25 September, 2003

When is a Manifold Curved: Covariant Derivatives and Curvature

I. Action of "directional derivatives" on Vectors: an Affine Connection

As one moves on a manifold, along a curve with tangent vector $\tilde{u}$, we write the derivative, in that direction, of a scalar function, $f \in \boldsymbol{\mathcal{F}}$, as $\tilde{u}(f) = u^i \, f_{,i}$. We now want to generalize this operator to ask how vectors, and other sorts of tensors, vary as we move along that same curve on the manifold. The problem is complicated because we have already chosen a basis field, $\{\widetilde{e}_\mu\}\big|_P$ in each of the vector spaces $\boldsymbol{\mathcal{T}}_P$ above the points, P, on the manifold. However, we have no real requirements on "the way these basis vectors point," in "nearby" vector spaces, except that they change in a continuous (smooth) way. Therefore the physics-related problem lies in ways to specify their change. This job is accomplished by a mapping called either "the covariant derivative" or "the affine connection." (Calling it the covariant derivative reminds us that we want some generalization of the derivative with the property that the rate of change of a tensor is again a tensor, i.e., it should vary covariantly. On the other hand, in general an affine relation is one that relates two geometric entities that can be smoothly "translated" one into the other, via a change in origin, which is surely what is going on when one moves from one vector space to another one.)

The following are reasonable requirements for an operator to be called the **covariant derivative of a vector in some direction, specified by another vector**, which reduces to our earlier notion of directional derivative when it acts only on functions:

$$\nabla \; : \; \boldsymbol{\mathcal{T}} \otimes \boldsymbol{\mathcal{T}} \longrightarrow \; \boldsymbol{\mathcal{T}} \quad , \tag{1.1}$$

$\nabla_{\tilde{u}}\widetilde{v}\big|_P \longleftarrow$ the (covariant) derivative, at the point P, of $\widetilde{v}$ in the direction $\widetilde{u}$.

(a) linear for addition: $\nabla_{\tilde{u}}(\widetilde{v} + \widetilde{w}) = \nabla_{\tilde{u}}\widetilde{v} \; + \; \nabla_{\tilde{u}}\widetilde{w} \quad ,$

(b) a derivation for products in the upper argument: $\nabla_{\tilde{u}}(f\widetilde{v}) = f\nabla_{\tilde{u}}\widetilde{v} + [\tilde{u}(f)]\,\widetilde{v} \quad ,$

(c) purely linear in the first (lower) argument: $\nabla_{f\tilde{u}+\tilde{w}}\widetilde{v} = f\nabla_{\tilde{u}}\widetilde{v} + \nabla_{\tilde{w}}\widetilde{v} \quad .$

Since this object is linear in its first argument, we could consider the quantity $\nabla \widetilde{v}$ as a [1,1]-tensor since, it is awaiting one more vector—which is the role of a 1-form—in order to give back the vector desired. Therefore, we can say that $\nabla \widetilde{v}$ is an element of $\Lambda \otimes \boldsymbol{\mathcal{T}}$, and re-write the important part of Eqs. (1.1) above—the part dealing with its behavior as a derivation—in the following simple form:

$$\nabla(f\widetilde{v}) = f\,\nabla \widetilde{v} \;+\; df \otimes \widetilde{v} \tag{1.2}$$

Since these requirements are insufficient to uniquely determine the covariant derivative, we need a way to further specify them. This is usually done by first specifying the behavior of the covariant derivative of the elements of a given basis set, $\{\widetilde{e}_\mu\}$. The directional derivative of a basis vector is again a tangent vector; therefore, it must be a linear combination of the original basis vectors. Depending on one's point of view, and noting that that derivative depends tensorially (i.e., linear with respect to both addition and scalar multiplication) on the direction in question, we may write down the defining equations for that derivative in several different ways:

$$\nabla \widetilde{e}_\mu = \underset{\sim}{\Gamma}{}^\lambda{}_\mu \otimes \widetilde{e}_\lambda \quad \in \Lambda \otimes \boldsymbol{\mathcal{T}},$$

or

$$\nabla_{\tilde{u}} \widetilde{e}_\mu = \{\underset{\sim}{\Gamma}{}^\lambda{}_\mu(\widetilde{u})\}\,\widetilde{e}_\lambda\ ,$$

or

$$\nabla_{\tilde{e}_\nu} \widetilde{e}_\mu \;\equiv\; \nabla_\nu \widetilde{e}_\mu \equiv \Gamma^\lambda{}_{\mu\nu}\,\widetilde{e}_\lambda \quad , \tag{1.3}$$

so that

$$\underset{\sim}{\Gamma}{}^\lambda{}_\nu(\widetilde{e}_\mu) = \Gamma^\lambda{}_{\nu\mu}\ , \text{or}\ \ \underset{\sim}{\Gamma}{}^\lambda{}_\mu \;= \Gamma^\lambda{}_{\mu\nu}\underset{\sim}{\omega}{}^\nu \quad ,$$

where the $\{\underset{\sim}{\omega}{}^\nu\}$ are a basis of 1-forms, **and** the 1-forms $\underset{\sim}{\Gamma}{}^\lambda{}_\mu$ are a collection of n^2 1-forms, collectively referred to as *"the connection 1-forms,"* originally introduced by Cartan. (Be sure and note the order of the two lower indices!)

The usual approach to specifying an affine connection is to give rules by which one determines the values of these 1-forms. Given these coefficients we can use linearity and the fact that the operator in question is a derivation to give us a complete formula for an arbitrary vector field, $\widetilde{v} \in \boldsymbol{\mathfrak{T}}$ and an arbitrary directional field, $\widetilde{u} \in \boldsymbol{\mathfrak{T}}$:

$$\nabla \widetilde{v} = \widetilde{e}_\lambda \otimes \left\{d\, v^\lambda + \underset{\sim}{\Gamma}^\lambda{}_\mu\, v^\mu\right\} = \widetilde{e}_\lambda \otimes \underset{\sim}{\omega}^\nu \left\{v^\lambda_{,\nu} + \Gamma^\lambda{}_{\mu\nu} v^\mu\right\} \;\equiv\; \widetilde{e}_\lambda \otimes \underset{\sim}{\omega}^\nu \{v^\lambda_{;\nu}\} \quad , \tag{1.4a}$$

or, for given $\widetilde{u}$

$$\nabla_{\tilde{u}}\tilde{v} = \tilde{e}_\lambda\{\tilde{u}(v^\lambda) + v^\nu \underset{\sim}{\Gamma}^\lambda{}_\nu(\tilde{u})\} = \tilde{e}_\lambda\, u^\nu\{v^\lambda{}_{,\nu} + \Gamma^\lambda{}_{\mu\nu} v^\mu\} \equiv \tilde{e}_\lambda\, u^\nu\, v^\lambda{}_{;\nu}\ , \tag{1.4b}$$

and the two subscripted symbols $v^\lambda_{,\nu}$ and $v^\lambda_{;\nu}$ are common abbreviations:

$$v^\lambda_{,\nu} \equiv \widetilde{e}_\nu(v^\lambda)\ , \qquad v^\lambda_{;\nu} \equiv v^\lambda_{,\nu} \;+\; \Gamma^\lambda{}_{\mu\nu}\, v^\mu\ . \tag{1.4c}$$

The first abbreviation, with the "comma," is simply a generalization of the usual symbol for partial derivatives so that it denotes the action of **any** basis of tangent vectors on functions, even though the basis vectors are no longer holonomic; the second abbreviation, with the "semi-colon," is referred to as "the components of the covariant derivative of the vector $\widetilde{v}$ in the direction specified by the ν-th basis vector, $\widetilde{e}_\nu$. When the v^λ are the components of a $[1,0]$ tensor, then the $v^\lambda{}_{;\nu}$ are the components of a $[1,1]$ tensor, as was originally desired.

II. Action of the covariant derivative on Differential Forms and other Tensors

We may extend this definition to also act on 1-forms by **requiring the covariant derivative to commute with the operation of 1-forms on tangent vectors**. Since the action of a 1-form on a tangent vector, say $\underset{\sim}{\alpha}(\widetilde{v})$, is a function, and we already know how to calculate directional derivatives of functions, we may write

$$\widetilde{u}\{\underset{\sim}{\alpha}(\widetilde{v})\} = \nabla_{\tilde{u}}\{\underset{\sim}{\alpha}(\widetilde{v})\} \;\equiv\; (\nabla_{\tilde{u}}\underset{\sim}{\alpha})\,(\widetilde{v}) + \underset{\sim}{\alpha}\,(\nabla_{\tilde{u}}\widetilde{v}) \quad . \tag{2.1}$$

Since every 1-form is a linear combination of the basis 1-forms, we now use Eq. (1.5) to determine the covariant derivative of the basis 1-forms by considering the special case when

$\widetilde{u} \to \widetilde{e}_\nu$, $\underset{\sim}{\alpha} \to \underset{\sim}{\omega}^\lambda$ and $\widetilde{v} \to \widetilde{e}_\mu$:

$$
\begin{aligned}
0 = \nabla_\nu \delta^\lambda_\mu = \nabla_\nu \left\{\underset{\sim}{\omega}^\lambda(\widetilde{e}_\mu)\right\} = \left\{\nabla_\nu \underset{\sim}{\omega}^\lambda\right\}(\widetilde{e}_\mu) + \underset{\sim}{\omega}^\lambda\left(\nabla_\nu \widetilde{e}_\mu\right) = \left\{\nabla_\nu \underset{\sim}{\omega}^\lambda\right\}(\widetilde{e}_\mu) + \Gamma^\lambda{}_{\mu\nu} \quad , \\
\Longrightarrow \left\{\nabla_\nu \underset{\sim}{\omega}^\lambda\right\}(\widetilde{e}_\mu) = -\Gamma^\lambda{}_{\mu\nu} \ , \qquad (2.2) \\
\Longrightarrow \ \nabla_\nu \underset{\sim}{\omega}^\lambda = -\Gamma^\lambda{}_{\mu\nu}\underset{\sim}{\omega}^\mu \quad \Rightarrow \ \nabla \underset{\sim}{\omega}^\lambda = -\underset{\sim}{\Gamma}^\lambda{}_\mu \otimes \underset{\sim}{\omega}^\mu \, .
\end{aligned}
$$

