Worldline for a Uniformly Accelerated Observer

I. Generalities concerning 4-acceleration

We are setting up to consider the trajectory of a brave young lady, who is going to go on a trip where the acceleration that she provides to her system is always constant, but non-zero. Ignoring the fact that this obviously will require large amounts of energy, we will see that this is quite an interesting trip. The majority of our measurements will be made by a standard observer, $\mathcal{O}$; the coordinates he uses will be called $\{x^\mu \,|\, \mu = 1,2,3,4\} \equiv \{x, y, z, ct\}$. In particular, his (timelike) worldline is simply the usual $\hat{t}$-axis.

For any other timelike curve, i.e., some physical person or object, we choose the parameter along that curve to be the proper time, τ, defined so that its tangent vector, $\widetilde{u}$, is such that $\widetilde{u}^2 = -c^2$; i.e., the definition of τ is such that for displacements along that curve, $(dx)^2 + (dy)^2 + (dz)^2 - (cdt)^2 \equiv -(cd\tau)^2$. Having this tangent vector, we may properly ask whether or not it varies along the curve; i.e., we may define the *acceleration 4-vector*, $\widetilde{a}$, and ask how it varies:

$$\widetilde{a} \quad \equiv \quad \frac{d}{d\tau}\widetilde{u} \,, \qquad \text{and } \ \widetilde{u} \ \text{ such that } \ u^\mu = \frac{dx^\mu}{d\tau} \ . \tag{1}$$

The derivative along the curve, of the normalization equation for $\widetilde{u}$, tells us that $\widetilde{a}$ is orthogonal to $\widetilde{u}$; since $\widetilde{u}$ is timelike, it must then be that $\widetilde{a}$ is spacelike:

$$\frac{d}{d\tau}\{\widetilde{u}^2 = -c^2\} \quad \Longrightarrow \quad \widetilde{a}\cdot\widetilde{u} = 0 \quad \Longrightarrow \quad \widetilde{a}^2 \equiv A^2 > 0 \ . \tag{2}$$

We are going to use this more general situation to describe the worldline of our brave lady, but will keep it more general for a little while yet. Since we need inertial observers to make valid observations, we must then also introduce the notion of a set of *instantaneously co-moving, inertial observers* associated to this arbitrary worldline, one for every value of time t, as measured by our standard observer $\mathcal{O}$. At any given time, t, the lady's worldline has some velocity $\vec{w}(t)$. We choose an inertial observer, $\mathcal{O}''(t)$ that is moving with that velocity, and therefore views her as at rest, at least instantaneously, at that single instant of time.

However, in the usual case, this inertial observer will notice that just a little bit later the lady's velocity is no longer zero; i.e., this observer will say that she has a non-zero acceleration. We wish to consider in particular that acceleration. Obviously just a little bit later, we will have to choose a different inertial observer, say $\mathcal{O}''(t + \delta t)$. When the particular value of t being considered is either arbitrary, we may just drop the extra indication of t-dependence and we refer to that particular instantaneously comoving, inertial observer by the symbol $\mathcal{O}''$. Since this comoving observer measures the lady's $\widetilde{u}$ to be just the 4-vector $(\vec{0}, c)$, the statement that $\widetilde{a}$ is perpendicular to $\widetilde{u}$ means that $a^4 = 0$, so that $A^2 = \widetilde{a}^2 = \vec{a}^2$. Therefore A has the physical interpretation as the magnitude of the 3-acceleration as measured in that frame, i.e., $A = |\vec{a}|_{\mathcal{O}''}$.

