Ideas Defining Generalized Kac-Moody Lie Algebras

I. Contragredient Lie Algebras

Let $\mathbf{A}$ be a square $n \times n$ matrix, and let $\mathcal{G}_{-1}$, $\mathcal{G}_0$, and $\mathcal{G}_1$ be n-dimensional vector spaces, where we take the sets $\{f_j\}_1^n$, $\{h_j\}_1^n$, and $\{e_j\}_1^n$ as sets of basis vectors for the three vector spaces, respectively. As well, let the vector spaces have a (joint) Lie algebra structure such that

a. the following constraint relations hold:

$$[e_i, f_j] = \delta_{ij} h_i \ , \quad [h_i, h_j] = 0 \ , \quad [h_i, e_j] = A_{ij} e_j \ , \quad [h_i, f_j] = -A_{ij} f_j \ , \tag{1.1a}$$

where we NOTE that there are NO sums on any of the indices;

b. and the free Lie algebra generated by the Lie product on these vector spaces creates a graded Lie algebra, $\mathcal{G}$, such that

$$\mathcal{G} \equiv \bigoplus_{i=-\infty}^{+\infty} \mathcal{G}_i \ , \qquad [\mathcal{G}_i, \mathcal{G}_j] \subseteq \mathcal{G}_{i+j} \ . \tag{1.1b}$$

The minimal Lie algebra generated by this structure is called a ***contragredient Lie algebra***, and the original structure, $\mathcal{G}_{-1} \oplus \mathcal{G}_0 \oplus \mathcal{G}_1$ is called *its local part.* The adjective *minimal* is defined via the following sequence of definitions:

i.) An ideal of $\mathcal{G}$ is homogeneous if all its elements are contained in a single element of the gradation.

ii.) If there are two graded Lie algebras, $\mathcal{G}$ and $\mathcal{G}'$, with isomorphic local parts, $\mathcal{G}$ is said to be *minimal* when the isomorphism can always be extended to an epimorphism (a map that is "onto") of $\mathcal{G}'$ onto $\mathcal{G}$.

iii.) Suppose there are homogeneous ideals that have no elements (other than 0) in common with the local part of $\mathcal{G}$; then there is a unique maximal such ideal, say $\mathcal{R}$, that contains all the others.

iv.) The factor algebra of $\mathcal{G}$ by this maximal ideal, i.e., $\mathcal{G}/\mathcal{R}$, is the minimal graded Lie algebra generated by this local part.

The matrix $\mathbf{A}$ is called **the Cartan matrix** for $\mathcal{G}$; however, this usage is rather more general than common parlance, as we will see below. To begin the discussion, we first note here that general graded algebras may be characterized in some reasonably simple ways:

1. A graded algebra is called *simple* if it contains no nontrivial homogeneous ideals.
2. A slightly weaker requirement is that a graded algebra be *transitive.* This means the following two (similar) requirements are satisfied:
   i.) if $\forall i \geq 0$, it is true that $[x, \mathcal{G}_{-1}] = 0$, then $x \equiv 0$;
   i.) if $\forall i \leq 0$, it is true that $[x, \mathcal{G}_{+1}] = 0$, then $x \equiv 0$.
3. All simple graded algebras are transitive.
4. Because of the constraining relations on the Lie product in the local part, it is clear that there are representations of $\mathcal{G}_0$ on each of the vector spaces $\mathcal{G}_{\pm 1}$, which we refer to as $\phi_{\pm 1} : \mathcal{G}_0 \ \rightarrow \ GL(\mathcal{G}_{\pm 1})$, where $GL(V)$ indicates the set of all general, linear transformations of a vector space, V, to itself. $GL(V)$ can of course be presented by the (full) group of invertible matrices over V.
5. A graded Lie algebra is referred to as *irreducible* when the representation ϕ_{-1} is irreducible.
6. A Cartan matrix, $\mathbf{A}$, is said to be *symmetrizable* if there exists a diagonal $n \times n$ matrix, $\mathbf{D}$, such that that their product, $\mathbf{DA}$, is a symmetric matrix.
7. The (Gel'fand-Kirillov) growth of a graded Lie algebra is given by the following, where d_j is defined as the (vector space) dimension of $\mathcal{G}_j$:

$$\textbf{growth of } \mathcal{G}\textbf{:} \ \ r(\mathcal{G}) \equiv \overline{\lim_{n\to\infty}} \left\{ \frac{\log \sum_{j=-n}^{n} d_j}{\log n} \right\} . \tag{1.2}$$

   A graded Lie algebra is said *to be of finite growth* when $r(\mathcal{G})$ is finite.

