

All simple Lie Algebras, over the complex field: due to Cartan
with detailed notes on those of ranks 1 and 2

I. All simple, complex Lie algebras: Cartan

The entire set of simple Lie algebras, over \mathbb{C} , was classified by Elie Cartan. It consists of 4 (countably) infinite sets plus, exceptionally, 5 individual others. All these are listed below; in each case the rank, r , of the algebra is given by the subscript. In addition, we give the formula for the dimension, the form for the Cartan matrix, A_{ij} , (which is always $r \times r$), and the Dynkin diagram. (Also note that each has several real distinct real forms.)

A. The sequence, $\mathbf{A}_r \equiv \mathfrak{sl}(r+1, \mathbb{C})$, $r = 1, 2, 3, \dots$, which have dimension $r(r+2)$;

$$A_{ij} = \begin{pmatrix} 2 & -1 & 0 & & \dots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & -1 & 2 \end{pmatrix} \quad \begin{array}{ccccccc} \bullet & \text{---} & \bullet & & \bullet & \dots & \bullet & \text{---} & \bullet \\ 1 & & 2 & & 3 & & r-1 & & r \end{array}$$

The matrix presentation of this algebra is the set of all $(r+1) \times (r+1)$ -dimensional, traceless matrices. The corresponding Lie group is the set of all matrices of that size with determinant $+1$.

B. The sequence, $\mathbf{B}_r = \mathfrak{so}(2r+1, \mathbb{C})$, $r = 2, 3, 4, \dots$, which have dimension $r(2r+1)$.

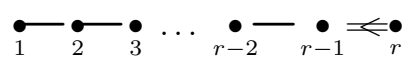
$$A_{ij} = \begin{pmatrix} 2 & -1 & 0 & & \dots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & -1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & -1 & 2 \end{pmatrix} \quad \begin{array}{ccccccc} \bullet & \text{---} & \bullet & & \bullet & \dots & \bullet & \text{---} & \bullet & \rightrightarrows & \bullet \\ 1 & & 2 & & 3 & & r-2 & & r-1 & & r \end{array}$$

The algebra $\mathbf{B}_1 = \mathfrak{so}(3, \mathbb{C})$ is isomorphic to \mathbf{A}_1 , and so is not listed again.

The matrix presentation is the set of all $(2r+1) \times (2r+1)$ -dimensional, skew-symmetric matrices. Equivalently, it may be described as the set of all matrices, T , of that size such that TA is skew-symmetric, where $A = \begin{pmatrix} 1 & & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{pmatrix}$, where I is the $r \times r$ identity matrix. (This second presentation allows one to choose the Cartan subalgebra as diagonal matrices.)

The corresponding Lie group is the set of all orthogonal transformations (connected to the identity) in this number of dimensions.

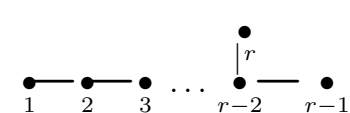
- C. the sequence $\mathbf{C}_r = \mathfrak{sp}(2r, \mathbb{C})$, $r = 3, 4, 5, \dots$, which have dimension $r(2r + 1)$.

$$A_{ij} = \begin{pmatrix} 2 & -1 & 0 & & \dots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & -2 & 2 \end{pmatrix}$$


The algebra $\mathbf{C}_1 = \mathfrak{sp}(2, \mathbb{C})$ is also isomorphic to \mathbf{A}_1 ; the algebra $\mathbf{C}_2 = \mathfrak{sp}(4, \mathbb{C})$ is isomorphic to $\mathbf{B}_2 = \mathfrak{so}(5)$. These two are therefore not listed.

The matrix presentation is the set of all $(2r) \times 2r$ matrices such that TA is skew-symmetric, where $A = \begin{pmatrix} 0I \\ -I0 \end{pmatrix}$, where I is the $r \times r$ identity matrix. The associated Lie group is the group of all symplectic transformations of this many dimensions.

- D. The sequence $\mathbf{D}_r = \mathfrak{so}(2r, \mathbb{C})$, $r = 4, 5, 6, \dots$, which have dimension $r(2r - 1)$.

$$A_{ij} = \begin{pmatrix} 2 & -1 & 0 & & \dots & 0 & 0 \\ -1 & 2 & -1 & 0 & & \dots & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & \dots & 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & \dots & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & \dots & 0 & 0 & -1 & 0 & 2 \end{pmatrix}$$


The algebra $\mathbf{D}_1 = \mathfrak{so}(2, \mathbb{C})$ is only one-dimensional and is therefore not terribly interesting.

The algebra $\mathbf{D}_2 = \mathfrak{so}(4, \mathbb{C})$ is isomorphic to $\mathbf{A}_1 \oplus \mathbf{A}_1$, and therefore not simple.

The algebra $\mathbf{D}_3 = \mathfrak{so}(6, \mathbb{C})$ is isomorphic to \mathbf{A}_3 and is therefore not included.

