Structure of the Root Spaces for Simple Lie Algebras

I. Introduction

A Cartan subalgebra, $\mathcal{H}$, of a Lie algebra, $\mathcal{G}$, is a subalgebra, $\mathcal{H} \subseteq \mathcal{G}$, such that

a. $\mathcal{H}$ is nilpotent, i.e., there is some n such that $(\mathcal{H})^n = \{0\}$, and

b. also it is its own normalizer, i.e., $\mathcal{N}(\mathcal{H}) \subseteq \mathcal{H}$:

The *normalizer*, $\mathcal{N}(\mathcal{P})$, of a subalgebra, $\mathcal{P} \subseteq \mathcal{G}$, is

$$\mathcal{N}(\mathcal{P}) \equiv \{x \in \mathcal{G} \mid [x, \mathcal{P}] \subseteq \mathcal{P}\} .$$

The normalizer is a subalgebra of $\mathcal{G}$, which contains $\mathcal{P}$ as an ideal within it; in fact, it is the largest subalgebra of $\mathcal{G}$ for which this is true.

All Lie algebras do contain a Cartan subalgebra.

All Cartan subalgebras contain a regular element (at least when the field over which they are defined is infinite). Moreover, any two Cartan subalgebras containing the same regular element are equivalent under inner automorphisms.

For a semisimple Lie algebra, any Cartan subalgebra may be shown to be a maximal, Abelian subalgebra that contains (at least) one *regular element.* Finding a regular element and then creating the maximal, Abelian subalgebra that contains it is a standard method for the construction of the Cartan subalgebra in this case.

A *regular element* is an element of $\mathcal{G}$ such that its adjoint representation has the minimum possible number of zero eigenvalues for that algebra.

We call that number the *rank*, r, of the algebra.

For any $g \in \mathcal{G}$, we have $\text{ad}(g)g = [g, g] = 0$;

therefore, we know that $\text{ad}(g)$ always has at least 1 zero eigenvalue.

The dimension of $\mathcal{H}$, as a vector space, is the same as the rank, r.

As $\mathcal{H}$ is Abelian, the adjoint representations of each element can be diagonalized simultaneously. Therefore, let us denote the general element by $h \in \mathcal{H}$, and write it as $h \equiv \lambda^i h_i$, where $\{h_i\}_1^r$ is a choice of basis for $\mathcal{H}$, and we intend that this "general element" has non-zero values for all the $\{\lambda^i\}_1^r$; therefore, $\mathrm{ad}(h)$ is an $m \times m$ matrix with the r different λ_i as parameters. We next hunt for the eigenvalues of $\mathrm{ad}(h)$. As all the elements can be simultaneously diagonalized, these eigenvalues will be linear in these parameters, that distinguish the different elements of $\mathcal{H}$. This then allows us to characterize them as linear functionals over the vector space $\mathcal{H}$, i.e., as elements of the dual vector space, usually denoted by $\mathcal{H}^*$. Extracting the 0 eigenvalues, there are $(m-r)$ different ones, possibly counting repetitions, although they are not all linearly independent since the space is only r-dimensional. We may then find a corresponding set of eigenvectors, i.e., determine x_α such that $\mathrm{ad}(h)\, x_\alpha = \alpha(h)\, x_\alpha$. Choosing them as a (possibly new) basis in $\mathcal{G}$, $\mathrm{ad}(h)$ will take a diagonal form, with r zero's and each of the other (non-zero) eigenvalues along the diagonal, in whatever order we have taken the basis. We will generically use the symbol α for any one of these non-zero, eigenvalue-determined linear functionals, or α_1, α_2, etc., if it is necessary to consider distinct ones, and will refer to them as *roots*,
and will denote the set of all the roots by $\boldsymbol{\Sigma}$:

$$\alpha \in \boldsymbol{\Sigma} \quad \Longleftrightarrow \quad \exists!\, x_\alpha \in \mathcal{G} \text{ such that } \mathrm{ad}(h)x_\alpha \equiv [h, x_\alpha] = \alpha(h)\, x_\alpha \;; \tag{1.1a}$$