Therefore the covariant derivative of a general 1-form, $\underset{\sim}{\alpha} \equiv \alpha_\mu \underset{\sim}{\omega}^\mu$, may be written

$$
\begin{aligned}
\nabla_{\tilde{u}}\underset{\sim}{\alpha} = \left\{\widetilde{u}(\alpha_\mu) - \alpha_\lambda \underset{\sim}{\Gamma}^\lambda{}_\mu(\widetilde{u})\right\}\underset{\sim}{\omega}^\mu = u^\nu\left\{\alpha_{\mu,\nu} - \alpha_\lambda \Gamma^\lambda{}_{\mu\nu}\right\}\underset{\sim}{\omega}^\mu \equiv u^\nu\left\{\alpha_{\mu;\nu}\right\}\underset{\sim}{\omega}^\mu \, , \\
\text{or, if still awaiting a tangent vector for the direction,} \qquad (2.3) \\
\nabla\alpha = \left\{d\alpha_\mu - \alpha_\lambda \underset{\sim}{\Gamma}^\lambda{}_\mu\right\} \otimes \underset{\sim}{\omega}^\mu \ \equiv \ \underset{\sim}{\omega}^\nu \otimes \underset{\sim}{\omega}^\mu\{\alpha_{\mu;\nu}\} \ \equiv \ \underset{\sim}{\omega}^\nu \otimes \underset{\sim}{\omega}^\mu\{\alpha_{\mu,\nu} - \alpha_\lambda \Gamma^\lambda{}_{\mu\nu}\} \, .
\end{aligned}
$$

where we see that the "difference" between the action of the covariant derivative of 1-forms and tangent vectors amounts to a difference in sign, and summing on the upper index of $\underset{\sim}{\Gamma}^\lambda{}_\mu$ instead of the lower one.

We may now easily extend the action of the covariant derivative operator to spaces of arbitrary tensors by simply requiring that **it satisfy the product rule** when it is asked to act **on tensor products**. Therefore, for example, we have

$$
\nabla(\widetilde{u} \otimes \widetilde{v}) \equiv \nabla\widetilde{u} \otimes \widetilde{v} + \widetilde{u} \otimes \nabla\widetilde{v} \, . \qquad (2.4)
$$

This causes the covariant derivative of a tensor product of two tangent vectors to have two positive terms with connection 1-forms, $\underset{\sim}{\Gamma}^\lambda{}_\mu$, the covariant derivative of a tensor product of a single tangent vector and a single 1-form to have one positive and one negative "Γ-term." A reasonably generic tensor—chosen arbitrarily to be of type [1,3], as is the Riemann curvature tensor—would have the components of its covariant derivative as follows:

$$
R^a{}_{bcd;e} = R^a{}_{bcd,e} + \Gamma^a{}_{ge}R^g{}_{bcd} - \Gamma^g{}_{be}R^a{}_{gcd} - \Gamma^g{}_{ce}R^a{}_{bgd} - \Gamma^g{}_{de}R^a{}_{bcg} \ . \qquad (2.5)
$$

III. Parallel Displacements of Functions and of Vectors, and the notion of Geodesics

Directional derivatives answer the question, "What is the rate of change of a function as it is moved to different places on the manifold, in some particular direction, as specified by moving it along some given curve?" Covariant derivatives answer the same question for vectors, 1-forms, and more complicated tensors. The usual Taylor series expansion is of course invoked to do this. **The purpose of doing this is, eventually, to learn some properties of the underlying manifold itself.**

We first do such an expansion for ordinary (scalar) functions. For $P \in U \subseteq M$, $f \in \boldsymbol{\mathfrak{F}}_{|U}$, and $\Gamma(\lambda)$ a curve such that $\Gamma(0) = P$, the Taylor expansion for f gives

$$f[\Gamma(\lambda)] = f[\Gamma(0)] + \lambda \left\{ \frac{d}{d\lambda} f[\Gamma(\lambda)] \right\}\Big|_{\lambda=0} + \tfrac{1}{2}\lambda^2 \left\{ \frac{d^2}{d\lambda^2} f[\Gamma(\lambda)] \right\}\Big|_{\lambda=0} + \ldots \equiv \qquad e^{\lambda \widetilde{u}} f_{|P} \ , \tag{3.1}$$

where we interpret the exponential function as simply its entire power series. You might notice the perhaps-easier but very similar expansion for functions of one real variable,

$$e^{a\partial_x} f(x) = \ldots = f(x+a) \ . \tag{3.1a}$$

The covariant derivative allows us to do the same sorts of things for vectors, 1-forms, etc.; i.e., we can actually talk about comparing the values of a vector field within the vector spaces at different points. However, in this section, we set up some apparatus for moving around on curves and first consider the case of functions. This will lead us to the notion of the *torsion* of a manifold; in the next section we will do the same thing for vectors, which will lead us to the notion of the *curvature* of a manifold. Firstly, we say that a vector field, $\widetilde{v}$, at a point P, is **parallelly propagated** along a curve, $\Gamma(\lambda)$ with tangent vector $\widetilde{u} \equiv \frac{d}{d\lambda}\Gamma(\lambda)$, if its covariant derivative in the direction $\widetilde{u}$, evaluated at the point P, is exactly zero—i.e., if it's not changing at that point, in that direction:

■ Definition for parallel propagation of a vector $\tilde{v}$ along a direction $\widetilde{u}$:

$$\nabla_{\tilde{u}} \widetilde{v} = 0 \ . \tag{3.2}$$

■ Definition of a **geodesic**:

We promulgate a definition of "a straight line," as a (local piece of a) curve with the property that its direction is always the same; we will refer to such a curve, locally, as **a geodesic**. Since the direction cannot change, the covariant derivative of the tangent vector, parallelly propagated along itself, must always be proportional to itself: $\nabla_{\tilde{u}}\widetilde{u} = \phi\,\widetilde{u}$, where ϕ is some (scalar) function, of proportionality, defined in the neighborhood under scrutiny. However, given such an equation, it is simple to determine a new choice of parameter for the curve such that the new function ϕ is simply zero. We refer to such parameters as *affine parameters.* If some parametrization has a non-zero function $\phi(\lambda)$, we may find a "better choice," $s = s(\lambda)$, by solving the equation $\frac{d^2 s}{d\lambda^2} = \phi(\lambda)\frac{ds}{d\lambda}$. When everything is transformed to this new variable the function ϕ will have been transformed to zero. Considering the equation to determine s, we can see that one may still change the affine parameter to another one by finding a different solution of $d^2s/dt^2 = 0$. The solution of this is then straightforward, and says that all affine parameters are related one to another via an equation of the form $s' = a\,s + b$, where a and b are constants, i.e., they simply determine a "choice of zero" and a "choice of constant scale length." Therefore, modulo these two constant "choices," the affine parameter is uniquely determined.

IV. Tensorial Tools to Measure the lack of Flatness of a Manifold

Two important tensors give reasonably specific information concerning the deviations of the manifold away from being just a flat $\mathbb{R}^n$. These are referred to as the *torsion tensor*, $\mathbf{T}$, and the *curvature tensor*, $\mathbf{R}$. These tensors depend on our choice of an affine connection, independent of a metric. For this reason, no metric has yet been introduced into the space (or spacetime) being considered. We are however "building up" to the point where we will be able to recognize the physical interaction of the metric and the connection, which may then relate the derivatives of the metric and the curvature.

The standard, classical version of Einstein's general relativity spends time studying the curvature tensor, because it will be seen to be a local, covariant measure of those motions of test particles that we usually ascribe to "tidal gravitational fields"—those gravitational fields that vary from one point to another. On the other hand, this same point of view "assumes" that the torsion tensor must surely be just identically zero; i.e., we don't attempt to measure it, but, instead, define its existence because of prior "philosophical or metaphysical" knowledge, concerning, perhaps, the way that functions should behave. In several other, more complicated theories of gravity, where plausible interactions of the gravitational field with local spinorial matter fields—classic examples are due to Cartan and A. Trautman—the torsion tensor couples to whatever spinor fields that may exist in the matter of the system under study. Therefore we will at least spend a little bit of extra time describing both of these objects, and describing how to use them to look at the structure of the manifold and its various tensor bundles.

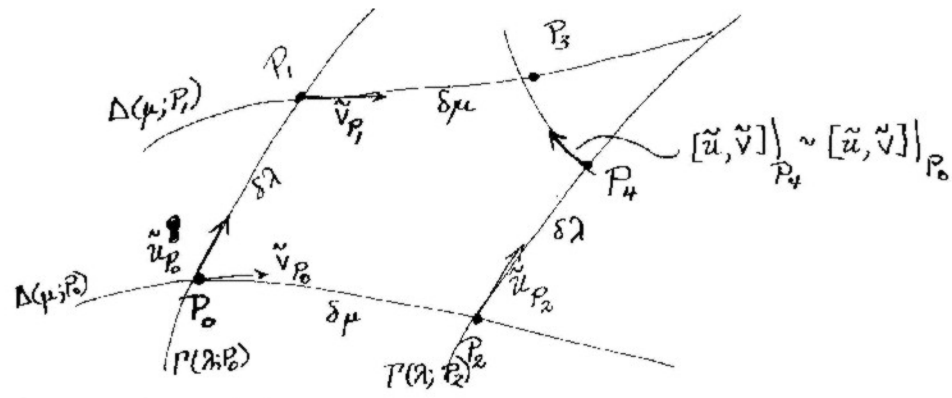
1. Preliminaries on a Geometrical Understanding of the Commutators of Vector Fields

A congruence of curves is a **continuous family** of ordinary curves, say $\Gamma_s(\lambda)$. The parameter λ varies **along** any individual curve, while the parameter s labels just which curve it is that we are considering. A trivial example is the set of curves over $\mathbb{R}^2$ given by $\Gamma_s(\lambda) = (\lambda, s) \in \mathbb{R}^2$. These are simply straight lines parallel to the $\hat{x}$-axis, in $\mathbb{R}^2$, intercepting the $\hat{y}$-axis at the value s.