II. Motion with (Locally-Measured) Acceleration Uniform

I propose to give here two distinct derivations of the important equations that determine the worldline of a uniformly-accelerated observer. The purpose, of course, of giving two different derivations is to hope that one can learn different things from each one of them. The first derivation is reasonably quick and straightforward, and is copied from your text, Griffiths' book, Example 12.10, pp. 516-7. It simply begins by saying that we want to consider the behavior of a particle of mass m that is acted on by a constant force, F, acting only in a single direction, for simplicity of the argument. As well, suppose that this particle is found at rest at the origin; i.e., at time $t = 0$ it has location $z = 0$, and we want to determine $z = z(t)$, the subsequent motion of the particle. We begin from (our relativistic version of) Newton's equation, remembering that everything is simply in one dimension, which I am taking to be the $\hat{z}$-direction:

$$\frac{dp}{dt} = F \quad \Longrightarrow \qquad p = Ft \, ,$$

where we have been able to perform the integral easily since F is a constant, and we have also dropped the possible constant of integration since we have set up the initial condition that our particle is at rest at time $t = 0$. We now recall the relationship between the momentum and the ordinary velocity, insert it here, and resolve that equation for the velocity:

$$\frac{m\,u}{\sqrt{1-(u/c)^2}} = p = Ft \quad \Longrightarrow \qquad u = \frac{At}{\sqrt{1+(At/c)^2}}\ , \qquad A \equiv F/m\ .$$

Remembering that the velocity is just $u = dz/dt$, we can set up an integral to obtain the motion as a function of the time:

$$z(t) = \int_0^t dt\,u(t) = A\int_0^t \frac{t\,dt}{\sqrt{1+(At/c)^2}} = \frac{c^2}{A}\sqrt{1+(At/c)^2}\Big|_0^t = \frac{c^2}{A}\left[\sqrt{1+(At/c)^2}-1\right]\ .$$

Remember that the graph of the location versus time for a non-relativistic particle moving under a constant acceleration is just the standard formula you learned, for instance, when studying falling under gravity quite near the Earth's surface, and is given by a parabola. This curve is quite different, and is in fact a hyperbola. Compare this equation to the one derived below, presented at Eq. (5b); this one is just the one obtained there, but with the special choice of parameters corresponding to $t_0 = 0$ and $z_0 = 0$.

I now proceed to the more detailed and careful derivation that I originally created, which answers more questions and considers more special cases, but which is rather more complicated and, perhaps, difficult to follow. Hopefully, it will be easier to follow after having worked through the simplified version presented just above.

We may now consider the very important special case where the acceleration is uniform, i.e., constant in both magnitude and direction. This means that A is a constant. However, we also need it to be in a constant direction, at least as measured in the $\mathcal{O}''$ frame. We choose our axes so that this constant direction is $\hat{z}''$. In that case, the equations $\widetilde{a}^2 = A^2$ and $\widetilde{a}\cdot\widetilde{u} = 0$ and $\widetilde{u}^2 = -c^2$ determine a relation between the coordinates of $\widetilde{a}$ and $\widetilde{u}$, as measured back in $\mathcal{O}$:

$$0 = \widetilde{a}\cdot\widetilde{u} = a^z\,u^z - a^t\,u^t \quad \Longrightarrow \qquad a^t = \frac{u^z}{u^t}a^z\ ,$$

$$A^2 = \widetilde{a}^2 = (a^z)^2 - (a^t)^2 = (a^z)^2\left[1-\left(\frac{u^z}{u^t}\right)^2\right] = (a^z)^2\frac{(u^t)^2-(u^z)^2}{(u^t)^2} = \left(\frac{ca^z}{u^t}\right)^2\ .$$

We then note that we want $u^t > 0$ and also $A > 0$, so that only one of the square roots is appropriate; i.e., the above equation should be re-written as

$$A = \frac{a^z}{u^t} c \, .$$

Insertion of this relation into the first line above gives us the second of the following two simple equations, the solutions of which then determine the worldline we are attempting to discover:

$$\frac{du^z}{d\tau} \equiv a^z = A\, u^t/c \, , \quad \frac{du^t}{d\tau} \equiv a^t = A\, u^z/c \, . \tag{3}$$

Inserting $u^z = dz/d\tau$ and $u^t = c\, dt/d\tau$ into the right-hand sides of these equations causes them to be perfect derivatives, which are then immediately integrated to give the following, including two constants of integration, b, ℓ:

$$\frac{d\, z}{d\tau} = u^z = A\, t + b \, , \quad \frac{d\, ct}{d\tau} = u^t = A\, z/c + \ell \, . \tag{4}$$