We may write $\alpha(h) = \lambda^i\, \alpha(h_i)$, so that we may present the vector, in $\mathcal{H}^*$, in terms of its (covariant) components, namely $(\alpha(h_1), \alpha(h_2), \ldots, \alpha(h_r))$. We define the linear span of all eigenvectors, for a given α, by $\mathcal{G}_\alpha$, and note that the Lie algebra decomposes into

$$\mathcal{G} = \mathcal{H} \oplus \bigoplus_{\alpha \in \boldsymbol{\Sigma}} \mathcal{G}_\alpha \;. \tag{1.1b}$$

As a vector space dual to $\mathcal{H}$, the space $\mathcal{H}^*$ must have dimension r; therefore the $m - r$ distinct eigenvalues, α_m, are far too many to be used to constitute a basis. Instead they form a *lattice* within $\mathcal{H}^*$, the structure of which we want to determine. Therefore it is important to determine a reasonable basis for $\mathcal{H}^*$. Using the current, non-canonical, choice of basis for

$\mathcal{H}$, $\{h_i\}$, there are two plausible ways to define a basis for $\mathcal{H}^*$. An obvious choice for a set of components for a given α would be to simply use the quantities $\alpha(h_i)$, which in fact we have already labelled as covariant components. A different but very plausible approach to the question of describing all the roots would be to simply give the value of $\alpha(h)$, i.e., the action of our eigenvalue cum functional when it acts on the general element of $\mathcal{H}$; however, with $h \equiv \lambda^i h_i$, this amounts to the same as choosing the set of $\{\alpha(h_i) \mid i = 1, \ldots, r\}$. A "cleverer" approach is seen by noting the existence of a scalar product on $\mathcal{H}^*$, and then using the basis dual to the $\{h_i\}$, relative to that scalar product. The scalar product in question can be defined on $\mathcal{H}^*$, and will be determined by the Killing form on $\mathcal{G}$, which we now describe.

II. The quadratic form on $\mathcal{G}$ due to Killing, and Cartan

For arbitrary $g, p \in \mathcal{G}$, the value of the Killing form on them is defined by

$$(g, p) \equiv \mathrm{tr}\ \{\mathrm{ad}(g)\mathrm{ad}(p)\}\ . \tag{2.1}$$

In general, this cannot be taken to be a scalar product on $\mathcal{G}$, since it may be both non-definite and degenerate. For example, for the 3-dimensional affine (or Euclidean) group in the plane only one entry is non-zero in the entire 3×3 matrix. However, we do have the following:

Cartan's Criterion: the Killing form of $\mathcal{G}$ is non-singular if and only if $\mathcal{G}$ is semi-simple.

For $\{z_i\}$ a basis for $\mathcal{G}$, this says that the matrix (z_i, z_j) is invertible $\Longleftrightarrow$ $\mathcal{G}$ is semi-simple.

For $t, h \in \mathcal{H}$, the Killing form, (t, h), is linear in its arguments. As $\mathcal{H}^*$ is the space of all linear functionals over $\mathcal{H}$, it follows that for every $t \in \mathcal{H}$ there is a unique $\alpha_t \in \mathcal{H}^*$ such that $\alpha_t(h) = (t, h)$. We however note that in general we will use this correspondence much more regularly in the opposite direction, defining a mapping from $\mathcal{H}^*$ to $\mathcal{H}$ as follows:

$$\forall \alpha \in \mathcal{H}^*, \quad \exists!\, t_\alpha \in \mathcal{H}\ \ni \quad \alpha(h) = (t_\alpha, h)\ . \tag{2.2}$$