The matrix presentation is the set of all $(2r) \times 2r$ matrices that are skew symmetric. Again an alternative, equivalent presentation is the set of all matrices, T , of this dimension such that TA is skew-symmetric, where $A = \begin{pmatrix} 0I \\ I0 \end{pmatrix}$, where I is the $r \times r$ identity matrix. The associated Lie group is the group of all orthogonal transformations, connected to the identity, of this many dimensions.

E. The single second rank, exceptional algebra is \mathbf{G}_2 , which has dimension 14:

$$A_{ij} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} \quad \bullet \rightleftarrows \bullet \\ 1 \quad 2$$

F. The single fourth rank, exceptional algebra is \mathbf{F}_4 , which has dimension 52:

$$A_{ij} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \quad \bullet - \bullet \rightleftarrows \bullet - \bullet \\ 1 \quad 2 \quad 3 \quad 4$$

G. Three exceptional algebras, $\{\mathbf{E}_6, \mathbf{E}_7, \mathbf{E}_8\}$, which have dimensions 78, 133, and 248:

$$E_6 : \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix} \quad \begin{array}{cccccc} & & & \bullet & & \\ & & & | & & \\ & & & 2 & & \\ \bullet & - & \bullet & - & \bullet & - & \bullet \\ 1 & & 3 & & 4 & & 5 & & 6 \end{array}$$

$$E_7 : \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix} \quad \begin{array}{cccccc} & & & \bullet & & \\ & & & | & & \\ & & & 2 & & \\ \bullet & - & \bullet & - & \bullet & - & \bullet \\ 1 & & 3 & & 4 & & 5 & & 6 & & 7 \end{array}$$

$$E_8 : \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix} \quad \begin{array}{cccccc} & & & \bullet & & \\ & & & | & & \\ & & & 2 & & \\ \bullet & - & \bullet & - & \bullet & - & \bullet & - & \bullet \\ 1 & & 3 & & 4 & & 5 & & 6 & & 7 & & 8 \end{array}$$

II. The rank-1 algebra

A. There is only one such algebra, when looked at over \mathbb{C} ; by mathematicians, that algebra is most often denoted by \mathbf{A}_1 , the name that Cartan gave it. However, it has several **isomorphic** complex forms, namely $\mathbf{A}_1 = \mathfrak{sl}(2, \mathbb{C}) \sim \mathbf{B}_1 = \mathfrak{so}(3, \mathbb{C}) \sim \mathbf{C}_1 = \mathfrak{sp}(2, \mathbb{C})$. In addition, there are many more distinct real forms, as described just below:

- 1.) $\mathfrak{su}(2)$, the three lin-ind. 2×2 , skew-Hermitean, traceless matrices,
- 2.) $\mathfrak{sl}(2, \mathbb{R}) \sim \mathfrak{sp}(2)$, the three lin-ind. 2×2 , real, traceless matrices,
- 3.) $\mathfrak{su}(1,1) \sim \mathfrak{su}^*(2)$, the three-(real)-parameter algebra given by

$$\begin{pmatrix} ia & b+ic \\ b-ic & -ia \end{pmatrix},$$

- 4.) $\mathfrak{so}(3, \mathbb{R})$, the three lin-ind., real, skew-symmetric matrices,
- 5.) $\mathfrak{so}(2,1)$, the algebra of all matrices of the form

$$\begin{pmatrix} 0 & -a & b \\ a & 0 & c \\ b & c & 0 \end{pmatrix},$$

B. Two reasonable choices for matrices to describe this algebra, i.e., \mathbf{A}_1 , are given by the following three sets. The first two are equivalent over the complex numbers, but not over the real numbers; i.e., there are similarity transformations that take one set into another set, but those transformations involve complex numbers. (Furthermore, we show how the first one is related to the Chevalley basis set, $\{h, e, f\}$, and how the second form is related to the usual Pauli matrices, $\{\sigma_x, \sigma_y, \sigma_z\}$.)

generators for $\mathfrak{sl}(2, \mathbb{R})$ over \mathbb{R} ;

$$\left\{ e \equiv t_+ \equiv \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \frac{1}{2}h \equiv t_0 \equiv \frac{1}{2} \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}, f \equiv t_- \equiv \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}, \quad (2.1)$$

$$[t_0, t_{\pm}] = \pm t_{\pm}, \quad [t_+, t_-] = 2t_0,$$

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h,$$

generators for $\mathfrak{su}(2, \mathbb{R})$ over \mathbb{R} ;

$$\left\{ t_x \equiv \frac{1}{2} \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = -\frac{i}{2}\sigma_x, t_y \equiv \frac{1}{2} \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix} = -\frac{i}{2}\sigma_y, t_z \equiv \frac{1}{2} \begin{pmatrix} -i & 0 \\ 0 & +i \end{pmatrix} = -\frac{i}{2}\sigma_z \right\},$$

$$[t_x, t_y] = t_z, \quad [t_y, t_z] = t_x, \quad [t_z, t_x] = t_y, \quad (2.2)$$

generators for $\mathfrak{so}(3, \mathbb{R})$ over \mathbb{R} ;