We want to integrate this pair of equations, which, unfortunately, appear coupled in an undesirable way. However, by adding and subtracting them we acquire

$$\frac{d\,(z+ct)}{d\tau} = A\,(z+ct)/c + b + \ell \, , \qquad \frac{d\,(z-ct)}{d\tau} = -A(z-ct)/c + b - \ell \, , \tag{4$'$}$$

where the equations are now nicely decoupled decoupled, and their solutions give exponentials. When these are then resolved for z and t separately—instead of $z \pm ct$—we acquire the desired solutions for the coordinates of the worldline, parametrized by τ, her proper time:

$$z = z_0 + \frac{c^2}{A}(\cosh A\tau/c - 1) \quad ; \qquad t = t_0 + \frac{c}{A}\sinh A\tau/c \quad , \tag{5a}$$

where we have chosen reasonable names for the integration constants, related to the earlier ones by $z_0 = (c - \ell)c/A$ and $t_0 = -b/A$. As well we have used the last constant of integration to make the zero of τ, i.e., what one might have called τ_0, so that it divides motion inward toward the origin—negative 3-velocity—and motion outward away from the origin—positive 3-velocity, which amounts to saying that $(dz/d\tau)_{|\tau=0} = 0$, which symmetrizes the past and the future as measured by the accelerated observer's own clock. At that point, the special observer $\mathcal{O}$ sees the accelerated observer at his coordinates (z_0, t_0), and momentarily at rest. We can eliminate the parameter from the equations, to discover the "shape" of the curve in spacetime, which gives us the following formula, which is the graph of a hyperbola, symmetric about the z-direction, coming nearest to the origin at $z = z_0, t = t_0$, and asymptotic to light rays—45°-lines—emitted and absorbed at an event with coordinates $\{z = z_0 - \frac{1}{A}, t = t_0\}$, which we label as the event $\mathcal{P}$:

$$\left[z - z_0 + \frac{c^2}{A}\right]^2 - [c(t-t_0)]^2 = \frac{c^4}{A^2} \; ,$$
$$\text{or} \;\; z - z_0 = \pm\frac{c^2}{A}\left[\sqrt{1 + \left(\frac{A(t-t_0)}{c}\right)^2} - 1\right] \; . \tag{5b}$$

Notice that we may now differentiate these equations to have explicit presentations of the world velocity and the world acceleration as functions of the proper time:

$$\begin{aligned} u^z &= c\sinh(A\tau/c) \; , \;\; u^t = c\cosh(A\tau/c) \quad \Longrightarrow \quad v = \tanh(A\tau/c) \; ; \\ a^z &= A\cosh(A\tau/c) \; , \;\; a^t = A\sinh(A\tau/c) \; . \end{aligned} \tag{5c}$$