For semi-simple algebras, the Killing form induces a true scalar product on the space of roots, $\mathcal{H}^*$, as we now show. We may choose a basis for $\mathcal{G}$ so that all the elements of $\mathcal{H}$ are diagonal; therefore, we can immediately note that the Killing form restricted to $\mathcal{H}$ gives

$$(h_i, h_j) = \sum_{\alpha\in\boldsymbol{\Sigma}} \alpha(h_i)\alpha(h_j) = \sum_{\alpha\in\boldsymbol{\Sigma}} (t_\alpha, h_i)(t_\alpha, h_j) \quad \Longrightarrow \quad (h_i, h_i) = \sum_{\alpha\in\boldsymbol{\Sigma}} (t_\alpha, h_i)^2 \ , \tag{2.3}$$

so that $(\cdot,\cdot)\Big|_{\mathcal{H}\otimes\mathcal{H}}$ is positive-definite, i.e., it is an ordinary scalar product. The induced quadratic form on $\mathcal{H}^*$ is then also a scalar product, defined via

$$\begin{gathered}\langle\alpha,\beta\rangle \equiv (t_\alpha, t_\beta) \ , \\ \text{where } \ \langle \ , \ \rangle : \mathcal{H}^* \times \mathcal{H}^* \to \mathbb{C} \ , \quad (\ , \ ) : \mathcal{H} \times \mathcal{H} \to \mathbb{C} \ .\end{gathered} \tag{2.4}$$

Having a scalar product, we may use it in the usual way to define angles between two elements $\alpha, \beta \in \mathcal{H}^*$:

$$\cos\theta_{(\alpha,\beta)} \equiv \frac{\langle\alpha, \beta\rangle}{\sqrt{\langle\alpha, \alpha\rangle\langle\beta, \beta\rangle}} \ . \tag{2.5}$$

To explicitly determine this mapping, i.e., to find the associated subalgebra member, t_α, we write out the content of Eq. (2.2) in terms of the basis for $\mathcal{H}$, i.e., write $t_\alpha \equiv \alpha^i t_i \in \mathcal{H}^*$, and then use the non-singularity of the Killing form:

$$t_\alpha \equiv \alpha^i h_i \quad \Longrightarrow \quad \alpha(h_k) = (t_\alpha, h_k) = \alpha^i(h_i, h_k) \implies \alpha^i = \{(h_i, h_k)\}^{-1}\alpha(h_k) \equiv g^{ik}\alpha(h_k) \ , \tag{2.6}$$

where the last entries involves matrix multiplication (using the Einstein summation convention on the indices) so that $\{(t_i, t_k)\}^{-1}$ indicates the elements of the matrix inverse to the matrix of the Killing form of the various basis elements. It is usual to use the matrix $(h_i, h_k) \equiv g_{ik}$ as a metric when it is non-degenerate, as it is in this case, and we write g^{ik} for the inverse metric.

These α^i's may then be thought of as the contravariant components of the functional/vector, $\alpha \in \mathcal{H}^*$. On the other hand, we may also better understand the quantities $\alpha(t_k)$ mentioned already by creating a dual basis for the space of functionals:

$$\tau^k(t_i) = (t_{\tau^k}, t_i) = \delta^k_i \quad \Longrightarrow \alpha = \{\alpha(t_k)\}\tau^k \ . \tag{2.7}$$

We are about to study the lattice structure of the roots in some detail. However, in that section we will need to know that the Killing form is *associative*, i.e., for $x, y, z \in \mathcal{G}$, we have

$$(x, [y, z]) = ([x, y], z) \ , \qquad \text{associativity for the Killing form.} \tag{2.8}$$