Given two *congruences* of curves, we will use them to create small closed paths in some neighborhood and then to propagate functions, vectors, and other tensors around these paths. Beginning at a particular point, $P_0 \in M$, we can specify a pair of curves through P_0, and then another pair of curves, with the same functional form of the first two, i.e., with their tangent vectors simply the representatives at some other points of the two tangent vector fields that generated the first pair of curves. This process then gives us an area bounded by some four curves, the idea being to outline something like a small "rectangle." Unfortunately, it may well be that these curves are not completely parallel—in general one would only expect this of curves of coordinate axes; therefore, as described below in a bit more detail, it may be necessary to select a "special" additional curve that "closes up" our rectangle into a closed area. It is claimed that this additional curve is precisely the curve whose tangent vector is the *Lie commutator* of the tangent vectors of the two curves, namely the vector given by the following definition of its action on functions:

$$[\widetilde{u},\widetilde{v}]\,f \quad \equiv \quad \widetilde{u}[\widetilde{v}(f)] - \widetilde{v}[\widetilde{u}(f)] \quad . \tag{4.0}$$

We now want to give a proof of this fact, which will be needed for further discussions. Therefore, we begin by re-naming the members of a congruence of curves, labelling them by the point at which they begin. More precisely, instead of writing the curves as $\Gamma_s(\lambda)$, for some particular value of s, we will label the particular curve that "begins" at $P_i \in M$ by the label $\Gamma(\lambda; P_i)$. We may then consider the two congruences $\Gamma(\lambda; P_i)$, with tangent vector $\widetilde{u}$, and $\Delta(\mu; P_j)$, with tangent vector $\widetilde{v}$, with $\widetilde{v}_P$ not parallel to $\widetilde{u}_P$ for any P in some neighborhood, U. This is sufficient to describe the picture given just below. Everything begins at some arbitrary point, $P_0 \in U \subseteq M$. We then take members of the two different congruences that begin at that point, $\Gamma(\lambda; P_0)$ and $\Delta(\mu; P_0)$, and consider moving away from P_0 along them. Following along the first curve, originally in the direction $\widetilde{u}_{P_0}$, until the parameter has increased by some (small) amount, $\delta\lambda$, we come to some point $P_1 \in M$; contrariwise, if we follow along the direction $\widetilde{v}_{P_0}$ until the parameter increases by the value $\delta\mu$, we come to a different point $P_2 \in M$. Then, at P_1, we follow the direction $\widetilde{v}_{P_1}$ until the parameter increases by the value $\delta\mu$, arriving at P_3; likewise, beginning at P_2, we may follow the direction $\widetilde{u}_{P_2}$ for until the parameter increases by the value $\delta\lambda$, arriving eventually at P_4. At this point, one wonders, aloud, as to whether or not P_3 and P_4 are the same place on the manifold! Were the manifold flat, **and** were the two curves straight, then this would surely be the case. On the other hand, in the generic case, it is surely not so.



To answer this question on the manifold, we select an arbitrary function, and use the sketch above to aid our visualization of the following scheme: $f \in \mathfrak{F}_{|U}$ and use Taylor series to propagate it from P_0 to P_3, and **also** from P_0 to P_4:

$$\begin{aligned} f(P_1) =& f(P_0) + (\delta\lambda)(\widetilde{u}(f))_{|P_0} + \tfrac{1}{2}(\delta\lambda)^2(\widetilde{u}[\widetilde{u}(f)])_{|P_0} + \ \cdots \ , \\ f(P_2) =& f(P_0) + (\delta\mu)(\widetilde{v}(f))_{|P_0} + \tfrac{1}{2}(\delta\mu)^2(\widetilde{v}[\widetilde{v}(f)])_{|P_0} + \ \cdots \ , \end{aligned} \tag{4.1}$$

This is sufficient algebra to allow us to determine the expression we actually want, namely what is $f(P_3) - f(P_4)$:

$$\begin{aligned} f(P_3) - f(P_4) &= [f(P_3) - f(P_1)] + [f(P_1) - f(P_0)] - [f(P_4) - f(P_2)] - [f(P_2) - f(P_0)] \\ &= \quad \{\text{Taylor series at } P_1\}\Big|_{\text{series from } P_0} + \text{ Taylor series at } P_0 \\ &\quad - \{\text{ Taylor series at } P_2\}\Big|_{\text{series from } P_0} - \text{ Taylor series at } P_0 \\ &= \quad \ldots \quad = (\delta\lambda\,\delta\mu)(\widetilde{u}[\widetilde{v}(f)])_{|P_0} - (\widetilde{v}[\widetilde{u}(f)])_{|P_0} + \ \ldots \ \equiv \ (\delta\lambda\,\delta\mu)[\widetilde{u},\widetilde{v}]_{P_0}(f) + \ \ldots \ , \end{aligned} \tag{4.2}$$

where it was necessary to insert the two Taylor series from Eqs. (4.1) into those expansions in Eqs. (4.2), so that everything was then evaluated at the original point, P_0. We see that in fact the two points are not the same, and that it is exactly the Lie commutator of the two vector fields that gives us the difference of the values of the function at the two points; i.e., it is this commutator that measures the lack of "matching up" of the two sets of curves, thereby justifying Eq. (4.0). To rephrase, relative to the above picture, we describe the following two trajectories:

1) first follow the curve with tangent vector $\widetilde{u}_{P_0}$ for $\delta\lambda$, to P_1, and then $\widetilde{v}_{P_1}$ for $\delta\mu$, to P_3, this will be equivalent to the second trajectory, described as

2) first follow the curve with $\widetilde{v}_{P_0}$ for $\delta\mu$, to P_2, and then $\widetilde{u}_{P_2}$ for $\delta\lambda$, to P_4, **and then, further,** a curve with tangent vector $[\widetilde{u},\widetilde{v}]_{P_0}$ for parameter distance $\delta\lambda\,\delta\mu$, which will finally bring us to P_3.

Since both these paths take us from P_0 to point P_3, we may create **a closed path** beginning and ending at P_0, by first following, say, the second one above, and returning along the negative of the first one. Having such an explicit description of an arbitrary closed curve, we will begin "dragging" various geometric objects around these curves to find out what happens to them. Although, on the manifold itself, we actually get back to the beginning, it is **not** the case that functions, vectors, etc., also return to themselves.

2. A pair of Preliminary Analytic Results concerning Commutators

To give good proofs of these important results we need some useful mathematical expressions concerning the the commutator of two vector fields, and the action of the exterior derivative of a 1-form on such a commutator. As the derivations are somewhat tedious, I will present them here in smaller type, **but** will highlight the important resulting formulae by giving their equation numbers on the left-hand side.

To discover a simple analytic formula for the commutator, we first note that the function $\widetilde{u}[\widetilde{v}(f)]$ is of course not the result of any tangent vector acting on f, since it involves second derivatives. However, it is straightforward to show that the "Lie commutator" of two tangent vectors is just again some other tangent vector, which we may describe in the following way, relative to some, perhaps non-holonomic basis set, $\{\widetilde{e}_\mu\}$:

$$[\widetilde{u},\widetilde{v}] = \widetilde{u}(v^\mu)\,\widetilde{e}_\mu - \widetilde{v}(u^\mu)\,\widetilde{e}_\mu + u^\mu v^\nu [\widetilde{e}_\mu\,,\widetilde{e}_\nu], \text{ or}$$

(4.3)
$$[\widetilde{u},\widetilde{v}] = \{\widetilde{u}(v^\mu) - \widetilde{v}(u^\mu) + u^\rho v^\sigma\, C_{\rho\sigma}{}^\mu\}\,\widetilde{e}_\mu = \left\{u^\nu\, v^\mu{}_{,\nu} - v^\nu\, u^\mu{}_{,\nu} + u^\rho v^\sigma\, C_{\rho\sigma}{}^\mu\right\}\widetilde{e}_\mu\ ,$$

where we have used the definition of the commutator of two (anholonomic) basis vectors, and the prescription for subscripts with commas given in Eqs. (1.4c).

Our next analytic step is to uncover a link between the commutator of two vectors and the action of the exterior derivative, $d\underset{\approx}{\alpha} \in \Lambda^2$ on the same pair of vectors. We want to show that for any $\alpha \in \Lambda$, there is an action, on $\widetilde{u}, \widetilde{v} \in \mathbf{\mathcal{T}}$ given by

$$d\underset{\approx}{\alpha}(\widetilde{u}, \widetilde{v}) = \widetilde{u}[\underset{\approx}{\alpha}(\widetilde{v})] - \widetilde{v}[\underset{\approx}{\alpha}(\widetilde{u})] - \underset{\approx}{\alpha}[\widetilde{u}, \widetilde{v}] \quad . \tag{4.4}$$

Before proving this, notice that in the simplest case, where we take $\widetilde{u}$ and $\widetilde{v}$ to be just two members of a commuting basis set, say $\{\partial_i, \partial_j\}$, then the equation simply says that $d\underset{\approx}{\alpha}(\partial_i, \partial_j) = \partial_i[\alpha(\partial_j)] - \partial_j[\alpha(\partial_i)] = \partial_i\alpha_j - \partial_j\alpha_i = 2\partial_{[i}\alpha_{j]}$, just exactly as one would expect!