$$\left\{ s_x \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & +1 & 0 \end{pmatrix}, s_y \equiv \begin{pmatrix} 0 & 0 & +1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, s_z \equiv \begin{pmatrix} 0 & -1 & 0 \\ +1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}, \quad (2.3)$$

$$[s_x, s_y] = s_z, \quad [s_y, s_z] = s_x, \quad [s_z, s_x] = s_y,$$

C. The structure of A_1 is not very complicated; nonetheless, one can use it to begin understanding the methods for searching for “roots” of a *Cartan subalgebra*, which is the general method used to determine the structure of arbitrary simple, and semi-simple, Lie algebras. Therefore I want to explain a little what these two concepts are, and how their understanding enlightens us about the structure of the originally given semi-simple Lie algebra.

For a semisimple Lie algebra, a Cartan subalgebra, \mathcal{H} , is a particular sort of maximal, commuting subalgebra of the Lie algebra, \mathcal{G} , being studied; usually this study is being made because of its relationship to the symmetries, of some sort or other, of a particular physical system. This subalgebra is not usually uniquely determined, so that there are often different methods of determining the vector space spanned by the roots; nevertheless, this vector space is unique, and determines the entire algebra itself. The *rank* of a semi-simple algebra is the number of elements in a basis for this maximal subalgebra.

Given such a maximal set of commuting “operators,” the idea is to find their “eigenvectors,” and associated “eigenvalues” from among the rest of the elements of the algebra, i.e., the (Lie) commutator of an element of \mathcal{H} with one of these eigenvectors just gives back some scalar multiple of the eigenvector again. For $h \in \mathcal{H}$, since $[h, e] = (\text{ad } h)e$, this says that the Lie algebra element e truly is an eigenvector for the operator $\text{ad}(h)$, so that we need to look at the adjoint representation of the algebra.

Without more ado, let’s first look at this question for A_1 . The choice of basis elements for it has already been made to prejudice the issue, i.e., to simplify the hunt for eigenvectors. By looking directly at the commutator table for the $\mathfrak{sl}(2, \mathbb{R})$ version of the algebra, we see that there is only one element in a maximal, commuting subalgebra. (Notice that we also know

there is only 1 element in \mathcal{H} since this algebra is given as being of rank 1, i.e., its name is A_1 . We quickly decide to take a basis for \mathcal{H} the element t_0 , since the eigenvectors are already displayed:

$$(\text{ad } t_0)t_+ = [t_0, t_+] = t_+, \quad (\text{ad } t_0)t_- = [t_0, t_-] = -t_-, \quad (2.4)$$

which demonstrates the 1-dimensional Cartan subalgebra, and the role of the rest of the algebra as eigenvectors for it, with roots, i.e., eigenvalues, of $+1$ and -1 . If this seems rather too easy, let us ask the same question for the same algebra, but let us first actually determine the 3×3 matrices that describe its adjoint representation; however, using the current basis set for the vector space, the adjoint representation of t_0 will already be diagonal, which is what makes it too easy. Therefore, we will use instead the basis set indicated for $\mathfrak{su}(2)$, namely $\{t_x, t_y, t_z\}$.

The adjoint representation is then easily calculated:

$$t_x \xrightarrow[\text{ad}]{} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & +1 & 0 \end{pmatrix}, \quad t_y \xrightarrow[\text{ad}]{} \begin{pmatrix} 0 & 0 & +1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad t_z \xrightarrow[\text{ad}]{} \begin{pmatrix} 0 & -1 & 0 \\ +1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (2.5)$$

which, interestingly, turns out to be the same matrices as the original presentation for the basis for $\mathfrak{so}(3)$. We now pick any one of these as the one basis element for $\text{ad}(\mathcal{H})$ and proceed to look for eigenvectors. To be different, let's use $\text{ad}(t_y)$:

$$\left| \begin{pmatrix} -\lambda & 0 & 1 \\ 0 & -\lambda & 0 \\ -1 & 0 & -\lambda \end{pmatrix} \right| = 0 \implies \lambda = 0, \pm i,$$

$$(\text{ad } t_y) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \xleftarrow[\text{ad}]{} (\text{ad } t_y)t_y; \quad (\text{ad } t_y) \begin{pmatrix} \mp i \\ 0 \\ 1 \end{pmatrix} = \pm i \begin{pmatrix} \mp i \\ 0 \\ 1 \end{pmatrix} \xleftarrow[\text{ad}]{} (\text{ad } t_y)(\mp i t_x + t_z).$$

This brief calculation tells us that if we begin with the choice $\{t_x, t_y, t_z\}$ as a basis for our Lie algebra, and also choose the Cartan subalgebra \mathcal{H} as spanned by t_y , then there are 3 roots, one of which is just zero, i.e., the statement that \mathcal{H} is abelian, while the other two roots are $\pm i$, and their associated eigenvectors are $\mp i t_x + t_z$.

