{r}$, and write the following, where as usual, the superscript ^{T} means the transpose of the matrix.:

$$\begin{pmatrix} \vec{r} \\ ct \end{pmatrix} \equiv \widetilde{r} = \Lambda(\vec{v})\widetilde{r}\,' = \Lambda(\vec{v})\begin{pmatrix} \vec{r}\,' \\ ct' \end{pmatrix} \ ,$$

$$\begin{aligned} \Lambda(\vec{v}) &= \begin{pmatrix} I_3 + (\gamma-1)\hat{v}\hat{v}^T & \gamma\vec{v} \\ \gamma\vec{v}^T & \gamma \end{pmatrix} \\ &= \begin{pmatrix} 1+(\gamma-1)(\hat{v})_x(\hat{v})_x & (\gamma-1)(\hat{v})_x(\hat{v})_y & (\gamma-1)(\hat{v})_x(\hat{v})_z & \gamma\, v_x/c \\ (\gamma-1)(\hat{v})_x(\hat{v})_y & 1+(\gamma-1)(\hat{v})_y(\hat{v})_y & (\gamma-1)(\hat{v})_y(\hat{v})_z & \gamma\, v_y/c \\ (\gamma-1)(\hat{v})_x(\hat{v})_z & (\gamma-1)(\hat{v})_y(\hat{v})_z & 1+(\gamma-1)(\hat{v})_z(\hat{v})_z & \gamma\, v_z/c \\ \gamma\, v_x/c & \gamma\, v_y/c & \gamma\, v_z/c & \gamma \end{pmatrix} \end{aligned}$$

Now that we have these structure, however, we may create other 4-vectors, all of which transform in the same way, and may also use those transformations to determine the relation to the transformations of the ordinary 3-vector quantities. A very important quantity involved with their definitions is the so-called proper time, τ, which is a scalar quantity. We know that any two points on the world line of some observer are "time-like separated." Therefore, for any two such points, A and B, we define the difference of their proper time by

$$(\Delta\tau)^2_{AB} \equiv (\Delta t)^2_{AB} - (\Delta\vec{r})^2_{AB}/c^2 \ ,$$

where the sign is chosen so that $\Delta\tau_{AB} \equiv \tau_B - \tau_A$ is positive when B is to the future of A, along the worldline in question. It should be noted that this is just the time shown on the "wrist-watch" of the observer whose worldline this is. Obviously, this quantity is an invariant one, having the same value no matter which observer happens to measure it, i.e., happens to be making measurements on this particular worldline's behavior. As well, we see that one may label uniquely points on a given worldline by their value of τ, relative to some chosen origin and some chosen scale of units, i.e., seconds, years, etc. Therefore we may take derivatives along that worldline with respect to the proper time:

1. The velocity 4-vector is given by

$$\frac{d}{d\tau}\widetilde{r} \equiv \widetilde{u} = \gamma_u \begin{pmatrix} \vec{u} \\ c \end{pmatrix} \ , \quad \vec{u} \equiv \frac{d}{dt}\vec{r}\,, \quad \frac{dt}{d\tau} = \gamma_u \ .$$

2. For a particle of mass m, moving with velocity $\vec{u}$, the energy-momentum 4-vector is given by

$$m\widetilde{u} \equiv \widetilde{p} \equiv \begin{pmatrix} \vec{p} \\ E/c \end{pmatrix} = m\gamma_u \begin{pmatrix} \vec{u} \\ c \end{pmatrix} \ .$$

3. The acceleration 4-vector is then given by

$$\widetilde{a} \equiv \frac{d}{d\tau}\widetilde{u} = \frac{d^2}{d\tau^2}\widetilde{r} = \gamma_u^2 \begin{pmatrix} \vec{a} + \gamma_u^2(\vec{u}\cdot\vec{a})\vec{u}/c^2 \\ \gamma_u^2(\vec{u}\cdot\vec{a})/c \end{pmatrix} \ , \quad \vec{a} \equiv \frac{d}{dt}\vec{u} = \frac{d^2}{dt^2}\vec{r}\,.$$

4. The 4-vector force is defined so that (the appropriate generalization of) Newton's Second Law is still true, where we also recall that $dE/dt = \vec{F}\cdot\vec{u}$:

$$\widetilde{K} \equiv \frac{d}{d\tau}\widetilde{p}\,, \qquad\qquad \widetilde{K} = \gamma_u \begin{pmatrix} \vec{F} \\ \frac{1}{c}dE/dt \end{pmatrix} \ ,$$