   For finite-dimensional Lie algebras the growth is zero.

   Affine algebras, as defined below, have finite growth.

   No algebra of infinite growth can be presented by finite-dimensional matrices.

8. A symmetric, bilinear form (,) on a Lie algebra $\mathcal{G}$ is said to be

i.) *invariant* if it satisfies the following equality for all x, y, and z contained in $\mathcal{G}$:

$$([x,y],z) = (x,[y,z]) \;, \tag{1.3}$$

ii.) and to be *degenerate* is there exist subsets $\mathcal{X} \subseteq \mathcal{G}$ and $\mathcal{Y} \subseteq \mathcal{G}$ such that for all $x \in \mathcal{X}$ and for all $y \in \mathcal{Y}$, it is true that $(x,y) = 0$.

This is the generalization of the usual Killing form on finite-dimensional Lie algebras. Every non-zero such form on a **simple** Lie algebra is (automatically) non-degenerate.

9. **If** an invariant form (,) is defined on the local part of a graded Lie algebra and, further, is such that $(G_i, G_j) = 0$ whenever $i + j \neq 0$, i.e., when considering elements from $\mathcal{G}_1$ either among themselves or with elements from $\mathcal{G}_0$ and the same is true for $\mathcal{G}_{-1}$,
then it can be uniquely extended to an invariant form on the entire Lie algebra, with the same property extending itself, i.e., such that $(G_i, G_j) = 0$ whenever $i + j \neq 0$.

II. Generalized Kac-Moody Lie Algebras

There are many plausible restrictions that may be applied to the Cartan matrix discussed above, in order to restrict the contragredient Lie algebras to a class that is somewhat more manageable. In at least most cases, one wants to (at least) restrict the entries so that the resulting algebra is of finite growth and allows the existence of a symmetric, bilinear form—a generalization of the Killing form—of the sort described just above, i.e., with the property that $(G_i, G_j) = 0$ whenever $i + j \neq 0$. Because of this desire to produce something which can be understood, classified, and used, there are various slightly-different definitions of various ones of these restrictions. I will not try to cover the most general possibilities.

At the very least, a ***generalized Cartan matrix*** is usually required to satisfy the following 5 constraints:

0.) its elements are integers;

1.) its diagonal elements are always 2;

2.) its off-diagonal elements are never positive;

3.) while not symmetric, if $A_{ij} = 0$, then the transposed element A_{ji} must also vanish;

4.) the matrix is not in block diagonal form, i.e., it is irreducible.

This last requirement of course is not so much a requirement, but simply a way of saying that we want to consider all the simple cases separately; they may then be put back together as direct sums, or even semi-direct sums.

These constraints are sufficient to allow the definition, and extension, of a symmetric, bilinear form, and to create a root structure very similar to that for finite-dimensional Lie algebras, including, in particular, that the elements of the local part satisfy the usual Serre relations:

$$(\mathrm{ad}\, e_i)^{-A_{ij}} e_j \neq 0 \ , \ (\mathrm{ad}\, e_i)^{1-A_{ij}} e_j = 0 \ ; \qquad (\mathrm{ad}\, f_i)^{1-A_{ij}} f_j = 0 \ , (\mathrm{ad}\, f_i)^{-A_{ij}} f_j \neq 0 \ . \tag{1.4}$$

In the case that the Cartan matrix is (also) invertible, then this simply re-generates the usual finite-dimensional algebras, which is a theorem originally proved by Chevalley. However, in the more general case one must proceed with somewhat more care.

In particular, if, in addition to the requirements above, one also requires that the Cartan matrix be symmetrizable, then one finds that there are three distinct cases:

1. The matrix may have rank n, i.e., be invertible, which gives back all the finite-dimensional, simple Lie algebras.
2. The matrix may have rank $n-1$, which gives the so-called Kac-Moody, or *affine* Lie algebras, which are quite well understood, these days.
3. The matrix may have rank less than $n-1$, which gives the so-called *hyperbolic* algebras. They have been classified, and studied, but are still not very well understood.