To proceed for a proof, we begin by expanding the right-hand side of the equation. We also do this for the special case that the basis set chosen is actually holonomic, i.e., the elements of the basis set all commute with each other; since the result is a completely tensorial equation, its validity will not depend upon any properties of a particular choice of basis, although the length of the proof will indeed be somewhat shorter as a result:

$$\begin{aligned}\widetilde{u}[\underset{\approx}{\alpha}(\widetilde{v})] - \widetilde{v}[\underset{\approx}{\alpha}(\widetilde{u})] - \underset{\approx}{\alpha}[\widetilde{u}, \widetilde{v}] &= u^\mu (\alpha_\nu\, v^\nu)_{\,,\mu} - v^\nu (\alpha_\mu\, u^\mu)_{\,,\nu} - \alpha_\nu\, u^\mu\, v^\nu{}_{,\mu} + \alpha_\mu\, v^\nu\, u^\mu{}_{,\nu} \\ &= u^\mu v^\nu \alpha_{\nu\,,\mu} - v^\nu u^\mu \alpha_{\mu\,,\nu} = 2\, u^\mu v^\nu\, \alpha_{[\nu\,,\mu]}\end{aligned}$$

However, we may now consider re-writing the left-hand side of our Eq. (4.3) as follows, using the basis of 1-forms, $\{\underset{\approx}{\omega}^\nu\}$:

$$\begin{aligned}d\underset{\approx}{\alpha}(\widetilde{u}, \widetilde{v}) &= \Big[d\,\{\alpha_\nu \underset{\approx}{\omega}^\nu\}\Big](\widetilde{u}, \widetilde{v}) = \Big[(\alpha_{\nu\,,\mu} \underset{\approx}{\omega}^\mu \wedge \underset{\approx}{\omega}^\nu\Big](\widetilde{u}, \widetilde{v}) = 2\,\alpha_{[\nu\,,\mu]}\Big[\underset{\approx}{\omega}^\mu \otimes \underset{\approx}{\omega}^\nu\Big](\widetilde{u}, \widetilde{v}) \\ &= 2\alpha_{[\nu\,,\mu]} \underset{\approx}{\omega}^\mu(\widetilde{u})\, \underset{\approx}{\omega}^\nu(\widetilde{v}) = 2\alpha_{[\nu\,,\mu]}\, u^\mu\, v^\nu \ .\end{aligned}$$

Both sides have now been brought to the same form; very important parts of the proof are the cancellation of two terms in the first set of equalities, and the skew symmetry imposed by the exterior derivative in the second set. We will be able to use this information to create simple and usable forms of the equations for both the torsion and the curvature tensors. A particularly useful application of this formula—as is often the case—is its application when all of the objects involved are (appropriate) basis vectors. Therefore, in particular, let us now re-write Eq. (4.4) for the case when we choose $\underset{\approx}{\alpha} \to \underset{\approx}{\omega}^\lambda$, $\widetilde{u} \to \widetilde{e}_\mu$, $\widetilde{v} \to \widetilde{e}_\nu$:

$$\begin{aligned} d\,\underset{\approx}{\omega}^\lambda(\widetilde{e}_\mu\,, \widetilde{e}_\nu) &= -\,\underset{\approx}{\omega}^\lambda([\widetilde{e}_\mu\,, \widetilde{e}_\nu]) = -C_{\mu\nu}{}^\lambda \\ \Longrightarrow \quad d\,\underset{\approx}{\omega}^\lambda &= -\tfrac{1}{2}\, C_{\mu\nu}{}^\lambda\, \underset{\approx}{\omega}^\mu \wedge \underset{\approx}{\omega}^\nu \ , \end{aligned} \tag{4.5}$$

where the last line gives us an explicitly-useful **expression for the exterior derivative of an arbitrary basis 1-form!**

3. The Torsion Tensor, $\mathbf{T}$

Following the analogy of the differences of our curves discussed above, we now define a related quantity that is equipped to deal with both functions and vectors, by virtue of involving the covariant derivative in its formulation. The **torsion tensor** is defined as the following operator, on two tangent vectors, giving a result which is a tangent vector, i.e., it is an element of $\Lambda^2 \otimes \mathfrak{T}$, or, if you prefer, a $[1,2]$ tensor:

$$\mathbf{T}(\widetilde{u},\widetilde{v}) \equiv (\nabla_{\tilde{u}}\widetilde{v} - \nabla_{\tilde{v}}\widetilde{u}) - [\widetilde{u}\,,\widetilde{v}] \quad . \tag{4.6}$$

To show that the torsion is actually a tensor of type [1,2] one needs to show that it just lets (scalar) functions pass through. Since this is not true of covariant derivatives, it is not obvious that it would be true for the torsion; however, the proof follows from our determination of what happens to functions when followed around a curve closed up by the commutator of its tangent vectors. More precisely, that calculation allows one to show the following:

$$\begin{aligned} &\mathbf{T}(\widetilde{u}, f\widetilde{v}) = f\mathbf{T}(\widetilde{u},\widetilde{v}) = \mathbf{T}(f\widetilde{u},\widetilde{v})\ , \\ &\text{or}\quad \Delta f\Big|_{P_0} = \delta\lambda\,\delta\mu\,\mathbf{T}(\tilde{u},\tilde{v})f\Big|_{P} + \text{terms cubic in } \delta\lambda \text{ and/or } \delta\mu, \end{aligned} \tag{4.6a}$$

where Δf as the difference between the two Taylor-series expansions of $f(P_3)$ determined by those two different routes. Since we probably still believe in "uniqueness" of Taylor series expansions, this of course gives us strong motivation for setting the Torsion tensor to zero. Nonetheless, let us study it a bit more.

Instead of applying the torsion to a function, if we simply write out the definition of $\mathbf{T}(\tilde{u},\tilde{v})$, one may get a general formula for it in terms of the connection coefficients, $\Gamma^j{}_{mn}$ and the commutation coefficients, $C_{mn}{}^j$:

$$\begin{aligned} \mathbf{T}(\tilde{u},\tilde{v}) &= \widetilde{e}_\mu\Big\{\widetilde{u}(v^\mu) + v^\nu\, \underset{\sim}{\Gamma}^\mu{}_\nu(\widetilde{u}) - \widetilde{v}(u^\mu) - u^\nu\, \underset{\sim}{\Gamma}^\mu{}_\nu(\widetilde{v}) - \widetilde{u}(v^\mu) + \widetilde{v}(u^\mu) - u^\lambda v^\nu C_{\lambda\nu}{}^\mu\Big\} \\ &= \widetilde{e}_\mu\Big\{v^\nu\, \underset{\sim}{\Gamma}^\mu{}_\nu(\widetilde{u}) - u^\nu\, \underset{\sim}{\Gamma}^\mu{}_\nu(\widetilde{v}) - u^\lambda v^\nu C_{\lambda\nu}{}^\mu\Big\} \\ &= -u^\mu v^\nu \widetilde{e}_\lambda\Big\{C_{\mu\nu}{}^\lambda - \Gamma^\lambda{}_{\nu\mu} + \Gamma^\lambda{}_{\mu\nu}\Big\}\ . \end{aligned} \tag{4.7}$$

However, the action of the tensor $\mathbf{T}$ on the two arbitrary vectors could also have just been written out explicitly, in terms of the components of the [1,2]-tensor; therefore we may conclude from the above that

$$\mathbf{T}(\widetilde{u},\widetilde{v}) \equiv\; u^\mu v^\nu \widetilde{e}_\lambda\, T^\lambda{}_{\mu\nu} \quad\Longrightarrow\qquad T^\lambda{}_{\mu\nu} + C_{\mu\nu}{}^\lambda \;+\; 2\Gamma^\lambda{}_{[\mu\nu]} = 0\ , \tag{4.8}$$

providing **a simple relation between the torsion, the commutation coefficients, and the (skew part of the) components of the connection.** One can also see that this also gives us a proof that $\mathbf{T}$ is indeed a tensor, although one surely could have given a much more direct proof.

There is still one more **(very useful)** variant of the action of the torsion tensor which is often referred to as Cartan's First Structure Equations. To see how it comes about, we return to Eq. (4.6) for $\mathbf{T}$ and expand that definition as follows:

$$\mathbf{T}(\widetilde{u},\widetilde{v}) = \widetilde{e}_\nu \left\{ \widetilde{u}(v^\nu) + v^\mu \underset{\sim}{\Gamma}{}^\nu{}_\mu(\widetilde{u}) - \widetilde{v}(u^\mu) - u^m u \underset{\sim}{\Gamma}{}^\nu{}_\mu(\widetilde{v}) - \underset{\sim}{\omega}{}^\nu([\widetilde{u},\widetilde{v}]) \right\} .$$

As we can always write $v^\nu = \underset{\sim}{\omega}{}^\nu(\widetilde{v})$, we can re-write the quantity in the large brace above, i.e., the coefficient of $\widetilde{e}_\nu$, as

$$\widetilde{u}[\underset{\sim}{\omega}{}^\nu(\widetilde{v})] - \widetilde{v}[\underset{\sim}{\omega}{}^\nu(\widetilde{u})] - \underset{\sim}{\omega}{}^\nu([\widetilde{u},\widetilde{v}]) + \underset{\sim}{\Gamma}{}^\nu{}_\mu(\widetilde{u})\, \underset{\sim}{\omega}{}^\nu(\widetilde{v}) - \underset{\sim}{\Gamma}{}^\nu{}_\mu(\widetilde{v})\, \underset{\sim}{\omega}{}^\nu(\widetilde{u})\ .$$

However, in the first three terms of the above, we recognize the right-hand side of the identity concerning exterior derivatives of 1-forms, given in Eq. (4.4), for the case that the 1-form $\underset{\sim}{\alpha}$, there, is chosen as a basis 1-form, $\underset{\sim}{\omega}{}^\nu$, so that we can replace those three terms by simply $d\underset{\sim}{\omega}{}^\nu(\widetilde{u},\widetilde{v})$. As well, we see that the last two terms of our current equation are simply the same pair of 1-forms acting first on $\widetilde{u},\widetilde{v}$, and then acting on $\widetilde{v},\widetilde{u}$, and with a minus sign between them; i.e., they are skew-symmetric in their action on this pair of tangent vectors, which is the hallmark of the action of a 2-form on a pair of vectors. Therefore, we may now put all this together as the following equations

$$\mathbf{T}(\widetilde{u},\widetilde{v}) = \widetilde{e}_\nu \left\{ d\underset{\sim}{\omega}{}^\nu(\widetilde{u},\widetilde{v}) + (\underset{\sim}{\Gamma}{}^\nu{}_\mu \wedge \underset{\sim}{\omega}{}^\mu)(\widetilde{u},\widetilde{v}) \right\} = \widetilde{e}_\nu \left\{ d\underset{\sim}{\omega}{}^\nu + \underset{\sim}{\Gamma}{}^\nu{}_\mu \wedge \underset{\sim}{\omega}{}^\mu \right\} (\widetilde{u},\widetilde{v})\ .$$

Since this is true for arbitrary tangent vectors, $\widetilde{u},\widetilde{v}$, we conclude that it must be true as an identity between tensors, of type [1,2], still awaiting those vectors to act upon.