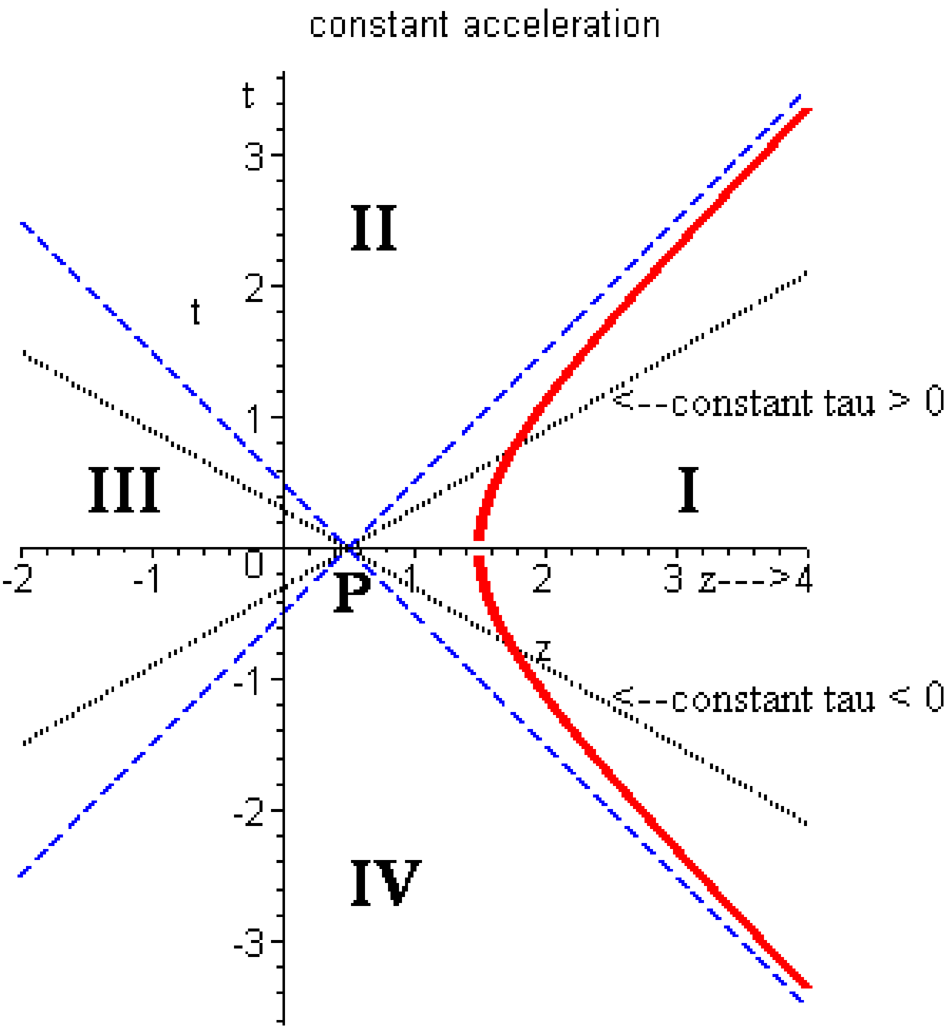
To get a "feel" for these quantities, consider, for example, an observer who accelerates at $A = g = 9.8$ m/sec^2, for ten of her years:

$$A\tau/c = 10.323 \; ; \qquad v = 1 - 2.2 \times 10^{-9} \; ;$$
$$z - z_0 = 1.40 \times 10^{20} \text{ meters } = 14,700 \text{ light-yrs}, \qquad t - t_0 = 14,730 \text{ years.}$$

It is reasonable to choose our accelerating observer as the co-moving observer for the particular time when he is observed by $\mathcal{O}$ to be at rest. This corresponds to choosing t to be zero when τ is zero, i.e., to choose the constant $t_0 = 0$. Since all motion is in the z-direction, it is also reasonable to suppose that the accelerated observer can maintain constant the directions of his $\hat{x}'$ and $\hat{y}'$ axes; we refer to this as arranging to have **a non-rotating frame**. On the other hand, her timelike basis vector, $\hat{t}'$ changes as his speed increases; therefore, her $\hat{z}'$-direction changes as well, being then a function of τ. The basis vector $\hat{z}'$ must be that direction which, along with $\hat{x}'$ and $\hat{y}'$, spans the 3-plane of simultaneity at each value of τ, i.e., the 3-plane perpendicular to $\hat{t}'$. Therefore, $\hat{z}'$ must be in the same direction as the 4-acceleration, $\widetilde{a}$, since it is perpendicular to $\widetilde{u}$, and our case is the one where $\widetilde{a}$ and $\widetilde{u}$ have no components in the x, y-plane, which is equal to the x', y'-plane. Since both are spacelike, and $|\widetilde{a}| = A$ while $|\hat{z}'| = +1$, we require that $\hat{z}' = (1/A)\widetilde{a}$. The trajectory equations, Eqs. (5c), then give us the following relationships:

$$\begin{aligned} \hat{t}' &= \frac{1}{c}\widetilde{u} = \sinh(A\tau/c)\,\hat{z} + \cosh(A\tau/c)\,\hat{t}\ , \\ \hat{z}' &= \frac{1}{A}\widetilde{a} = \cosh(A\tau/c)\,\hat{z} + \sinh(A\tau/c)\,\hat{t} \quad . \end{aligned} \tag{6}$$