This is therefore a good time to go ahead and provide the proof of that in detail:

$$\begin{aligned}(x, [y, z]) &= \mathrm{tr}\,\{\mathrm{ad}(x)\mathrm{ad}([y, z])\} = \mathrm{tr}\,\{\mathrm{ad}(x)[\mathrm{ad}(y), \mathrm{ad}(z)]\} \\ &= \mathrm{tr}\,\{\mathrm{ad}(x)\mathrm{ad}(y)\mathrm{ad}(z) - \mathrm{ad}(x)\mathrm{ad}(z)\mathrm{ad}(y)\} \\ &= \mathrm{tr}\,\{\mathrm{ad}(x)\mathrm{ad}(y)\mathrm{ad}(z) - \mathrm{ad}(y)\mathrm{ad}(x)\mathrm{ad}(z)\} = \mathrm{tr}\,\{[\mathrm{ad}(x), \mathrm{ad}(y)]\mathrm{ad}(z)\} \\ &= \mathrm{tr}\,\{\mathrm{ad}([x, y])\mathrm{ad}(z)\} = ([x, y], z) \ ,\end{aligned} \tag{2.9}$$

where we have used the very useful property of the adjoint operator that it creates a representation of the algebra, i.e., preserves (Lie) products:

$$\begin{aligned}\mathrm{ad}([b, c])\, x \equiv [[b, c], x] =& [x, [c, b]] = [c, [x, b]] - [b, [c, x]] \\ &= -\, [c, [b, x]] + [b, [c, x]] = [\mathrm{ad}(b), \mathrm{ad}(c)]\, x \ .\end{aligned} \tag{2.10}$$

A not-too-complicated example of this is probably useful at this point. We will use $\boldsymbol{su}(3)$, a particular real form of $\boldsymbol{sl}(3,\mathbb{C})$. It is 8 dimensional, with a Cartan subalgebra of dimension 2. Following Cahn we define an arbitrary element of the algebra in the form

$$g = aT_+ + bT_- + cT_z + dU_+ + eU_- + fV_+ + gV_- + hY \ . \tag{2.11}$$

Cahn chooses the basis for the Cartan subalgebra as $\{T_z, Y\}$. Using the commutators given on p. 11 of his book, after fixing the typo's for $[v_+, u_-] = t_+$ and $[u_-, v_+] = -t_+$, we can write the adjoint representation for any one of these, for simplicity writing it for the arbitrary element noted above:

$$\mathrm{ad}(g) \Longrightarrow \begin{array}{c|cccccccc|} & T_+ & T_- & T_z & U_+ & U_- & V_+ & V_- & Y \\ \hline T_+ & +c & 0 & -a & 0 & f & -e & 0 & 0 \\ T_- & 0 & -c & b & -g & 0 & 0 & d & 0 \\ T_z & -2b & 2a & 0 & e & -d & -g & f & 0 \\ U_+ & 0 & -f & \frac{1}{2}d & h-\frac{1}{2}c & 0 & b & 0 & -d \\ U_- & g & 0 & -\frac{1}{2}e & 0 & \frac{1}{2}c-h & 0 & -a & e \\ V_+ & -d & 0 & -\frac{1}{2}f & a & 0 & h+\frac{1}{2}c & 0 & -f \\ V_- & 0 & e & \frac{1}{2}g & 0 & -b & 0 & -h-\frac{1}{2}c & g \\ Y & 0 & 0 & 0 & -\frac{3}{2}e & \frac{3}{2}d & -\frac{3}{2}g & \frac{3}{2}f & 0 \end{array} \; . \qquad (2.12)$$

An example shows how to use this matrix:

> If we just wanted $\mathrm{ad}(V)_+$, we would set $f = 1$, and all the other parameters to zero. Thinking of the 8 basis vectors above as basis vectors for 8-dimensional column vectors, then, for instance, U_- would correspond to a column vector with a $+1$ in the fifth position and the others zero, while T_+ would be presented as a column vector with a $+1$ in the first position and all the others zero. Doing all this, we note that it does say that the action of $\mathrm{ad}(V)_+$ on U_- is indeed T_+, as was noted above as a special case.