For both the cases above when the rank is less than the dimension of the matrix, this is the same as saying that there are 0 eigenvalues for the matrix. For the root system this fact causes a countable infinity of additional roots, which "have zero length." The names of some interesting simple examples are, for instance, $A_1^{(1)}$ and $A_2^{(2)}$, which are the only ones that have rank 2, the minimal rank possible. Their Cartan matrices are given by the following:

$$A_1^{(1)} \rightarrow \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} ; \qquad A_2^{(2)} \rightarrow \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix} . \tag{2.1}$$

III. Affine Algebras as Loop Algebras

The simplest approach to creation of a Kac-Moody algebra is via a so-called *loop algebra*; when a loop algebra is extended via the addition of a (single) central element, then it is referred to as an *affine algebra*. There is then a constructive theorem showing that these are the same as the so-called *"untwisted"* Kac-Moody algebras.

A loop algebra is simply the algebra of all maps from a given, finite-dimensional Lie algebra, $\mathcal{G}$, into the complex plane, realized as Laurent polynomials on the Riemann sphere. The elements of the loop algebra are then the set of all $g\,P(z)$, where $g \in \mathcal{G}$ and $P(z)$ is an arbitrary (Laurent) polynomial over $\mathbb{C}$, i.e., a function of z of the form $\{\sum_{j=-\infty}^{+\infty} c_j z^j\}$, where all but a finite number of the c_j vanish, which is usually denoted by the symbol $\mathbb{C}[z, z^{-1}]$. The Lie product for this (much larger) set is then given by

$$\forall g, r \in \mathcal{G} \;\; [g\,P, r\,Q] \equiv [g, r]\,PQ \;, \tag{3.1}$$

where P and Q are arbitrary Laurent polynomials in z. If, for instance, $\{g_i\}_1^n$ is a basis for $\mathcal{G}$, then a basis of the loop algebra is given by the set $\{g_i\, z^m \mid i = 1, \ldots, n\,;\, m = -\infty, \;\ldots\;, +\infty\}$.

In order for this to be isomorphic to the desired (untwisted) Kac-Moody algebra we must add either one or two more elements. The first one is the (1-dimensional) center, a basis for which I will call $\mathbf{c}$. The second one is by now also customarily included in the definition, although at first it was not, and was then slowly added to the definition because of its great utility; this is a derivation on the algebra, which I will denote by the symbol $\mathbf{d}$, and which does not appear in the derived algebra, i.e., the ideal generated by the set of all possible commutators of the original algebra. The addition of $\mathbf{c}$ requires us to re-define the commutation relationship given above—so that, of course, it will now end up being the appropriate one for the actual Kac-Moody algebra, rather than the one for just the loop algebra. The correct one is then

$$\text{for the K-M algebra } \widetilde{\mathcal{G}} \equiv (\mathcal{G}\otimes\mathbb{C}[z,z^{-1}])\oplus\mathbb{C}\mathbf{c}\oplus\mathbb{C}\mathbf{d}\,,$$

$$[g_i z^m, g_j z^n] \equiv [g_i,g_j]z^{m+n} + m\delta_0^{m+n}(g_i,g_j)\mathbf{c}\,, \tag{3.2}$$

$$[\mathbf{c},\widetilde{\mathcal{G}}]=0\,,\qquad [d, g_i z^m] = m\, g_i z^m\,,$$

where we recall, for instance, that the symbol $\mathbb{C}\mathbf{c}$ means the (1-dimensional) vector space of all possible complex multiples of $\mathbf{c}$, while (g_i, g_j) is just the value of the Killing form for those elements, in the original, finite-dimensional Lie algebra.

The Cartan subalgebra of this algebra is now

$$\widetilde{\mathcal{H}} \equiv (\mathcal{H}\otimes z^0)\oplus\mathbb{C}\mathbf{c}\oplus\mathbb{C}\mathbf{d}\,, \tag{3.3}$$

where $\mathcal{H}$ is the Cartan subalgebra for $\mathcal{G}$. On the other hand, as already suggested, the status of $\mathbf{d}$ is a little bit different than the others. In particular the usual Cartan matrix for $\widetilde{\mathcal{G}}$ does not include $\mathbf{d}$ although it does indeed include $\mathbf{c}$. One then determines the dual space for $\widetilde{\mathcal{H}}$, labelled as usual by $\widetilde{\mathcal{H}}^*$, by appending two additional (basis) elements, δ and Λ, that are

a. orthogonal to each other and also to all the roots for $\mathcal{H}\otimes z^0$, which is of course just isomorphic to $\mathcal{H}$,

b. $t_\delta = \mathbf{d}$ and $t_\Lambda = \mathbf{c}$,

c. for all eigenvectors

$$X_\alpha \in \mathcal{G}\,,\text{and } \forall h \in \widetilde{\mathcal{H}}\,,\quad [h, X_\alpha\, z^m] = \{\alpha(h) + m\delta(h)\}X_\alpha\, z^m\,, \tag{3.4a}$$

d. and also

$$\forall n \neq 0\,,\quad [h, h_\alpha\, z^n] = n\delta\, h_\alpha\, z^n\,. \tag{3.4b}$$