It is this relationship which is originally due to Cartan:

Cartan's First Structure Equations

$$\mathbf{T} = \widetilde{e}_\lambda \otimes \left\{ d\underset{\sim}{\omega}^\lambda - \underset{\sim}{\omega}^\mu \wedge \underset{\sim}{\Gamma}^\lambda{}_\mu \right\}$$
$$\text{or} \quad \mathbf{T} = \widetilde{e}_\lambda \otimes \left\{ d\underset{\sim}{\omega}^\lambda - \Gamma^\lambda{}_{\mu\nu}\, \underset{\sim}{\omega}^\mu \wedge \underset{\sim}{\omega}^\nu \right\} = \widetilde{e}_\lambda \otimes \left\{ d\underset{\sim}{\omega}^\lambda - \Gamma^\lambda{}_{[\mu\nu]}\, \underset{\sim}{\omega}^\mu \wedge \underset{\sim}{\omega}^\nu \right\} \ . \tag{4.9a}$$

Inserting the form for $d\underset{\sim}{\omega}^\lambda$ from Eqs. (4.5), we may re-write the torsion tensor in yet one more way, also somewhat useful:

$$\mathbf{T} = -\widetilde{e}_\lambda \otimes \left\{ \tfrac{1}{2} C_{\mu\nu}{}^\lambda \ + \Gamma^\lambda{}_{[\mu\nu]} \right\} \underset{\sim}{\omega}^\mu \wedge \underset{\sim}{\omega}^\nu \ . \tag{4.9b}$$

4. The Curvature of an Affine Connection

The commutator of two tangent vectors gives us enough geometrical information to "close-up" an area bounded by two members of two congruences of curves, therefore creating a "closed loop" as the *boundary of that area*; the torsion measures what happens to the values of a function taken around a closed loop. However, *the curvature* tells us what happens to the values of a vector as it is taken around a closed loop.

As an operator, the (affine) curvature takes the following form,

$$\mathbf{R} : \mathcal{T} \otimes \mathcal{T} \otimes \mathcal{T} \longrightarrow \mathcal{T} \,, \qquad \mathbf{R}(\tilde{u}, \tilde{v}) \equiv \nabla_{\tilde{u}} \nabla_{\tilde{v}} - \nabla_{\tilde{v}} \nabla_{\tilde{u}} - \nabla_{[\tilde{u},\tilde{v}]} \quad , \tag{4.10}$$

and is an obvious extension, to act on vectors, of the torsion operator. On the other hand, also like the torsion tensor, it is a tensor, multilinear in all of its arguments. From that point of view, one may work through the definition above, and determine its components, as a [1,3]-tensor, relative to a chosen pair of (reciprocal) bases:

$$\mathbf{R} = \widetilde{e}_\lambda \otimes \underset{\sim}{\omega}^\mu \otimes \{ d\underset{\sim}{\Gamma}^\lambda{}_\mu + \underset{\sim}{\Gamma}^\lambda{}_\rho \wedge \underset{\sim}{\Gamma}^\rho{}_\mu \} \equiv \tfrac{1}{2} R^\lambda{}_{\mu\rho\sigma} \underset{\sim}{\omega}^\mu \otimes (\underset{\sim}{\omega}^\rho \wedge \underset{\sim}{\omega}^\sigma) \otimes \widetilde{e}_\lambda \,, \tag{4.11}$$

$$\tfrac{1}{2} R^\lambda{}_{\mu\rho\sigma} = \Gamma^\lambda{}_{\mu[\sigma\,,\rho]} + \Gamma^\nu{}_{\mu[\sigma} \Gamma^\lambda{}_{\nu\rho]} - \tfrac{1}{2} \Gamma^\lambda{}_{\mu\nu} C_{\rho\sigma}{}^\nu \quad , \tag{4.12}$$

$$\mathbf{R}(\widetilde{u}, \widetilde{v}) \widetilde{w} = 2 \widetilde{e}_\lambda \left(u^\rho v^\sigma\, w^\lambda{}_{;[\rho\sigma]} \right) = \widetilde{e}_\lambda \left(u^\rho v^\sigma\, R^\lambda{}_{\mu\rho\sigma} w^\mu \right) \ . \tag{4.13}$$

$$\Longrightarrow \ \text{the very useful} \ \ [\nabla_\sigma, \nabla_\rho] w^\lambda = w^\lambda{}_{;[\rho\sigma]} \ = \ \tfrac{1}{2} R^\lambda{}_{\mu\rho\sigma} w^\mu \qquad .$$

or it may be rephrased as **Cartan's Second Structural Equations**

$$\mathbf{R} \equiv \widetilde{e}_\lambda \otimes \underset{\sim}{\omega}^\mu \otimes \underset{\sim}{\Omega}^\lambda{}_\mu \quad \Longrightarrow \quad \underset{\sim}{\Omega}^\lambda{}_\mu \ = d\, \underset{\sim}{\Gamma}^\lambda{}_\mu + \underset{\sim}{\Gamma}^\lambda{}_\rho \wedge \underset{\sim}{\Gamma}^\rho{}_\mu \ . \tag{4.14}$$

The curvature components satisfy a number of useful identities:

First Bianchi Identities :

$$R^a{}_{bcd}\,\underset{\approx}{\omega}^b \wedge \underset{\approx}{\omega}^c \wedge \underset{\approx}{\omega}^d = \Omega^a{}_b \wedge \underset{\approx}{\omega}^b = d\{\underset{\approx}{\omega}^a(\mathbf{T})\} \underset{torsion-free}{\longrightarrow} 0\ , \tag{4.15a}$$

or in components $R^a{}_{bcd} + R^a{}_{cdb} + R^a{}_{dbc} = T^a{}_{cd,b} + T^a{}_{db,c} + T^a{}_{bc,d} \underset{torsion-free}{\longrightarrow} 0$, (4.15b)

Second Bianchi Identities :

$$D\underset{\approx}{\Omega}^a{}_b \quad \equiv \quad d\underset{\approx}{\Omega}^a{}_b + \underset{\approx}{\Gamma}^a{}_c \wedge \underset{\approx}{\Omega}^c{}_b - \underset{\approx}{\Gamma}^e{}_b \wedge \underset{\approx}{\Omega}^a{}_e = 0\ , \tag{4.16a}$$

$$\text{or in components} \qquad R^a{}_{bcd;e} + R^a{}_{bde;c} + R^a{}_{bec;d} = 0\ . \tag{4.16b}$$

V. (Pseudo)-Riemannian Geometry: Introduction of a Metric Tensor, g; Its Relation to the Affine Connection

1. Functions and Properties of a metric tensor, possibly indefinite

The metric is originally introduced as a (real-valued) measure for scalar products of tangent vectors with themselves; it should therefore be a tensor of type [0,2], i.e., an element of the space $\Lambda \otimes \Lambda$:

$$\mathbf{g} \in \Lambda^1 \otimes \Lambda^1 \text{ or } \mathbf{g} : \mathcal{T} \otimes \mathcal{T} \longrightarrow \boldsymbol{\mathfrak{F}} \equiv \Lambda^0 \ : \qquad \text{i.e., } \ \mathbf{g} = g_{\mu\nu}\,\underset{\approx}{\omega}^\mu \otimes \underset{\approx}{\omega}^\nu\ , \tag{5.1}$$

$$\text{and the definitions} \qquad g_{\mu\nu} = \mathbf{g}(\widetilde{e}_\mu\,, \widetilde{e}_\nu) \implies \mathbf{g}(\widetilde{u}, \widetilde{v}) \ = \ g_{\mu\nu}\,u^\mu\,v^\nu\ . \tag{5.2}$$

It is very useful to often view the set of components $g_{\mu\nu}$ as constituting the elements of a matrix, $G \equiv ((g_{\mu\nu}))$, that of course describes the metric relative to the (previously-chosen) basis of 1-forms.

In almost all cases, it is desirable that this matrix be invertible, so that the matrix G^{-1} exists; it is customary to denote its elements by the symbol $g^{\mu\nu}$; i.e., $G^{-1} = ((g^{\mu\nu}))$, and to treat it as the components of a tensor of type [2,0], i.e., an element of $\mathfrak{T}\otimes\mathfrak{T}$, which can generate scalar products of 1-forms.

$$
\begin{gathered}
(G^{-1}G)^{\mu}{}_{\lambda} = g^{\mu\nu} g_{\lambda\nu} = \delta^{\mu}_{\lambda} = g_{\lambda\nu} g^{\nu\mu} = (GG^{-1})_{\lambda}{}^{\mu} \ , \\
\mathbf{g}^{-1} \in \mathfrak{T}\otimes\mathfrak{T} \text{ or } \ \mathbf{g^{-1}} : \Lambda^1 \otimes \Lambda^1 \longrightarrow \mathfrak{F} \equiv \Lambda^0 \text{ so that } \ \mathbf{g^{-1}}(\underset{\sim}{\alpha}, \underset{\sim}{\beta}) \equiv g^{\mu\nu} \alpha_\mu v_\beta \ .
\end{gathered}
\tag{5.3}
$$

Because we use the symbols $g^{\mu\nu}$ for the components of G^{-1}, it turns out that the components that one might think of as "the metric with one index up and one index down" are just the components of the identity matrix, i.e., $g^{\mu\nu}\, g_{\nu\lambda} = \delta^{\mu}_{\lambda}$.