We want to emphasize that the choice of \mathcal{H} is not unique, and, one should presume, the resulting determination of eigenvectors is not unique; nonetheless, the values of the roots is

unique. To see this we re-do the calculation, taking t_z as the basis element for this choice of \mathcal{H} . (Since none of the elements commutes with each other, so that the Cartan subalgebra is 1-dimensional, obviously one may choose any single element.)

$$\left| \begin{pmatrix} -\lambda & -1 & 0 \\ +1 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{pmatrix} \right| = 0 \implies \lambda = 0, \pm i,$$

$$(\text{ad } t_z) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \xleftarrow[\text{ad}]{} (\text{ad } t_z)t_z; (\text{ad } t_z) \begin{pmatrix} \pm i \\ 1 \\ 0 \end{pmatrix} = \pm i \begin{pmatrix} \pm i \\ 1 \\ 0 \end{pmatrix} \xleftarrow[\text{ad}]{} (\text{ad } t_z)(\pm i t_x + t_y),$$

in agreement with our statements above.

III. The rank-2 algebras

A. There are 4 such algebras, provided we go ahead and include the semi-simple one, \mathbf{D}_2 , the others being $\mathbf{A}_2 \sim \mathbf{C}_2$, \mathbf{B}_2 , and \mathbf{G}_2 . One can characterize them also by their various real forms:

- 1.) for \mathbf{A}_2 , we have $\mathfrak{sl}(3, \mathbb{R})$, $\mathfrak{su}(3)$, $\mathfrak{sp}(2)$, with 8 dimensions, while
- 2.) for \mathbf{B}_2 , we have $\mathfrak{so}(5)$, with 10 dimensions, and
- 3.) \mathbf{D}_2 , which has real form $\mathfrak{so}(4) \sim \mathfrak{so}(3) \oplus \mathfrak{so}(3)$, with 6 dimensions, and
- 4.) \mathbf{G}_2 , which still remains as an exceptional algebra, with 14 dimensions; it is the only finite, simple Lie algebra with a -3 in its Cartan matrix.

B. Beginning as before with $\mathfrak{sl}(3)$, we can take a basis for all 3×3 traceless matrices in the following order, where the naming is the same as that given by R. Cahn for his “complexification” of the basis elements for $\mathfrak{su}(3)$:

$$T_+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, T_- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, T_z = \frac{1}{2} \begin{pmatrix} +1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, U_+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$U_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, V_+ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, V_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, Y = \frac{1}{3} \begin{pmatrix} +1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \quad (3.1)$$

Using the commutators given on p. 11 of his book, after fixing the typo for $[v_+, u_-] = t_+$ and $[u_-, v_+] = -t_+$ instead of what he has, we can write the adjoint representation for any one of these. For simplicity, I choose to do them all at once by defining an arbitrary element of the algebra in the form

$$g = aT_+ + bT_- + cT_z + dU_+ + eU_- + fV_+ + gV_- + hY. \quad (3.2)$$

Its adjoint representation can then be read off from the table, where $((\text{ad } g))^k_j = [g, X_j]^k$ as follows:

$$(\text{ad } g) \implies \begin{pmatrix} +c & 0 & -a & 0 & f & -e & 0 & 0 \\ 0 & -c & b & -g & 0 & 0 & d & 0 \\ -2b & 2a & 0 & e & -d & -g & f & 0 \\ 0 & -f & \frac{1}{2}d & h - \frac{1}{2}c & 0 & b & 0 & -d \\ g & 0 & -\frac{1}{2}e & 0 & \frac{1}{2}c - h & 0 & -a & e \\ -d & 0 & -\frac{1}{2}f & a & 0 & h + \frac{1}{2}c & 0 & -f \\ 0 & e & \frac{1}{2}g & 0 & -b & 0 & -h - \frac{1}{2}c & g \\ 0 & 0 & 0 & -\frac{3}{2}e & \frac{3}{2}d & -\frac{3}{2}g & \frac{3}{2}f & 0 \end{pmatrix}, \quad (3.3)$$

where the matrix has its basis set ordered in the same order as the elements in the definition of g , i.e., in the order $[X_j]_1^8 \equiv [T_+, T_-, T_z, U_+, U_-, V_+, V_-, Y]$. Therefore, for instance, the matrix just above tells us that

$$(\text{ad } g)T_+ = [g, T_+] = cT_+ - 2bT_z + gU_- - dV_+,$$

since T_+ is the 1st entry in the basis set, so we are reading down the first column, which has the entries for $((\text{ad } g))^1_1 = c$, $((\text{ad } g))^3_1 = [g, T_+]^3 = -2b$, etc.

