Some Notes for Lie algebras

0. A one-parameter subgroup of a continuous group is a mapping from a single real variable, into the group, say $Z : \mathbb{R} \to G$, so that $Z(t)$ is a group element for every (appropriate) value of t. We can think of the subgroup as a "curve" or "path" within the group itself, i.e., a 1-dimensional (continuous) subset. It is usual to parametrize the path so that it passes through the identity; one may then take the value of t at that point to be zero.

0a.) An example is the set of all (3-dimensional) rotations about a fixed axis, $\hat{n}$, $\{R(\hat{n};\theta) \mid 0 \le \theta < 2\pi\}$.

1. The generator of the one-parameter subgroup, $Z(t)$, is the tangent vector to the curve, at the identity; it therefore measures how fast the points on the curve are changing. A formula for the generator is then given by the following, where the exponential map is noted as the function which allows the generator to do its "generating":

$$\zeta \equiv \frac{d}{dt} Z(t)\Big|_{t=0} , \qquad Z(t) = e^{t\zeta} .$$

1a. A quite common "notation" for the generator of a simple, one-parameter subgroup is simply d/dt, where the symbol for that parameter is t.

1b. The simplest example is the case of translations acting on functions of one variable. The group of translations of that variable, say x, may be presented as shown below:

$$\{f(x) \to f(x+a) \mid a \in \mathbb{R}\} \quad \Longrightarrow \quad \{e^{a\frac{d}{dx}} f(x) = f(x+a) \mid a \in \mathbb{R}\} .$$

2. For an arbitrary Lie group, i.e., continuous group, with, say, n degrees of freedom, there would be n distinct, independent one-parameter subgroups passing through the origin, called, say, X_n. The set of all the generators for these subgroups could then be thought of as a vector space, of dimension n, with the $\{X_n\}$ as a basis. That this is a vector space can be seen since any scalar multiple, or linear combination of those n generators could also be thought of as generating a one-parameter subgroup passing through the origin. Because of the properties of the products of exponentials of operators, as expressed through the Baker-Campbell-Hausdorff theorem, the product of any two group elements may be expressed in terms of the generators of those two elements **and** an infinite sequence of multiple commutators of those elements. Therefore, abstractly, the set of all generators for the group is both a vector space and possesses a product map that satisfies the properties of commutators, i.e., is skew-symmetric and satisfies the Jacobi relation. Such a space is called a Lie algebra, and the particular such set of generators for some Lie group G is referred to as the Lie algebra for G, and denoted by the symbol $\mathcal{G}$.

2a. The Lie product—often simply called the commutator product—of two elements can be denoted simply by $a \bullet b$, but is much more commonly written as if it were a commutator, i.e., $[a, b]$. Remember that it is **not associative**, so that $a \bullet (b \bullet c) \neq (a \bullet b) \bullet c$, but does satisfy the Jacobi identity, $a \bullet (b \bullet c) + b \bullet (c \bullet a) + c \bullet (a \bullet b) = 0$.

2b. The Lie product of any two of the basis elements must of course be a linear combination of the basis elements:

$$[X_i\,,\ X_j] \equiv C_{ij}{}^k X_k\ .$$

The quantities $C_{ij}{}^k$ are called the commutation coefficients and,
to within choice of basis, characterize that Lie algebra.

- i.) The skew symmetry of the Lie product tells us that the commutation coefficients are skew symmetric on the lower indices, i.e., $C_{ij}{}^k = -C_{ji}{}^k$.
- ii.) The Jacobi identity requires that the commutation coefficients satisfy the following (cyclic) quadratic identity:

$$C_{ij}{}^{\ell} C_{k\ell}{}^m + C_{jk}{}^{\ell} C_{i\ell}{}^m + C_{ki}{}^{\ell} C_{j\ell}{}^m = 0\ .$$

3. A more abstract definition of a Lie algebra:

A Lie algebra, $\mathcal{L}$, is

- a.) a (finite- or infinite-dimensional) vector space (over a field K, such as $\mathbb{R}$, $\mathbb{C}$, etc.,)
- b.) with a rule of composition, $(X, Y) \longrightarrow [X, Y] \in L$ such that $\forall X, Y, Z \in L$, and $\forall \alpha, \beta \in K$:
    - i. $[\alpha X + \beta Y, Z] = \alpha[X, Z] + \beta[Y, Z]\,,$ {linearity}
    - ii. $[X, Y] = -[Y, X]\ ,$ {antisymmetry}
    - iii. $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0\ ,$ {Jacobi identity}

4. A subspace $\mathcal{N} \subset \mathcal{L}$ is a *subalgebra* if $[\mathcal{N}, \mathcal{N}] \subset \mathcal{N}$, i.e., if it is closed.