The existence of an invertible metric induces various other important mappings, which map tangent vectors into 1-forms, and vice versa. These mappings are often referred to as "raising" and "lowering" of indices; thus, the existence of an invertible metric "obscures" the differences between the two kinds of vectors that we use, namely tangent vectors and 1-forms. We describe the most fundamental of these induced mappings below; where we take $\widetilde{u} = u^\mu \widetilde{e}_\mu$, $\widetilde{v} = v^\lambda \widetilde{e}_\lambda \in \mathfrak{T}$ and $\underset{\sim}{\alpha} = \alpha_\nu \underset{\sim}{\omega}^\nu$, $\underset{\sim}{\beta} = \beta_\rho \underset{\sim}{\omega}^\rho \in \Lambda^1$ as arbitrary tangent vectors, and 1-forms, respectively.

$$
\begin{gathered}
\mathbf{g}_* : \mathfrak{T} \longrightarrow \Lambda^1 \text{ so that } \ \{\mathbf{g}_*(\widetilde{u})\}(\widetilde{v}) \equiv \mathbf{g}(\widetilde{u}, \widetilde{v}) \ , \text{ or } \ \mathbf{g}_*(\widetilde{u}) = (g_{\mu\nu} u^\nu) \underset{\sim}{\omega}^\nu \equiv u_\nu \, \underset{\sim}{\omega}^\nu \in \Lambda^1 \ , \\
\mathbf{g}^* : \Lambda^1 \longrightarrow \mathfrak{T} \text{ so that } \ \underset{\sim}{\beta}\{\mathbf{g}^*(\underset{\sim}{\alpha})\} \equiv \mathbf{g}(\underset{\sim}{\beta}, \underset{\sim}{\alpha}) \ , \text{ or } \ \mathbf{g}^*(\underset{\sim}{\alpha}) = (g^{\mu\nu} \alpha_\mu) \tilde{e}_\nu \equiv \alpha^\nu \, \widetilde{e}_\nu \in \mathfrak{T} \quad .
\end{gathered}
\tag{5.4}
$$

This is then the general process of "raising" and "lowering" indices on a vector and on a 1-form. It should be clear that the process can easily be extended to operate on any other sorts of tensors one might desire.

2. Determination of the Affine Connection

Having introduced a metric into our system, we may now use the affine connection that we already have to ask how that metric varies as we proceed from one vector space to some other nearby one, i.e., we should consider the type [0,3] tensor, $\nabla\mathbf{g}$. In the standard version of Einstein's theory of general relativity, one assumes that physical reasons cause us to require this tensor to be zero. For the moment, I will go ahead and simply give this extra tensor a name, and see how it would enter into our calculations, being aware that there are in fact alternative theories of gravity where it is presumed to be an interesting physical quantity. As before, since this is a minor variation, I will put this section in small print, and label important results on the left.

We define the set of 1-forms, referred to as the **"metrizability coefficients"**, $\underset{\sim}{Q}^{\mu\nu}$:

$$\nabla g^{\mu\nu} \equiv \underset{\sim}{Q}^{\mu\nu} = Q_\lambda{}^{\mu\nu}\,\underset{\sim}{\omega}^\lambda\,, \Longrightarrow \nabla g_{\mu\nu} = -\underset{\sim}{Q}_{\mu\nu}\,, \tag{5.5}$$

$$\Longrightarrow \quad g_{\mu\nu\,;\lambda} = \partial_\lambda g_{\mu\nu} - \Gamma^\eta{}_{\mu\lambda} g_{\eta\nu} - \Gamma^\eta{}_{\nu\lambda} g_{\mu\eta} = \, - \, Q_{\lambda\mu\nu} \tag{5.6a}$$

$$\text{or} \quad g_{\mu\nu\,,\lambda} = \Gamma_{\nu\mu\lambda} + \Gamma_{\mu\nu\lambda} - Q_{\lambda\mu\nu} = 2\Gamma_{(\nu\mu)\lambda} - Q_{\lambda\mu\nu} \tag{5.6b}$$

They are symmetric on their indices, since $g_{\mu\nu}$ is symmetric on its indices, and the other relations follow from the usual (inverse) relation for the metric, $g^{\mu\nu}g_{\nu\lambda} = \delta^\mu_\lambda$, and the expression for the covariant derivative in terms of the components of the (affine) connection.

This finally defines all the necessary geometrical quantities needed to determine an equation that gives the components of the connection in terms of physically more relevant quantities; i.e., we may now contemplate justifying our "choice" for the connection 1-forms, $\Gamma^\lambda{}_{\mu\nu}$. They can be determined in terms of

(1) the ordinary derivatives of the components of the metric, i.e., $g_{\mu\nu\,,\lambda}$,

(2) the metrizability coefficients, $Q_{\lambda\mu\nu}$,

(3) the torsion coefficients $T^\lambda{}_{\mu\nu}$, and

(4) the commutativity coefficients, $C_{\mu\nu}{}^\lambda$.

Since these various objects come, *a priori*, with indices in different sorts of places, it is most useful to first use the metric to "raise and/or lower" indices so that we have all of them on the same level. We now suppose that this has been done—with the ordering of the indices **definitely unchanged** by this process. The algebraic process for solving for the connection coefficients is begun by re-writing our last equation, Eq. (5.6), three times, one under the other, each time permuting the names of the indices (cyclicly), and multiplying the last copy by -1:

$$g_{\mu\nu\,,\lambda} = \Gamma_{\nu\mu\lambda} + \Gamma_{\mu\nu\lambda} - Q_{\lambda\mu\nu}$$

$$g_{\nu\lambda\,,\mu} = \Gamma_{\lambda\nu\mu} + \Gamma_{\nu\lambda\mu} - Q_{\mu\nu\lambda}$$

$$-g_{\lambda\mu\,,\nu} = -\Gamma_{\mu\lambda\nu} - \Gamma_{\lambda\mu\nu} + Q_{\nu\lambda\mu}$$

Adding these three equations gives the rather lengthy, but quite useful, equation

$$
\begin{aligned}
g_{\mu\nu}{}_{,\lambda} + & g_{\nu\lambda}{}_{,\mu} - g_{\lambda\mu}{}_{,\nu} \\
&= 2\Gamma_{\nu(\mu\lambda)} + 2\Gamma_{\mu[\nu\lambda]} + 2\Gamma_{\lambda[\nu\mu]} + Q_{\nu\lambda\mu} - Q_{\mu\nu\lambda} - Q_{\lambda\mu\nu} \\
&= 2\Gamma_{\nu\mu\lambda} - 2\Gamma_{\nu[\mu\lambda]} + 2\Gamma_{\mu[\nu\lambda]} + 2\Gamma_{\lambda[\nu\mu]} + Q_{\nu\lambda\mu} - Q_{\mu\nu\lambda} - Q_{\lambda\mu\nu}\ .
\end{aligned} \tag{5.7}
$$

This equation can be solved for the desired connection coefficients, namely $\Gamma_{\nu\mu\lambda}$, provided we know their skew part. However, one form of Cartan's first structure equations, Eqs. (4.8), gives that part of the connection coefficients in terms of the torsion and the commutativity coefficients. It is sufficiently important that I now re-write it here, with all indices lowered:

$$
2\Gamma_{\lambda[\mu\nu]} = -T_{\lambda\mu\nu} - C_{\mu\nu\lambda}\ . \tag{5.8}
$$

At this point the algebra in question is clearly straight-forward, if lengthy, and gives the following result:

$$
\begin{aligned}
\Gamma_{\mu\nu\lambda} &= \tfrac{1}{2}(-g_{\nu\lambda}{}_{,\mu} + g_{\mu\nu}{}_{,\lambda} + g_{\mu\lambda}{}_{,\nu}) \\
&\quad + \tfrac{1}{2}(-Q_{\mu\nu\lambda} + Q_{\lambda\mu\nu} + Q_{\nu\mu\lambda}) \\
&\quad\quad + \tfrac{1}{2}(-T_{\mu\nu\lambda} + T_{\lambda\mu\nu} + T_{\nu\mu\lambda}) \\
&\quad\quad\quad + \tfrac{1}{2}(-C_{\nu\lambda\mu} + C_{\mu\nu\lambda} + C_{\mu\lambda\nu})\ ,
\end{aligned} \tag{5.9}
$$

where the four distinct entries in the equation come from the four distinct sorts of geometric contributions already mentioned, above. Of course, once we have the connection coefficients, they can be used to create the connection 1-forms, $\underset{\sim}{\Gamma}{}^{\lambda}{}_{\mu}$, and, from there, the curvature 2-forms, $\underset{\sim}{\Omega}{}^{\lambda}{}_{\mu}$, or the (equivalent) Riemann curvature tensor components, $R^{\lambda}{}_{\mu\rho\sigma}$, from Eqs. (4.11). Our "choice" for the connection coefficients, which connect quantities in "adjoining" vector spaces, has now been rephrased in terms of more physical quantities.

3. the Levi-Civita Connection

It is unlikely that one would ever want to use all parts of the general formula for an affine connection, given in Eqs. (5.9). Nonetheless, one may use different portions of it in different places. We will now concentrate **only** on the version that corresponds to Einstein's "official" theory of general relativity, which uses a **metric-compatible, torsion-free connection**; i.e., Einstein's general relativity assumes explicitly that the metrizability coefficients and the torsion components are exactly zero!

We have already discussed why it is plausible, at least, to set the torsion tensor to zero. At this point, let me note two very important things that happen, to our notation, **when we set the torsion tensor to zero**: This causes **Cartan's First Structure Equations**, Eqs. (4.9), to have useful content. They now tell us explicitly a relationship between the components of the affine connection and the commutation coefficients that describe the lack of commutativity of the (non-holonomic) basis we are using for tangent vectors:

$$
\begin{aligned}
d\underset{\sim}{\omega}^\lambda = \underset{\sim}{\omega}^\mu \, \underset{\sim}{\Gamma}^\lambda{}_\mu \ &= \Gamma^\lambda{}_{\mu\nu} \, \underset{\sim}{\omega}^\mu \wedge \underset{\sim}{\omega}^\nu = \Gamma^\lambda{}_{[\mu\nu]} \, \underset{\sim}{\omega}^\mu \wedge \underset{\sim}{\omega}^\nu \\
\Longrightarrow \quad \Gamma^\lambda{}_{[\mu\nu]} &= -\tfrac{1}{2} C_{\mu\nu}{}^\lambda \ ,
\end{aligned}
\tag{5.10}
$$

where the second line should be compared with Eqs. (4.9b), and clearly follows from there once we have set the torsion (tensor) to zero.

