# Physics 538

## Suggested Homework #1

**1.** Please consider the following set of 8 matrices, which it is claimed constitute a (simple) Lie algebra. For notation, let the symbol $\omega$ be defined as a particular cube-root of $+1$, namely $\omega \equiv e^{2\pi i/3}$:

$$A^+ \equiv \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \; A^- \equiv \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \; B^+ \equiv \begin{pmatrix} 0 & \omega^2 & 0 \\ 0 & 0 & 1 \\ \omega & 0 & 0 \end{pmatrix}, \; B^- \equiv \begin{pmatrix} 0 & 0 & \omega^2 \\ \omega & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$C^+ \equiv \begin{pmatrix} 0 & \omega & 0 \\ 0 & 0 & 1 \\ \omega^2 & 0 & 0 \end{pmatrix}, \; C^- \equiv \begin{pmatrix} 0 & 0 & \omega \\ \omega^2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \; D^+ \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \; D^- \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}.$$

a. First, show that it is indeed a Lie algebra by calculating an explicit table showing the values of the usual matrix commutator. Note that $\omega$ satisfies an algebraic equation that follows from its property as a third-root of unity, i.e., $\omega^2 + \omega + 1 = 0$.

b. Show that any of the pairs of matrices $\{Q^+, Q^-\}$, where $Q$ may be taken to be either $A$, $B$, $C$, or $D$, may be taken as a Cartan subalgebra for this algebra. Can you also determine a pair of commuting matrices in this algebra that are not acceptable as a Cartan subalgebra?

c. Choose two different ones of the possible Cartan subalgebras noted in (b) and find all their roots.

d. In each of the cases you considered in part (c), choose a subset of positive roots, and a (further) subset of simple roots. Calculate the Killing form for the entire algebra, determine its restriction onto your Cartan subalgebra (in each of your two cases), and create the mapping onto the space of roots, which is a scalar product on that space. Restrict that scalar product further to involve just the simple roots, and re-normalize it so that you determine the Cartan matrix for this algebra.

**2.** Consider the Cartan matrix

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}.$$

It describes a Lie algebra of rank 3. Label the (three+three) simple roots as $\{e_i\}_1^3$ and $\{f_i\}_1^3$, and determine the remaining roots, their lengths, and the lattice structure.