On a 2-dimensional graph, where the x- and y-directions are suppressed, as usual, the direction of $\hat{z}'$ tells us the direction of a "line of simultaneity" for $\mathcal{O}'$, just as the direction of $\hat{t}' = (\widetilde{u})/c$ tells us the direction (at that moment) of the worldline of $\mathcal{O}'$. Continuing to view things in terms of the usual, Minkowski diagram for $\mathcal{O}$, consistently with the statement that the velocity is $v = c\tanh(A\tau/c)$, then the slope of the worldline is just $c/v = \coth A\tau/c$, while the slope of the line of simultaneity, for that τ, is just $v/c = \tanh A\tau/c$. These lines of simultaneity, i.e. lines of constant τ, are then simply straight lines with slope $\tanh A\tau/c$, in $\mathcal{O}$'s Minkowski diagram, all going through the same event $z = z_0 - c^2/A, t = 0$, which we have labelled P. Therefore we may write them as the straight lines defined by the equation $ct = (\tanh A\tau/c)\,(z - z_0 + c^2/A)$.



As the accelerated observer moves the lines of simultaneity rotate "upward" as τ increases from zero, and "downward" as it decreases. The light cone at $\mathcal{P}$, two parts of which are the asymptotes of our accelerated observer's hyperbolic motion, separates spacetime into four quadrants. Quadrant **I** is the right-hand spacelike one, which contains the trajectory of our accelerated observer; the future timelike one is labelled **II**, the past timelike one is **III**, and the left-hand spacelike one is **IV**.

Before continuing to try to understand the physics involved in all this, let's agree to make some simplifying numerical assignments. We have already taken t_0 to be zero; therefore, let's also choose $z_0 = c^2/A$, and then agree to measure everything in units of inverse acceleration, which amounts to also choosing the acceleration, $A = 1$. **Notice** that this chooses the special point P, to be the (common) origin of all our observers, except for the accelerated observer himself, whose origin is now at the point at which he was observed to be at rest, namely at $\mathcal{O}$'s coordinates $(1, 0)$.

We then pick an arbitrary event **in quadrant I**, with coordinates $\{x, y, z, t\}$, as measured by $\mathcal{O}$. It corresponds to some specific, positive proper time, τ, corresponding to the line of simultaneity of $\mathcal{O}'$ that passes through this event, and also consider the co-moving observer, $\mathcal{O}''(\tau)$ for which the accelerated observer is (momentarily) at rest when he observe this event. [Notice that since every specific value of τ, the lady's "wrist-watch time," corresponds to a specific value of t, we may also label the various co-moving observers by the value of τ instead of t.] To understand $\mathcal{O}''(\tau)$ in more detail, we first note that the event at which $\mathcal{O}$ measures our accelerating observer to be instantaneously at rest, i.e., $(1, 0)$ in her coordinates, is in our co-moving observer's past, relative to when he meets with the observer at rest, $\mathcal{O}$. In fact, from the co-moving observer's point of view, his origin, i.e., the event of coordinating his origin with $\mathcal{O}$, is simultaneous with his observation that the accelerating observer is, momentarily, at rest. To see that this last statement is so, let us note that the co-moving observer is, always, moving with velocity $v = c \tanh(A\tau_0/c)$, in the positive direction. As well, let's agree to reduce our considerations to one spatial and one temporal dimension, momentarily ignoring the x- and y-directions, in which nothing interesting is happening. Therefore, his worldline, in $\mathcal{O}$'s coordinates, is given by $t = z/v = \coth(A\tau_0/c)\, z/c$, while his line of simultaneity for the origin consists of all those points along the line $t = v\, z/c^2 = \tanh(A\tau_0/c)\, z/c$. However, when the accelerating observer has velocity v, and therefore $\tau = \tau_0$, his coordinates are $z = (c^2/A)\cosh(A\tau_0)$ and $t = (c/A)\sinh(A\tau_0)$, which do indeed lie on the line of simultaneity just described above. We can then ask, further, what are the co-moving observer's measurements for the accelerated observer at that time when he is at rest. We have just learned above that the time in question is $t'' = 0$. To determine the distance, we have several possible approaches.