There is however an additional, very pleasant thing that happens to the notation when the torsion vanishes, since we now have an explicit relation between the commutation coefficients and the skew-symmetric part of the connection 1-forms. We consider the exterior derivative of a 1-form, and also a 2-form, in an arbitrary basis (while an arbitrary p-form follows in exactly the analogous way):

$$
\begin{aligned}
d\underset{\sim}{\alpha} =& (d\alpha_\mu) \wedge \underset{\sim}{\omega}^\mu + \alpha_\mu \, d\underset{\sim}{\omega}^\mu = (d\alpha_\nu - \alpha_\mu \underset{\sim}{\Gamma}^\mu{}_\nu) \ \wedge \underset{\sim}{\omega}^\nu \\
&= (\alpha_{\nu,\lambda} - \alpha_\mu \, \Gamma^\mu{}_{\nu\lambda}) \, \underset{\sim}{\omega}^\lambda \wedge \underset{\sim}{\omega}^\nu = (\alpha_{\nu;\lambda}) \, \underset{\sim}{\omega}^\lambda \wedge \underset{\sim}{\omega}^\nu \ ; \\
d\underset{\sim}{\beta} =& (d\beta_{\mu\nu} \wedge \underset{\sim}{\omega}^\mu \wedge \underset{\sim}{\omega}^\nu + \beta_{\mu\nu} \, d\underset{\sim}{\omega}^\mu \wedge \underset{\sim}{\omega}^\nu - \beta_{\mu\nu} \, \underset{\sim}{\omega}^\mu \wedge d\underset{\sim}{\omega}^\nu = (d\beta_{\lambda\eta} - \beta_{\mu\eta} \underset{\sim}{\Gamma}^\mu{}_\lambda - \beta_{\lambda\nu} \underset{\sim}{\Gamma}^\nu{}_\eta) \wedge \underset{\sim}{\omega}^\lambda \wedge \underset{\sim}{\omega}^\eta \\
&= (\beta_{\lambda\eta\sigma} - \beta_{\mu\eta} \Gamma^\mu{}_{\lambda\sigma} - \beta_{\lambda\nu} \Gamma^\nu_{\eta\sigma}) \underset{\sim}{\omega}^\sigma \wedge \underset{\sim}{\omega}^\lambda \wedge \underset{\sim}{\omega}^\eta = (\beta_{\lambda\eta;\sigma}) \underset{\sim}{\omega}^\sigma \wedge \underset{\sim}{\omega}^\lambda \wedge \underset{\sim}{\omega}^\eta \ .
\end{aligned}
$$

What this says is that

if we insert the components of the covariant derivative into an exterior derivative, of a p-form, then this automatically includes effects from the exterior derivatives of the basis forms.

We now want to note the "(physical) advantages" to setting the metrizability coefficients to zero. (The usual "language" for such a process is referred to as insisting that the connection to "metric compatible.") Metric compatibility has the great advantage that we don't have to worry particularly when raising and lowering indices, or when calculating scalar products. If, contrariwise, the connection were not metric compatible, then would have the following behavior for an "ordinary" scalar product as she moved it from one vector space to a nearby one, using the usual product rule:

$$\nabla_{\tilde{w}}\{\mathbf{g}(\widetilde{u},\widetilde{v})\} = \{\nabla_{\tilde{w}}\mathbf{g}\}(\widetilde{u},\widetilde{v}) + \mathbf{g}(\nabla_{\tilde{w}}\widetilde{u},\widetilde{v}) + \mathbf{g}(\widetilde{u},\nabla_{\tilde{w}}\widetilde{v}) \quad . \tag{5.5$'$}$$

The first term on the right-hand side of the equation is just the metrizability tensor; by setting it equal to zero, we arrange our theory so that the "scalar product," i.e., the metric, just commutes through the covariant derivative operation, so that it then seems to be the same—in functional form—at every point on our manifold, thereby making the physical meaning of the scalar product much clearer!

This particular choice of affine connection was actually first made by Levi-Civita, the person who first invented "tensor calculus," back in the latter parts of the 19th century; therefore it is usually referred to as **the Levi-Civita connection**. From now on, we will not worry further about any other connection. However, we need to consider, in some little detail, how to actually calculate the Levi-Civita connection for a given choice of (1) a metric, and (2) a basis of 1-forms (or of tangent vectors). Therefore we now re-write Eqs. (5.10) as specialized for the Levi-Civita connection:

the Levi-Civita Connection

$$\begin{aligned}\underset{\sim}{\Gamma}_{\mu\nu} \equiv g_{\mu\eta}\,\underset{\sim}{\Gamma}^{\eta}{}_{\nu} = \Gamma_{\mu\nu\lambda}\,\underset{\sim}{\omega}^{\lambda} = = \Big\{ &\tfrac{1}{2}(-g_{\nu\lambda\,,\mu} + g_{\mu\nu\,,\lambda} + g_{\mu\lambda\,,\nu}) \\ &+ \tfrac{1}{2}(C_{\lambda\nu\mu} + C_{\mu\nu\lambda} + C_{\mu\lambda\nu})\Big\}\,\underset{\sim}{\omega}^{\lambda}\ ,\end{aligned} \tag{5.10LC}$$

Also note that the triplet of terms involving derivatives of the metric is obviously symmetric under the interchange of the indices ν and λ, and therefore does **not contribute** to that part of the connection that is skew-symmetric on those indices. As well, $C_{\lambda\nu\mu}$ is skew-symmetric on those indices, so that this equation is completely consistent—as it surely must be—with Eqs. (5.10), as well as Eqs. (5.8) and Eqs. (4.9b).

The first triplet of terms, involving the partial derivatives of the components of the metric, comes with its own name, the **Christöffel symbol**; it is said to be of the *first kind* when all of its indices are lowered, and of the *second kind* when the "first" index is raised. The following notation is very common, although not quite universally used, for the two kinds of Christöffel symbols:

$$[\mu;\nu\lambda] \equiv \tfrac{1}{2}(-g_{\nu\lambda\,,\mu} + g_{\mu\nu\,,\lambda} + g_{\mu\lambda\,,\nu})$$

$$\left\{ {\mu \atop \nu\lambda} \right\} \equiv g^{\mu\eta}[\eta;\nu\lambda] = \tfrac{1}{2}\, g^{\mu\eta}(-g_{\nu\lambda\,,\eta} + g_{\eta\nu\,,\lambda} + g_{\eta\lambda\,,\nu}) \; .$$

By its definition, one easily notes that the Christöffel symbol is symmetric in its second pair of indices, implying of course similar symmetry for that part of the connection coefficients that are created from this contribution. Secondly, we recall that if one makes the choice to use a holonomic basis set for our vector spaces—one where the basis vectors for tangent vectors are just the ordinary partial derivatives of the coordinates—then the commutation coefficients $C_{\mu\nu\lambda}$ would vanish, and this last choice, of vector basis, would have now completely determined the connection as simply that given by the Christöffel symbols. This approach was once used in all books by physicists on this subject. It is still being used by Carroll, for example.

On the other hand, there is quite a different approach, originally invented by Elie Cartan, for the determination of the Levi-Civita connection that has become much more common and popular in these days, and is in fact used by most working relativists. In this mode, one chooses a non-holonomic basis set for the tangent vector spaces with the property that the components of the metric are constants, just as one might in flat space. (It is a characteristic of curved spaces that one may **not** require both that the metric coefficients be constant and that the basis set be holonomic!) Having made such a choice, of course the partial derivatives of the metric coefficients are all zero, thereby eliminating that "half" of the contributions to the connection. In this case all the contributions come from the commutation coefficients, causing the connection coefficients to have quite different symmetry properties. If we re-write the equation for the metrizability coefficients, Eqs. (5.5), one last time, we have the following completely general form for it:

$$\nabla g_{\mu\nu} \;=\; d\,g_{\mu\nu} - 2\underset{\sim}{\Gamma}^{\lambda}{}_{(\mu}\, g_{\lambda\nu)} = d\,g_{\mu\nu} - 2\underset{\sim}{\Gamma}_{(\nu\mu)}\;. \tag{5.5''}$$

Since the Levi-Civita connection is metric compatible, the left-hand side of this equation is zero, so that we have a very simple form for the symmetric part of the connection 1-forms:

$$\text{for metric compatible connection:} \qquad d\,g_{\mu\nu} = 2\underset{\sim}{\Gamma}_{(\mu\nu)}\;. \tag{5.11LC}$$

Therefore, having chosen a (non-holonomic) basis so that the metric components are constant, it becomes immediately apparent that the various connection 1-forms are skew-symmetric. In our 4-dimensional spacetime, this indicates that there are only 6 independent 1-forms to be determined. Looking at the defining equation for the curvature 2-forms, $\underset{\sim}{\Omega}_{\lambda\mu}$, we see that the same skew-symmetry condition applies there, as well, giving us only 6 independent curvature 2-forms, also! Therefore while the general Levi-Civita connection forms require considerable calculation, these two distinct modes of calculating them simplify the process greatly:

$$\textbf{a choice of } \{\tilde{e}_a\} \textbf{ as} \quad \begin{cases} \text{holonomic} \Rightarrow \Gamma_{abc} = \Gamma_{a(bc)}, & \text{since } C_{ab}{}^{c} \equiv 0, \\ \qquad \text{for } \tfrac{1}{2}N^2(N+1) \underset{\text{for } N=4}{\longrightarrow} & \text{40 independent components,} \\ & \\ \text{a tetrad} \Rightarrow \Gamma_{abc} = \Gamma_{[ab]c}, & \text{since } g_{ab,c} \equiv 0, \\ \qquad \text{for } \tfrac{1}{2}N^2(N-1) \underset{\text{for } N=4}{\longrightarrow} & \text{24 independent components} \end{cases} \tag{5.12}$$