Using these definitions, and the fact that each of them is a 4-vector and therefore transforms very simply by multiplication by $\Lambda(\vec{v})$, we may work out the Lorentz transformations of the associated 3-vectors, which are, in general, as expected, not very nice, except for the 3-momentum and energy/c, which transform exactly the same way as does the 3-location and c(time):

1. The 3-velocity, $\vec{u}$, and its associated function γ_u:

$$\vec{u}_\parallel = \frac{\vec{u}'_\parallel + \vec{v}}{1+\vec{v}\cdot\vec{u}'/c^2}\ ,\ \vec{u}_\perp = \frac{\gamma_v^{-1}\,\vec{u}'_\perp}{1+\vec{v}\cdot\vec{u}'/c^2}\ ,$$

$$\text{or}\ \ \vec{u} = \frac{\gamma_v^{-1}\vec{u}' + (1-\gamma_v^{-1})(\hat{v}\cdot\vec{u}')\hat{v} + \vec{v}}{1+\vec{v}\cdot\vec{u}'/c^2}\ ,$$

$$\text{and}\ \ \frac{1}{\sqrt{1-(u/c)^2}} \equiv \gamma_u = \gamma_v(1+\vec{v}\cdot\vec{u}'/c^2)\gamma_{u'} = \gamma_v\frac{(1+\vec{v}\cdot\vec{u}'/c^2)}{\sqrt{1-(u'/c)^2}}\ ,\ .$$

2. The 3-momentum, $\vec{p}$ and the energy E:

$$\vec{p}_\parallel = \gamma_v(\vec{p}'_\parallel + E'\vec{v}/c)\ ,\quad \vec{p}_\perp = \vec{p}'_\perp\ ,\quad E = \gamma_v(E' + \vec{v}\cdot\vec{p}')\ ,$$

$$\text{or}\ \ \vec{p} = \vec{p}' + (\gamma_v - 1)(\vec{p}'\cdot\hat{v})\hat{v} + \gamma_v E'\vec{v}/c^2\ .$$

3. The 3-acceleration, $\vec{a}$:

$$\vec{a}_\parallel = \frac{\gamma_v^{-3}}{(1+\vec{v}\cdot\vec{u}'/c^2)^3}\,\vec{a}'_\parallel\ ,\quad \vec{a}_\perp = \frac{\gamma_v^{-2}}{(1+\vec{v}\cdot\vec{u}'/c^2)^3}\left\{\vec{a}'_\perp + \vec{v}\times(\vec{a}'\times\vec{u}')/c^2\right\}$$

$$= \frac{1-(v/c)^2}{(1+\vec{v}\cdot\vec{u}'/c^2)^3}\left\{\vec{a}'_\perp - \frac{\vec{v}\cdot\vec{a}'/c}{1+\vec{v}\cdot\vec{u}'/c^2}\frac{\vec{u}'_\perp}{c}\right\}$$

$$\text{or}\ \ \vec{a} = \frac{\gamma^{-3}}{(1+\vec{v}\cdot\vec{u}'/c^2)^3}\left\{\vec{a}'_\parallel + \gamma\,\vec{a}'_\perp + \gamma\,\vec{v}\times(\vec{a}'\times\vec{u}')/c^2\right\}\ .$$

4. The 3-force, and its associated quantity $\dot{E} = dE/dt = \vec{F}\cdot\vec{u}$, when it is acting on an object with velocity $\vec{u}$:

$$\vec{F}_\parallel = \frac{\vec{F}'_\parallel + (\vec{u}'\cdot\vec{F}')\vec{v}/c^2}{1+\vec{v}\cdot\vec{u}'/c^2}\ ,\qquad \vec{F}_\perp = \frac{\gamma^{-1}\vec{F}'_\perp}{1+\vec{v}\cdot\vec{u}'/c^2}\ ,$$

$$\text{or}\quad \vec{F} = \frac{\gamma^{-1}\vec{F}' + (1-\gamma^{-1})(\hat{v}\cdot\vec{F}')\,\hat{v} + (\vec{u}'\cdot\vec{F}')\vec{v}/c^2}{1+\vec{v}\cdot\vec{u}'/c^2}\ ,$$

and a related set of equations, showing the true relationships between the 3-force and the 3-acceleration when objects are moving quite fast:

$$\vec{F} = \gamma_u m[\vec{a} + \gamma_u^2(\vec{u}\cdot\vec{a})\vec{u}/c^2]\ ,$$

$$\text{or}\ \ \vec{F}_{\parallel\vec{u}} = \gamma_u^3\,m\,\vec{a}_{\parallel\vec{u}}\ ,\quad \vec{F}_{\perp\vec{u}} = \gamma_u\,m\,\vec{a}_{\perp\vec{u}}\ .$$