We must now choose elements for the Cartan subalgebra. In principle, the first step is to choose in some way specific numerical values for the arbitrary coefficients in the general adjoint matrix, Eq. (3.3), and to find the eigenvalues for that matrix. There will always be one 0 eigenvalue since the matrix commutes with itself. In general, for a rank r algebra, the minimum number of zero eigenvalues must be r , since we know the Cartan subalgebra must be of that dimension; of course, particular elements may have more. The definition of a

regular element of H is that its adjoint representation must have a minimum possible number of zero eigenvalues, i.e., it should commute with the smallest possible number of elements of the algebra. Cartan's criterion for \mathcal{H} is that it must have at least one regular element. If we immediately have an element with only r zero eigenvalues, then we can start adding on the others; if we have none, then we can try particular linear combinations of interesting ones until one is found. At that point, we pick, from these eigenvalues, another algebra element with which the chosen one commutes; i.e., we pick an eigenvector corresponding to one of the other zero eigenvalues, and write an arbitrary linear combination of the two. Next we find the eigenvalues for this linear combination. If the rank is greater than two, this process needs to be repeated until one has such a linear combination of r distinct elements, all commuting. The eigenvalues of the resultant matrix will be the roots for the algebra, and the associated eigenvectors define the "grading" of the rest of \mathcal{G} , associated with that choice of \mathcal{H} .

However, the matrix given above, which is the adjoint representation of an arbitrary element of the algebra, parametrized by 8 scalars, we find that its characteristic polynomial has a reasonably special form:

- a. it always has (at least) two zero roots, i.e., two of the roots are zero, for arbitrary values of those 8 parameters;
- b. the remainder of the polynomial is of course a sextic polynomial, but, in fact, is actually a cubic polynomial in $y \equiv \lambda^2$:

$$-16y^3 + Ay^2 + By + C$$

where A is quadratic in the parameters, B is quartic, and C is sextic. That it only involves λ^2 is because of the general fact that the roots always come in pairs, one negative and one positive.

It is therefore generally clear that there are regular elements available to us. There would be many ways to search for them. For instance, if one begins with the basis vectors given above, one at a time, it is found that all of them, individually, have eight 0 eigenvalues, except for

$\text{ad}(T_z)$, which has only two, and $\text{ad}(Y)$, which has four. As well, many combinations, two at a time, still have eight 0 eigenvalues, such as $\text{ad}(aT_+ + bU_{\pm})$. On the other hand, $\text{ad}(T_+ + Y)$ has only four, while $\text{ad}(V_+ + V_-)$ has just two, so that it together with $\text{ad}(-2T_z + Y)$ will also form a Cartan subalgebra. Generically the eigenvectors are reasonably “nasty” so that I won’t display them.

Sort of following Cahn, let us first look in more detail at just the following four elements:

$$(\text{ad } T_z) = \begin{pmatrix} +1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & +\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & +\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (3.4)$$

$$(\text{ad } (T_+ + T_-)) = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & 0 & 0 & 0 & 0 & 0 \\ -2 & +2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (3.5)$$

$$(\text{ad } Y) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (3.6)$$

$$(\text{ad } (2T_z + Y)) = \begin{pmatrix} +2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & +2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (3.7)$$

The first two matrices, Eqs. (3.4-5), have two zero eigenvalues, while the second two, Eqs. (3.6-7), each have four columns with only zeros, i.e., four zero eigenvalues. Our rank 2

algebra will have its Cartan subalgebra 2 dimensional, so that the adjoint representation of any element of \mathcal{H} will have at least two zero eigenvalues, indicating that it commutes with itself and also with the other basis element for \mathcal{H} . The requirement of regularity insists that at least one element of \mathcal{H} must have only two columns of zeros! Therefore, in our case, it must contain at least one of the first two matrices displayed above, i.e., either T_z or $T_+ + T_-$, or, perhaps, yet some other combination that we have not looked at in detail, but which could be determined from the general adjoint matrix. The roots are then the eigenvalues of a general linear combination of our choice of 2 elements. Following standard tradition, Cahn chooses the Cartan subalgebra to be spanned by T_z and Y , so that the general element can be written as $cT_z + hY$. On his p. 13, he then considers the eigenvalues and eigenvectors of $\text{ad}(cT_z + hY)$. In this basis he finds this matrix to be diagonal, so that its eigenvectors are simply

eigenvalue	eigenvector
c	T_+
$-c$	T_-
0	T_z
$h - \frac{1}{2}c$	U_+
$\frac{1}{2}c - h$	U_-
$\frac{1}{2}c + h$	V_+
$-\frac{1}{2}c - h$	V_-
0	Y

(3.8)

It is very interesting that the eigenvalue calculation for $cT_z + hY$ gives back eigenvalues that are linear in c and h . This is surely quite unusual. One would expect an arbitrary 8×8 matrix to give back eigenvalues for such a matrix that were something like eighth-roots of eighth-order polynomials in these quantities, which would surely not be nice at all. If you don't believe this, calculate, for example, the eigenvalues of $\text{ad}\{a(T_+ + T_-) + cT_z\}$, which include $\pm\sqrt{4a^2 + c^2}$, or the eigenvalues of $\text{ad}(aT_+ + dU_+ + gV_-)$ which are functions of $\sqrt[3]{adg}$, and, lastly, that all the eigenvalues of $\text{ad}(bT_- + dU_+ + gV_-)$ are zero. That this linearity is always true for the general element of a Cartan subalgebra is a very powerful and important theorem.