i.) We could notice that one often uses calibration hyperbolae to determine measurements on different worldlines: The relativistic interval is invariant with respect to different observers. Therefore, in particular, the hyperbola $z^2 - (ct)^2 = 1$ is the same as the hyperbola $(z'')^2 - (ct'')^2 = 1$. This, however, is the hyperbola on which our accelerating observer is travelling; moreover, when $\mathcal{O}$ observes him to be at rest, he is at $z = 1$, consistent with that hyperbola. Therefore, when $\mathcal{O}''(\tau_0)$ observes him at $t'' = 0$, at rest, he will also be at $z'' = 1$.

ii.) We could use the Lorentz transformation that converts measurements of $\mathcal{O}$ to those of $\mathcal{O}''(\tau_0)$:

$$\begin{pmatrix} z'' \\ t'' \end{pmatrix} = \begin{pmatrix} \cosh(A\tau_0) & -\sinh(A\tau_0) \\ -\sinh(A\tau_0) & \cosh(A\tau_0) \end{pmatrix} \begin{pmatrix} \cosh(A\tau_0) \\ \sinh(A\tau_0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} .$$

The figure just below is quite useful to help visualize some of this. The (yellow) line through the origin at 45° is a fiduciary light ray, just to help us keep track of slopes; however it is also the asymptote to our accelerated observer's worldline, which is the (thick, red) hyperbola leaving $z = 1$ at time $t = 0$. The uppermost (black) straight line through the origin is the worldline for this particular co-moving observer that we have been discussing, while the (black) line parallel to that, but lower down so that it is tangent to the accelerated worldline, is some observation station of his, at a distance $z'' = 1$ away. That station makes measurements and notices that the accelerated observer passes here, instantaneously at rest, simultaneously with the coincidence of the origins of his frame and that of $\mathcal{O}$. Lastly, the other two straight lines are lines of simultaneity for our particular co-moving observer. The (blue) uppermost one goes through the origin and also the point where the co-moving observer sees the accelerated observer at rest. The (green) lower one goes through the event when the resting observer sees the accelerated one at rest, showing the intersection with $\mathcal{O}$'s worldline, in her past.

Since the points involved were arbitrary, this tells us that every line of simultaneity for our accelerating observer passes through the (common) origin of all our inertial observers! This is indeed a very important observation. [Technically a complete calculation would have shown that all the accelerating observer's lines of simultaneity go through the point with coordinates

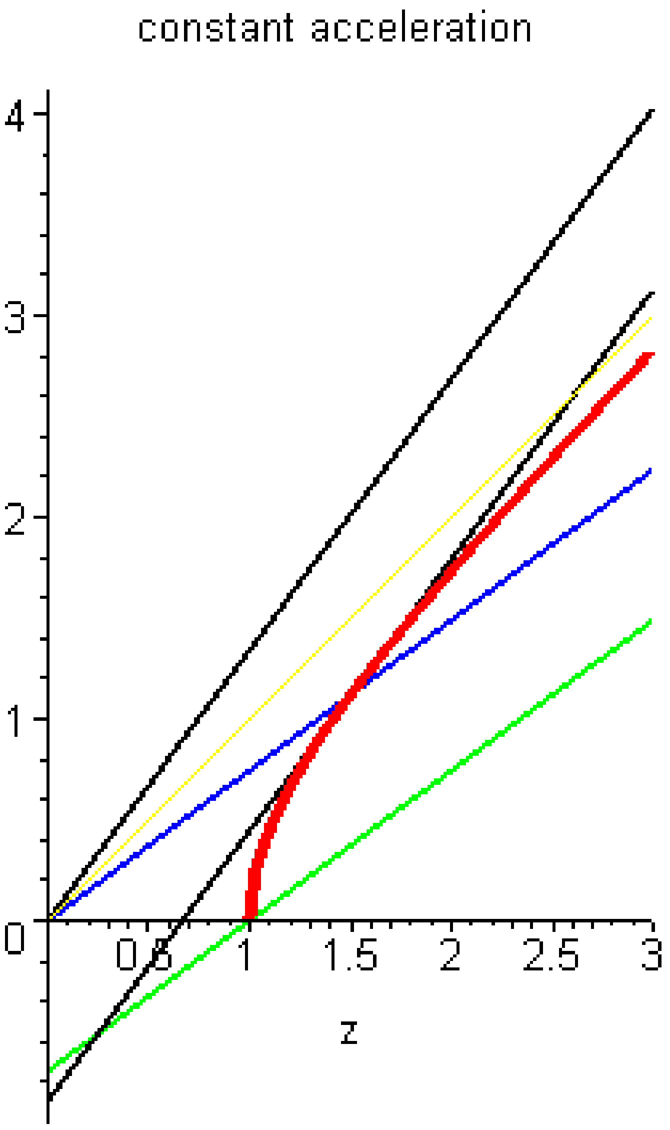
$(z_0 - c^2/A, 0)$, as also do its asymptotes; however, we have normalized this point to the origin by choice of z_0 and A.]