4a. An *ideal* is a subalgebra $\mathcal{N}$ such that $[\mathcal{N},\mathcal{L}]\subset\mathcal{N}$, i.e., such that $\{[n,\ell]\mid n\in\mathcal{N},\ell\in\mathcal{L}\}\subset\mathcal{N}$.

4b. The *center* of $\mathcal{L}$ is the maximal Abelian ideal $\mathcal{N}$, i.e., such that $[\mathcal{L},\mathcal{N}]=0$.

4c. If $\mathcal{N}$ is an ideal of $\mathcal{L}$, then $[\mathcal{N},\mathcal{N}]$ is also an ideal of $\mathcal{L}$.

Therefore, in particular, the commutator subalgebra $[\mathcal{L},\mathcal{L}]$ is an ideal of $\mathcal{L}$.

5. There are two (particular) useful sequences of ideals of an algebra $\mathcal{L}$:

$$\mathcal{L}^{(n)} \text{ defined as follows: } \mathcal{L}^{(0)}\equiv\mathcal{L}\,,\ \mathcal{L}^{(1)}\equiv[\mathcal{L},\mathcal{L}]\,,\ \dots\,,\quad \mathcal{L}^{(n+1)}\equiv[\mathcal{L}^{(n)},\mathcal{L}^{(n)}]\ ;$$

$$\mathcal{L}^{n} \text{ defined as follows: } \mathcal{L}^{1}\equiv\mathcal{L}\,,\ \mathcal{L}^{2}\equiv[\mathcal{L},\mathcal{L}]\,,\ \dots\ \mathcal{L}^{n+1}\equiv[\mathcal{L}^{n},\mathcal{L}]\,.$$

(The $\mathcal{L}^n$ are often called "powers"; sometimes the notation $\mathcal{L}_{(n-1)}$ is used for them.)

5a. The algebra $\mathcal{L}$ is *solvable* if there exists some integer n such that $\mathcal{L}^{(n)}=0$. Every solvable Lie algebra is isomorphic to a subalgebra of some Lie algebra of upper triangular matrices.

5b. The algebra $\mathcal{L}$ is *nilpotent* if for some integer n, the n-th power is 0, i.e., $\mathcal{L}^n=0$.
Every nilpotent algebra is solvable.

6. A Lie algebra $\mathcal{L}$ is *simple* if it has no non-trivial ideals and it is not Abelian.
[Trivial ideals are $\{0\}$ and $\mathcal{L}$. Proper is also used to mean the same as non-trivial.]

6a. A *semisimple* Lie algebra is one which is a direct sum of simple Lie algebras.

7. A representation of a Lie algebra $\mathcal{L}$ is a (continuous) linear mapping to a set of linear operators acting on a vector space, V, which preserves the Lie product. The product for the operators is their commutator product.
One also then says that V is an $\mathcal{L}$-module. (The difference in the two statements is that the representation is often construed as implying a specific set of matrices and, therefore, a specific choice of basis. Nonetheless, obviously the language above need not be construed that way.)

8. The adjoint representation of a Lie algebra $\mathcal{L}$ is the one given by its action on itself via its Lie product:

$$\text{for } Y\in\mathcal{L},\quad \operatorname{ad}Y(X)\equiv[Y,X]\,,\quad \forall X\in\mathcal{L}\ .$$

8a. To determine a presentation for this action in terms of matrices, we first choose a basis $\{X_n\}$ and agree that for an arbitrary element $Y\in\mathcal{G}$, we write $Y=Y^kX_k$ and agree to present those components Y^k as the elements of a column vector. Our task, then, is to determine the elements of the matrix that presents $\operatorname{ad}Y$.