It is also instructive to choose a different Cartan subalgebra and perform the same calculation. A second, reasonable choice, from the ones suggested in Eqs. (3.4-5) would be for \mathcal{H}_2 to be taken as $a(T_+ + T_-) + hY$. We then find that the eigenvalues for the adjoint representation of this combination are the ones given in the following table:

eigenvalue	eigenvector
0	$T_+ + T_-$
0	Y
$a - h$	$V_- - U_-$
$a + h$	$U_+ V_+$
$-a - h$	$U_- + V_-$
$-a + h$	$-U_+ + V_+$
$2a$	$-T_+ + T_- + 2T_z$
$-2a$	$T_+ - T_- + 2T_z$

(3.9)

Although this Cartan subalgebra is different, and the structure of the eigenvectors is different, by mapping a to $\frac{1}{2}c$ we see that the eigenvalues are the same. Also, as expected, they are linear with respect to the coefficients of a vector lying in the vector space that is the Cartan subalgebra.

- C. The algebra, $\mathbf{B}_2 = \mathfrak{so}(5)$ has 10 generators, i.e., a basis set for the space of all 5×5 skew-orthogonal matrices. An easy choice for them is written in the form

$$\begin{aligned}
g_{12} &= \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ +1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, & g_{13} &= \begin{pmatrix} 0 & 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, & g_{14} &= \begin{pmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ +1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\
g_{15} &= \begin{pmatrix} 0 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}, & g_{23} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, & g_{24} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
g_{25} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 & 0 \end{pmatrix}, \quad g_{34} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad g_{35} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix}, \\
g_{45} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & +1 & 0 \end{pmatrix}. \tag{3.10}
\end{aligned}$$

The commutator table of these matrices is calculated—via Maple, say—and I give it below. However, knowing that it is skew-symmetric, we only show the entries on and above the diagonal:

	g_{12}	g_{13}	g_{14}	g_{15}	g_{23}	g_{24}	g_{25}	g_{34}	g_{35}	g_{45}
g_{12}	0	$-g_{23}$	$-g_{24}$	$-g_{25}$	$+g_{13}$	$+g_{14}$	$+g_{15}$	0	0	0
g_{13}		0	$-g_{34}$	$-g_{35}$	$-g_{12}$	0	0	$+g_{14}$	$+g_{15}$	0
g_{14}			0	$-g_{45}$	0	$-g_{12}$	0	$-g_{13}$	0	$+g_{15}$
g_{15}				0	0	0	$-g_{12}$	0	$-g_{13}$	$-g_{14}$
g_{23}					0	$-g_{34}$	$-g_{35}$	$+g_{24}$	$+g_{25}$	0
g_{24}						0	$-g_{45}$	$-g_{23}$	0	$+g_{25}$
g_{25}							0	0	$-g_{23}$	$-g_{24}$
g_{34}								0	$-g_{45}$	$+g_{35}$
g_{35}									0	$-g_{34}$
g_{45}										0

(3.11)

Then it is straightforward to create the adjoint representation matrix for the generic element of \mathcal{G} , which we label as

$$g = a g_{12} + b g_{13} + c g_{14} + d g_{15} + e g_{23} + f g_{24} + g g_{25} + h g_{34} + i g_{35} + j g_{45} \tag{3.12}$$

$$(\text{ad } g) = \begin{pmatrix} 0 & e & f & g & -b & -c & -d & 0 & 0 & 0 \\ -e & 0 & h & i & a & 0 & 0 & -c & -d & 0 \\ -f & -h & 0 & j & 0 & a & 0 & b & 0 & -d \\ -g & -i & -j & 0 & 0 & 0 & a & 0 & b & c \\ b & -a & 0 & 0 & 0 & h & i & -f & -g & 0 \\ c & 0 & -a & 0 & -h & 0 & j & e & 0 & -g \\ d & 0 & 0 & -a & -i & -j & 0 & 0 & e & f \\ 0 & c & -b & 0 & f & -e & 0 & 0 & j & -i \\ 0 & d & 0 & -b & g & 0 & -e & -j & 0 & h \\ 0 & 0 & d & -c & 0 & g & -f & i & -h & 0 \end{pmatrix}. \tag{3.13}$$

For just $e = 1$ and the others zero, i.e., for g_{23} , the adjoint matrix has 4 zero eigenvalues and 3 each of $\pm i$, the zero eigenvalues corresponding to the 4 eigenvectors $\{g_{23}, g_{14}, g_{15}, g_{45}\}$. Each of the other, individual ones that I try has exactly the same structure for its eigenvalues, namely four zeros, 3 repetitions of $+i$, and 3 repetitions of $-i$. As a first choice for \mathcal{H} , I consider linear combinations of g_{14} and g_{23} , which commute with each other. One finds that $\text{ad}(\gamma g_{14} + \epsilon g_{23})$ has only its elements for zero eigenvectors, and the following root structure

eigenvalue	eigenvector	
$\pm i\gamma$	$\mp i g_{15} + g_{45}$.
$\pm i\epsilon$	$g_{35} \mp i g_{25}$	
$\pm i(\gamma + \epsilon)$	$g_{12} \pm i g_{13} + g_{34} \mp i g_{15}$	
$\pm i(\gamma - \epsilon)$	$-g_{12} \pm i g_{13} + g_{34} \pm i g_{24}$	