We now compare in detail the coordinates of an arbitrary event, at, say (z_1, t_1) as measured by $\mathcal{O}$, with the coordinates as measured by $\mathcal{O}'$. Of course, those coordinates should actually be the coordinates measured by the appropriate inertial observer that measures the accelerating observer to be at rest, except that those coordinates should be translated by the appropriate amount for the relationship between those two observers. At a proper time τ, the relation between measurements made by $\mathcal{O}'$ and $\mathcal{O}''(\tau)$ is given by

$$z' = z'' - 1 \ , \quad t' = t'' + \tau = \tau \ ,$$

where the last equality comes because, as we have seen, t'' is always zero, i.e., $\mathcal{O}''(\tau)$ always observes the accelerating observer simultaneously with his meeting with the observer at rest, $\mathcal{O}$. On the other hand, the relation between the coordinates of $\mathcal{O}$ and those of $\mathcal{O}''(\tau)$ is given by the appropriate Lorentz transformation:

$$\begin{aligned} z &= z'' \cosh(\tau) + t'' \sinh(\tau) = (z' + 1) \cosh(\tau) \ , \\ t &= t'' \cosh(\tau) + z'' \sinh(\tau) = (z' + 1) \sinh(\tau) \ . \end{aligned} \tag{6}$$



IV. The Neighborhood Where This Transformation is Valid

Before studying in detail the geometry described by these coordinates, we first consider over what neighborhood of spacetime they are valid. It is reasonable that they are not valid for the entirety of space-time, since there are points in spacetime that

(a) can never send information to $\mathcal{O}'$,

(b) some which can never receive information from $\mathcal{O}'$, and

(c) some that fit both of these descriptions. Mathematically, the difficulties with the transformation are of two sorts.

a. Firstly, since all the lines of simultaneity meet at the point $\mathcal{P}$, there is a singularity at that point in the determination of a value of τ for that event.

b. Secondly, for events in quadrants **II** and **III** there are no lines of simultaneity that pass through them; i.e., there are no solutions to the equation that determines the value of τ.

c. Thirdly, for events in quadrant **IV**, as the value of t increases the associated value of τ would decrease, so that the normal ordering given by the progression of time is reversed. This suggests that we have already passed into an unacceptable region. Only **I** is correctly treated by these coordinates.

Note that an equivalent, more elementary approach to the problem is obtained by considering the light ray sent out by $\mathcal{O}$ from her origin. As that light ray is asymptotic to our accelerated observer's worldline, it will intersect that worldline only after infinite proper time. Therefore, no information acquired by $\mathcal{O}$ at any positive times can ever be communicated to the accelerated observer. There are large parts of the spacetime as viewed by $\mathcal{O}$ that he can never observe. We can refer to this particular light ray as **a horizon for our observer**, since it has the property that no light rays can ever pass through it and be received by our accelerated observer.

An approach toward a more general statement, concerning the range of validity of the coordinates of an arbitrarily accelerated observer would lead us to some sort of an approximate statement, at least, that they may not be valid over a neighborhood larger than "of size c^2/A." An interesting additional test one could make would be to consider an observer who moves always at constant velocity **except** for some small time period during which he accelerates from one constant velocity to another. The two associated inertial coordinate systems are locally well-defined, but overlap at larger distances, of the order of c^2/A, in such a way that events outside some neighborhood have two, differing sets of coordinates. If we were to try to ascribe to our semi-accelerated observer some single set of coordinates that covered all of spacetime, it is clear that we would be led into contradictions.