I almost always use the approach that uses a non-holonomic choice of vector basis with constant metric coefficients since this not only makes an understanding of vector components more intuitive but also reduces the number of independent connection and curvature coefficients that are needed. Workers in the field usually characterize that case by using the special word **tetrad** to indicate a choice of non-holonomic basis set such that the components of the metric are constant. While any constant choice would satisfy the criteria for being a tetrad, it turns out there are really only two further choices that occupy much space in the research literature on the 4-dimensional space-times for general relativity, which I now mention.

a. **Orthonormal tetrads** are those that are commonly used by true, physical observers, since they are the "obvious" generalization of the simplest sort of basis set in our ordinary 3-dimensional space; therefore, they are often referred to in the literature as "physical basis sets, " or "physical tetrads." For our choice of signature, such a tetrad would appear as follows:

$$\begin{aligned} \text{an orthonormal tetrad}: \ \mathbf{g} &= \underset{\approx}{\omega}^1 \otimes \underset{\approx}{\omega}^1 + \underset{\approx}{\omega}^2 \otimes \underset{\approx}{\omega}^2 + \underset{\approx}{\omega}^3 \otimes \underset{\approx}{\omega}^3 - \underset{\approx}{\omega}^4 \otimes \underset{\approx}{\omega}^4 \\ \text{or} \ \mathbf{g} &\approx (\underset{\approx}{\omega}^1)^2 + (\underset{\approx}{\omega}^2)^2 + (\underset{\approx}{\omega}^3)^2 - (\underset{\approx}{\omega}^4)^2 \,, \\ \text{with } ((g_{\mu\nu})) &\Longrightarrow \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} , \end{aligned} \tag{5.13}$$

where the symbol $\approx$ is used to "copy" a very common mode of writing in the literature that doesn't "bother" to write the actual tensor product symbols, and "presumes" that the metric is symmetric. More precisely, when considering a metric tensor, which is actually an element of $\Lambda^1 \otimes \Lambda^1$, it is very common to just write $dx\, dy$, when what is really meant is $\frac{1}{2}\{dx \otimes dy + dy \otimes dx\}$. This form of writing is "sloppy" but very common, and of course convenient.

b. **Null tetrads** A different form for a constant tetrad comes from the fact that one often studies physical fields moving with the speed of light, such as electromagnetic or gravitational radiation. These have directions associated with them which are "null," i.e., of zero length; therefore it is also quite common to use a system of 4 null vectors as a choice of tetrad. In special relativity, it is easy to see, for instance, that $\hat{z} \pm \hat{t}$ constitute a pair of (linearly-independent) null vectors that describe "light rays" either outgoing and incoming along the $\hat{z}$-direction. On the other hand, another linearly-independent pair of null rays does not exist; nonetheless, one should never let a simple bothersome fact like nonexistence deter one from doing what "needs to be done." Therefore, the standard approach to resolving this difficulty is to introduce complex, null-length basis vectors in, for instance, the plane of the wave-front. Again, in special relativity, this would correspond to the pair of basis vectors, $\hat{x} \pm i\hat{y}$. I note that it is "somewhat" customary to use the symbols $\{\underset{\sim}{\theta}^\alpha\}_1^4$ for the elements of a null basis, and also $\nu_{\alpha\beta}$ for the components of a metric made from a null tetrad, just as it is customary to use the symbols $\eta_{\mu\nu}$ for the components of a metric made from an orthonormal tetrad:

$$\begin{gathered}
\text{a null tetrad}: \ \mathbf{g} \equiv \boldsymbol{\nu} = \underset{\sim}{\theta}^1 \otimes \underset{\sim}{\theta}^2 + \underset{\sim}{\theta}^2 \otimes \underset{\sim}{\theta}^1 + \underset{\sim}{\theta}^3 \otimes \underset{\sim}{\theta}^4 + \underset{\sim}{\theta}^4 \otimes \underset{\sim}{\theta}^3 \\
\text{or} \ \ \boldsymbol{\nu} \ \approx \ 2\,\underset{\sim}{\theta}^1 \underset{\sim}{\theta}^2 + 2\,\underset{\sim}{\theta}^3 \underset{\sim}{\theta}^4 \ , \\
\text{with } ((\nu_{\alpha\beta})) \Longrightarrow \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}
\end{gathered} \tag{5.14}$$

(Other relative signs are used in the literature as well.)

It is worth going ahead just a bit more with this null tetrad. Taking $\{\underset{\sim}{\omega}^\mu\}_1^4$ as a representative orthonormal tetrad, and $\{\underset{\sim}{\theta}^\alpha\}_1^4$ as the associated null tetrad, we may write explicitly the transformation matrix between them:

$$\begin{gathered}
\underset{\sim}{\theta}^\alpha = A^\alpha{}_\mu \, \underset{\sim}{\omega}^\nu \ , \\
\begin{pmatrix} \underset{\sim}{\theta}^1 \\ \underset{\sim}{\theta}^2 \\ \underset{\sim}{\theta}^3 \\ \underset{\sim}{\theta}^4 \end{pmatrix} = \begin{pmatrix} +1 & +i & 0 & 0 \\ +1 & -i & 0 & 0 \\ 0 & 0 & +1 & -1 \\ 0 & 0 & +1 & +1 \end{pmatrix} \begin{pmatrix} \underset{\sim}{\omega}^1 \\ \underset{\sim}{\omega}^2 \\ \underset{\sim}{\omega}^3 \\ \underset{\sim}{\omega}^4 \end{pmatrix} \ .
\end{gathered} \tag{5.15}$$

The matrix A then has determinant $-i$, consistent with the fact that the determinant of $\nu_{\alpha\beta}$ is $+1$ while the determinant of $\eta_{\mu\nu}$ is -1. We could then show that the value of the metric, in this basis is indeed $\nu_{\alpha\beta}$ as stated above, by using the inverse of this matrix A to transform H into N, i.e., the matrix

$$\nu_{\alpha\beta} = (A^{-1})^\mu{}_\alpha (A^{-1})^\nu{}_\beta \, \eta_{\mu\nu} \ . \tag{5.16}$$

This allows us to determine the volume form tensor (or Levi-Civita tensor) for this basis:

$$\eta^{\alpha\beta\gamma\delta} = A^{\alpha}{}_{\mu} A^{\beta}{}_{\nu} A^{\gamma}{}_{\lambda} A^{\delta}{}_{\eta}\, \epsilon^{\mu\nu\lambda\eta} \quad \Longrightarrow \quad \eta^{1234} = -i = \eta_{1234}\ . \tag{5.17}$$

As this tensor is, more often than not, involved in the calculations of (Hodge) duals, let us use it, now, to re-calculate the duals of the basis of 2-forms in this particular basis set. I first remind you what it is in an orthonormal basis, $\{\underset{\sim}{\omega}^{\mu}\}_1^4$:

$$\Lambda^2 \leftrightarrow \Lambda^2 : * \begin{pmatrix} \underset{\sim}{\omega}^1 \wedge \underset{\sim}{\omega}^2 \\ \underset{\sim}{\omega}^2 \wedge \underset{\sim}{\omega}^3 \\ \underset{\sim}{\omega}^3 \wedge \underset{\sim}{\omega}^1 \end{pmatrix} \quad = \quad -i \begin{pmatrix} \underset{\sim}{\omega}^3 \wedge \underset{\sim}{\omega}^4 \\ \underset{\sim}{\omega}^1 \wedge \underset{\sim}{\omega}^4 \\ \underset{\sim}{\omega}^2 \wedge \underset{\sim}{\omega}^4 \end{pmatrix}\ . \tag{5.18}$$

Then I note the following for our null tetrad basis:

$$\Lambda^2 \leftrightarrow \Lambda^2 : * \begin{pmatrix} \underset{\sim}{\omega}^1 \wedge \underset{\sim}{\omega}^2 \\ \underset{\sim}{\omega}^2 \wedge \underset{\sim}{\omega}^3 \\ \underset{\sim}{\omega}^3 \wedge \underset{\sim}{\omega}^1 \\ \underset{\sim}{\omega}^1 \wedge \underset{\sim}{\omega}^4 \\ \underset{\sim}{\omega}^2 \wedge \underset{\sim}{\omega}^4 \end{pmatrix} \quad = \quad \begin{pmatrix} -\underset{\sim}{\omega}^3 \wedge \underset{\sim}{\omega}^4 \\ +\underset{\sim}{\omega}^2 \wedge \underset{\sim}{\omega}^3 \\ -\underset{\sim}{\omega}^3 \wedge \underset{\sim}{\omega}^1 \\ +\underset{\sim}{\omega}^1 \wedge \underset{\sim}{\omega}^4 \\ -\underset{\sim}{\omega}^2 \wedge \underset{\sim}{\omega}^4 \end{pmatrix}\ . \tag{5.19}$$

One can see that this allows us to very easily pick out the (two) subspaces of Λ^2 that correspond to *self-dual* and *anti-self-dual* 2-forms, and describe a basis set for them:

$$\begin{aligned} \underset{SD}{\Lambda^2} : \quad \text{basis is} \quad & \begin{pmatrix} 2\,\underset{\sim}{\omega}^2 \wedge \underset{\sim}{\omega}^3 \\ \underset{\sim}{\omega}^1 \wedge \underset{\sim}{\omega}^2 - \underset{\sim}{\omega}^3 \wedge \underset{\sim}{\omega}^4 \\ 2\,\underset{\sim}{\omega}^1 \wedge \underset{\sim}{\omega}^4 \end{pmatrix}\ , \\ \underset{aSD}{\Lambda^2} : \quad \text{basis is} \quad & \begin{pmatrix} 2\,\underset{\sim}{\omega}^1 \wedge \underset{\sim}{\omega}^3 \\ \underset{\sim}{\omega}^1 \wedge \underset{\sim}{\omega}^2 + \underset{\sim}{\omega}^3 \wedge \underset{\sim}{\omega}^4 \\ 2\,\underset{\sim}{\omega}^2 \wedge \underset{\sim}{\omega}^4 \end{pmatrix}\ , \end{aligned} \tag{5.20}$$

There is no particular obvious reason at the moment for the factors of 2, but they do correspond to something useful later on.