(3.13a)

A different choice, namely $\text{ad}(\zeta g_{24} + \mu g_{35})$ also has the two (trivial) zero eigenvalues, and the following eigenvectors:

eigenvalue	eigenvector	
$\pm i\mu$	$\mp i g_{13} + g_{15}$.
$\pm i\zeta$	$\mp i g_{12} + g_{14}$	
$\pm i(\zeta + \mu)$	$\mp i g_{23} + g_{25} - g_{34} \pm i g_{45}$	
$\pm i(\mu - \zeta)$	$g_{23} \pm i g_{25} + g_{45} \pm i g_{34}$	

(3.13b)

Therefore, either one of these qualifies as an appropriate choice for Cartan subalgebra and an associated set of roots. The reader might like to pick out a pair of simple roots, showing explicitly that the other two are the appropriate combinations.

D. The (exceptional) algebra \mathbf{G}_2 can be visualized as a particular subalgebra of $\mathfrak{so}(7)$, as pointed out by Humphreys. It is 14 dimensional, and of course of rank 2, and in fact the 7-dimensional description is its smallest representation. We may span its Cartan subalgebra by the pair labelled as $\{h_1, h_2\}$. The remaining 12 elements are then given names, again by Humphreys, as are seen in the table of Lie products (commutators) as presented just below.

	h_1	h_2	g_1	g_2	g_3	g_{-1}	g_{-2}	g_{-3}	$g_{1,-2}$	$g_{2,-3}$	$g_{3,-1}$	$g_{2,-1}$	$g_{3,-2}$	$g_{1,-3}$
h_1	0	0	$-g_1$	$+g_2$	0	g_{-1}	$-g_{-2}$	0	$2g_{1,-2}$	$-g_{2,-3}$	$-g_{3,-1}$	$-2g_{2,-1}$	$g_{3,-2}$	$g_{1,-3}$
h_2		0	0	$-g_2$	g_3	0	g_{-2}	$-g_{-3}$	$-g_{1,-2}$	$2g_{2,-3}$	$-g_{3,-1}$	$g_{2,-1}$	$-2g_{3,-2}$	$g_{1,-3}$
g_1			0	$2g_{-3}$	$-2g_{-2}$	$2h_1 + h_2$	$3g_{2,-1}$	$3g_{3,-1}$	g_2	0	0	0	0	g_3
g_2				0	$2g_{-1}$	$3g_{1,-2}$	$-h_1 + h_2$	$3g_{3,-2}$	0	g_3	0	g_1	0	0
g_3					0	$3g_{1,-3}$	$3g_{2,-3}$	$-h_1 - 2h_2$	0	0	g_1	0	g_2	0
g_{-1}						0	$2g_3$	$-2g_2$	0	0	$-g_{-3}$	$-g_{-2}$	0	0
g_{-2}							0	$2g_1$	$-g_{-1}$	0	0	0	$-g_{-3}$	0
g_{-3}								0	0	$-g_{-2}$	0	0	0	$-g_{-1}$
$g_{1,-2}$									0	$g_{1,-3}$	$-g_{3,-2}$	h_1	0	0
$g_{2,-3}$										0	$g_{2,-1}$	0	h_2	0
$g_{3,-1}$											0	0	0	$-h_1 - h_2$
$g_{2,-1}$												0	$-g_{3,-1}$	$g_{2,-3}$
$g_{3,-2}$													0	$-g_{1,-2}$
$g_{1,-3}$														0

The generic adjoint matrix is then created by allowing a generic element of \mathbf{G}_2 to be given as a sum of the 14 basis elements, each multiplied, in order, by a lower-case letter from the alphabet; i.e., our generic algebra element, g is denoted by

$$g \equiv a h_1 + b h_2 + c g_1 + d g_2 + e g_3 + f g_{-1} + g g_{-2} + h g_{-3} + i g_{1,-2} + j g_{2,-3} \\ + k g_{3,-1} + \ell g_{2,-1} + m g_{3,-2} + n g_{1,-3} ,$$

and then its adjoint representation is the following matrix:

$$\begin{pmatrix} 0 & 0 & -2f & g & h & 2c & -d & -e & -\ell & 0 & n & i & 0 & -k \\ 0 & 0 & -f & -g & 2h & c & d & -2e & 0 & -m & n & 0 & j & -k \\ c & 0 & -a & -\ell & -k & 0 & -2h & 2g & 0 & 0 & e & d & 0 & 0 \\ -d & d & -i & a-b & -m & 2h & 0 & -2f & c & 0 & 0 & 0 & e & 0 \\ 0 & -e & -n & -j & b & -2g & 2f & 0 & 0 & d & 0 & 0 & 0 & c \\ -f & 0 & 0 & -2e & 2d & a & i & n & -g & 0 & 0 & 0 & 0 & -h \\ g & -g & 2e & 0 & -2c & \ell & b-a & j & 0 & -h & 0 & -f & 0 & 0 \\ 0 & h & -2d & 2c & 0 & k & m & -b & 0 & 0 & -f & 0 & -g & 0 \\ -2i & i & 0 & -3f & 0 & 3d & 0 & 0 & 2a-b & 0 & 0 & 0 & n & -m \\ j & -2j & 0 & 0 & -3g & -3g & 3e & 0 & 0 & 2b-a & 0 & -n & 0 & \ell \\ k & k & -3h & 0 & 0 & 0 & 0 & 3c & 0 & 0 & -a-b & m & -\ell & 0 \\ 2\ell & -\ell & -3g & 0 & 0 & 0 & 3c & 0 & 0 & -k & j & b-2a & 0 & 0 \\ -m & 2m & 0 & -3h & 0 & 0 & 0 & 3d & k & 0 & -i & 0 & a-2b & 0 \\ -n & -n & 0 & 0 & -3f & 3e & 0 & 0 & -j & i & 0 & 0 & 0 & a+b \end{pmatrix}$$

As actual matrices the above objects are given by Humphreys, as follows, where we use the notation E_{jk} to mean a matrix which has **only one non-zero entry**, namely it has the entry of +1 at the j -th row and k -th column:

First the root vectors corresponding to the “long roots”,

root vector name	matrix representation
$g_{1,-2} = g_{2,-1}^T$	$E_{23} - E_{65}$
$g_{1,-3} = g_{3,-1}^T$	$E_{24} - E_{75}$
$g_{2,-3} = g_{3,-2}^T$	$E_{34} - E_{76}$

and then those corresponding to the “short roots,”

root vector name	matrix representation
$g_1 = -g_{-1}^T$	$\sqrt{2}(E_{12} - E_{51}) - (E_{37} - E_{46})$
$g_2 = -g_{-2}^T$	$\sqrt{2}(E_{13} - E_{61}) + (E_{27} - E_{45})$
$g_3 = -g_{-3}^T$	$\sqrt{2}(E_{14} - E_{71}) - (E_{26} - E_{35})$

along with a choice of basis for the Cartan subalgebra:

root vector name	matrix representation
h_1	$E_{22} - E_{33} - E_{55} + E_{66} = \text{diag}(0, +1, -1, 0, -1, +1, 0)$
h_2	$E_{33} - E_{44} - E_{66} + E_{77} = \text{diag}(0, 0, +1, -1, 0, -1, +1)$

On the other hand, a naming more in line with our approach via the Chevalley generators would be given by simply beginning with the basis for the 2-dimensional $\mathcal{G}_{\pm 1}$ subalgebras:

$$E_1 = g_{3,-2} = E_{43} - E_{67} = (F_1)^T, \quad E_2 = g_3 = \sqrt{2}(E_{14} - E_{71}) - (E_{26} - E_{35}) = (F_2)^T.$$

We may generate the entire algebra from these, beginning with the following, where $\text{diag}(a, \dots, b)$ indicates a diagonal matrix with elements beginning with a and ending with b :

$$[E_1, F_1] = H_1 = \text{diag}(0, 0, -1, +1, 0, +1, -1),$$

$$[E_2, F_2] = H_2 = \text{diag}(0, +1, +1, -2, -1, -1, +2),$$

$$\text{along with } [E_1, F_2] = 0.$$

As these correspond to the (Chevalley) simple roots, one may then calculate the vectors corresponding to the remaining roots, which amount to

$$E_{12} \equiv [E_1, E_2] = -E_{21}, \quad E_{221} \equiv [E_2, E_{21}], \quad E_{2221} \equiv [E_2, E_{221}], \quad \text{and} \quad E_{12221} \equiv [E_1, E_{2221}],$$

while one verifies that $[E_1, E_{12}]$, $[E_2, E_{2221}]$, and $[E_1, E_{221}]$ all vanish, as they should. Of course there are the analogous relations for the F_j 's.

Naming the (2-dimensional) roots, via the rows of the Cartan matrix, as

$$\alpha_1 = [2, -3], \quad \alpha_2 = [-1, 2],$$

where this means that

$$[H_j, E_1] = (\alpha_1)_j E_1, \quad [H_j, E_2] = (\alpha_2)_j E_2,$$

then we can see the following correspondences:

$$E_{21} \quad \text{corresponds to the root } \alpha_1 + \alpha_2,$$

$$E_{221} \quad \text{corresponds to the root } \alpha_1 + 2\alpha_2,$$

$$E_{2221} \quad \text{corresponds to the root } \alpha_1 + 3\alpha_2,$$

$$E_{12221} \quad \text{corresponds to the root } 2\alpha_1 + 3\alpha_2,$$

along with the corresponding comments for the F_j 's and the negatives of these various roots.

[The mapping between these names and those used by Humphreys, which recall the origin of the matrices within the defining representation of $\mathfrak{so}(7)$, is left to the reader.]