Therefore, we may now characterize the root space for this algebra as

$$\widetilde{\mathcal{H}}^* = \{g_\alpha z^n \mid \alpha\in\boldsymbol{\Sigma},\ n\in\mathbb{Z}\}\oplus\{\mathcal{H}\, z^m \mid m\in\mathbb{Z}/0\}\,. \tag{3.5}$$

We now extend the Killing form on the original $\mathcal{G}$ to one defined on this algebra, $\widetilde{\mathcal{G}}$, as follows, where X and Y are arbitrary elements of $\mathcal{G}$:

$$
\begin{aligned}
&(X\,z^m, Y\,z^n) \equiv \delta_0^{m+n}\,(X,Y)\;, \qquad (X\,z^m, \mathbf{c}) = 0 = (X\,z^m, \mathbf{d}) \\
&(\mathbf{c},\mathbf{c}) = 0\;, \quad (\mathbf{c},\mathbf{d}) = +1\;, \quad (\mathbf{d},\mathbf{d}) = 0\;.
\end{aligned}
\tag{3.6}
$$

To map this into the usual form of a Kac-Moody algebra, generated by the Chevalley generators, we first remember that it is in fact the derived algebra that will satisfy this mapping requirement, i.e., the one without $\mathbf{d}$. we first take the standard basis sets $\{h_j\}_1^n$, $\{e_j\}_1^n$, and $\{f_j\}_1^n$ that generate the original, finite-dimensional Lie algebra, $\mathcal{G}$, and append (just) one more triplet. To determine the eigenvector elements of this extra triplet, namely e_0 and f_0, we first denote by θ the *highest root* of $\mathcal{G}$, and choose scalar multiples of e_θ and f_θ to accomplish the following mapping:

$$
e_0 \propto f_\theta\,z\;, \quad f_0 \propto e_\theta\,z^{-1} \quad \text{such that} \quad (e_0, f_0) = 1\;. \tag{3.6a}
$$

It then follows that their commutator should define the still-desired h_0; using the commutation relation above, we find that

$$
h_0 \equiv [e_0, f_0] = c - h_\theta\,z^0\;, \tag{3.6b}
$$

while the associated roots, from Eq. (3.4a), the definitions, and the fact that $(\mathbf{d},\mathbf{c}) = 1$, are

$$
\pm\alpha_0 \equiv \pm(\delta - \theta)\;. \tag{3.6c}
$$

One can then proceed and show that this does define the desired mapping, so that the standard notation for $\widetilde{\mathcal{G}}$ is $\mathcal{G}^{(1)}$. For those that are simple, one may put here for $\mathcal{G}$ any of the standard, finite-dimensional, simple Lie algebras. If it is true that there is an idempotent outer automorphism of $\mathcal{G}$, i.e., a mapping $\tau : \mathcal{G} \to \mathcal{G}$ such that there is some integer q such that τ^q is the identity map, then one may also utilize this to generate other K-M algebras from finite-dimensional ones. When $q = 2$ these are referred to by the symbol $\mathcal{G}^{(2)}$, and there are in fact an infinite number of them, namely the sequences $\boldsymbol{A}_\ell^{(2)}$ and $\boldsymbol{D}_\ell^{(2)}$, plus $\boldsymbol{E}_6^{(2)}$. (It is perhaps worth noting that this collection, namely those algebras labelled by $\boldsymbol{A}$, $\boldsymbol{D}$, or $\boldsymbol{E}$, are those that have all their roots the same length. They are often referred to as being *simply-laced.*) Finally, when $q = 3$ there is only one, this being $D_4^{(3)}$.

I note that there is an incredible amount of literature on the structure, and the infinite-dimensional representations, of these algebras. The representation theory eventually led to the notion of ***vertex algebras***, which are a considerably larger class of infinite-dimensional Lie algebras, which will not be described here.