To do this, we first label the commutation coefficients ${C_{ij}}^k$ of the basis, i.e., that $[X_i, X_j] = {C_{ij}}^k X_k$. Then we have the following calculation:

$$Y^i Z^j {C_{ij}}^k X_k = [Y, Z] = W = W^k X_k \quad \Longrightarrow \quad W^k = Y^i Z^j {C_{ij}}^k = ({C_{ij}}^k Y^i) Z^j \ .$$

$$\text{define the matrix presenting } \mathrm{ad}(Y) \text{ as} \qquad [(\mathrm{ad}Y)]^k{}_j Z^j = [(\mathrm{ad}Y)Z]^k \ ,$$

$$\text{or, supressing indices } \ (\mathrm{ad}Y)Z = W \ ,$$

$$\Longrightarrow \quad [(\mathrm{ad}\, Y)]^k{}_j = {C_{ij}}^k Y^i \ .$$

The representation of the algebra via these matrices is given in the expected/usual way, namely for $Z, Y, Q \in \mathcal{G}$, and, as above, the basis as $\{X_j\}_1^n$ we have

$$\mathrm{ad}Z\{\mathrm{ad}YQ\} \ \Longrightarrow \quad (\mathrm{ad}Z)^k{}_n (\mathrm{ad}Y)^n{}_i Q^i = Z^m {C_{mn}}^k Y^\ell {C_{\ell i}}^n Q^i \ .$$

The Jacobi identity, as given explicitly for the commutation coefficients in §2b(ii), then arranges for this to be a representation, i.e., for $\mathrm{ad}([X, Y]) = \{\mathrm{ad}(X)\}\{\mathrm{ad}(Y)\} - \{\mathrm{ad}(Y)\}\{\mathrm{ad}(X)\}$.

9. The *Killing form* is the *symmetric* map $\mathcal{L} \otimes \mathcal{L} \to K$ via

$$(X, Y) \equiv \mathrm{tr}\,\{\mathrm{ad}(X)\mathrm{ad}(Y)\} = {C_{mk}}^i {C_{si}}^k X^m Y^s \equiv g_{ms} X^m Y^s \ .$$

9a. The Jacobi identity allows us to show that the Killing form has the property referred to as *associativity*: $(X, [Y, Z]) = ([X, Y], Z)$.

9b. The symmetric matrix $g_{ms} = (X_m, X_s)$ can be used as a metric on the vector space if it is non-degenerate.

9c. A Lie algebra $\mathcal{L}$ is semisimple if and only if its Killing form is non-degenerate, i.e., if and only if $(X, Y) = 0$ for all $X \in \mathcal{L}$ implies $Y = 0$.

10. A *Casimir operator* for a matrix Lie algebra is a quadratic (or higher) associative product of these matrices that commutes with the entire algebra.

10a. When the Casimir operator is evaluated in a given, irreducible representation, it is a multiple of the identity matrix in that representation.

11. Every Lie algebra, over $\mathbb{R}$ or $\mathbb{C}$, has a maximal solvable ideal in the sense that it contains any other solvable ideal; this maximal solvable ideal is called the *radical* of the algebra.

11a. If $\mathcal{N}$ is the radical of $\mathcal{L}$, then the quotient algebra $\mathcal{L}/\mathcal{N}$ is semisimple.

12. ***The semidirect sum:***

Let $\mathcal{G}$ and $\mathcal{M}$ be Lie algebras, and let $\sigma : \mathcal{M} \to GL(\mathcal{G})$, i.e., there is a representation of $\mathcal{M}$ in $\mathcal{G}$, treated as a vector space.

Then the semidirect sum of the two algebras is denoted by $\mathcal{G} \oplus\!\!\!\!\!\!\supset \mathcal{M}$.

Denoting the elements by (g, m), where $g \in \mathcal{G}$ and $m \in \mathcal{M}$, we may define the Lie product in the sum via the following:

$$[(g, m), (g', m')] \equiv \big([g, g'] + \sigma(m)g' - \sigma(m')g \ , \ [m, m']\big) \ .$$

A (standard) example is the Lie algebra of the affine group of the plane, $\mathbb{R}^2$; that group is $\boldsymbol{so}(2)$, spanned by the (single) rotation in the plane, plus the two translations in the plane, $\boldsymbol{t}(2)$, where the action on a typical vector, $x \in \mathbb{R}^2$, is given by $(R, a)x \equiv Rx + a$, where R is a rotation and a a translation vector. Then we have for the product $(S, b)(R, a)x = SRx + Sa + b$.

Theorem of Ado

Every (finite-dimensional) Lie algebra over the field of complex numbers is isomorphic to some (finite-dimensional) matrix algebra.

Levi-Malcev Theorem

Let $\mathcal{L}$ be a Lie algebra over $\mathbb{R}$ or $\mathbb{C}$, with radical $\mathcal{N}$. Then there exists a semisimple subalgebra $\mathcal{S} \subset \mathcal{L}$ such that $\mathcal{L}$ can be written as the semidirect sum of $\mathcal{N}$ and $\mathcal{S}$, with $\mathcal{S}$ having an action on $\mathcal{N}$.