Then the arbitrary element of the Cartan subalgebra corresponds to maintaining c and h arbitrary, and setting the others to zero, so that the general element of $\mathcal{H}$ is given by $h = c\,T_z + h\,Y$, so that, from Eq. (2.12) above, we have $\mathrm{ad}(h) = \mathrm{diag}(c, -c, 0, h - \frac{1}{2}c, -h + \frac{1}{2}c, h + \frac{1}{2}c, -h - \frac{1}{2}c, 0)$, i.e., the diagonal 8×8 matrix with these elements on the diagonal. That this results in a diagonal matrix tells us that we have already chosen the basis for the algebra as eigenvectors of this choice for a Cartan subalgebra. There are 6 non-zero roots, three of which are the negatives of the other three; arbitrarily we choose three of them and refer to them as "the positive roots":

$$\alpha_1(h) = c\ , \quad \alpha_2(h) = -c/2 + h\ , \quad \alpha_3(h) = c/2 + h\ . \qquad (2.13)$$

The 3 elements of $\mathcal{H}$ that correspond to these 3 roots are just linear combinations of elements of $\mathcal{H}$, i.e., of the form $a\,T_z + b\,Y$; therefore we write down equations to determine those, noting that $\alpha_3 = \alpha_1 + \alpha_2$, so that linearity says we don't need that equation as well:

$$\begin{aligned} c = \alpha_1(T_z) &\equiv (t_{\alpha_1}, T_z) = a_1(T_z, T_z) + b_1(Y, T_z) \ , \\ 0 = \alpha_1(Y) &\equiv (t_{\alpha_1}, Y) = a_1(T_z, Y) + b_1(Y, Y) \ , \\ & \\ -1/2 = \alpha_2(T_z) &\equiv (t_{\alpha_2}, T_z) = a_2(T_z, T_z) + b_2(Y, T_z) \ , \\ 1 = \alpha_2(Y) &\equiv (t_{\alpha_2}, Y) = a_2(T_z, Y) + b_2(Y, Y) \ . \end{aligned} \tag{2.14}$$

We may easily resolve these 4 equations for the 4 desired unknowns, were we to know the actual values of the Killing form, at least as restricted to $\mathcal{H}$. Therefore, we next determine those:

$$\begin{aligned} (T_z, T_z) &= \text{trace}[\text{ad}(T_z)\text{ad}(T_z)] = 1 + 1 + 0 + \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{4} + 0 = 3 \ , \\ (Y, T_z) = (T_z, Y) &= \text{trace}[\text{ad}(T_z)\text{ad}(()Y)] = 0 + 0 + 0 - \tfrac{1}{2} - \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + 0 = 0 \ , \\ (Y, Y) &= \text{trace}[\text{ad}(()Y)\text{ad}(()Y)] = 0 + 0 + 0 + 1 + 1 + 1 + 1 + 0 = 4 \ . \end{aligned} \tag{2.15}$$

Given this (diagonal) matrix, we may resolve our equations above, and also work out the scalar products between them:

$$\begin{aligned} t_{\alpha_1} &= \tfrac{1}{3} T_z \ , \\ t_{\alpha_2} &= -\tfrac{1}{6} T_z + \tfrac{1}{4} Y \ , \\ \langle \alpha_1, \alpha_1 \rangle &\equiv (t_{\alpha_1}, t_{\alpha_1}) = (\tfrac{1}{3} T_z, \tfrac{1}{3} T_z) = \tfrac{1}{9}(3) = \frac{1}{3} \ , \\ \langle \alpha_1, \alpha_2 \rangle &\equiv (t_{\alpha_1}, t_{\alpha_2}) = (\tfrac{1}{3} T_z, -\tfrac{1}{6} T_z + \tfrac{1}{4} Y) = -\tfrac{1}{18}(3) + 0 = -\frac{1}{6} \ , \\ \langle \alpha_2, \alpha_2 \rangle &\equiv (t_{\alpha_2}, t_{\alpha_2}) = (-\tfrac{1}{6} T_z + \tfrac{1}{4} Y, -\tfrac{1}{6} T_z + \tfrac{1}{4} Y) = \tfrac{1}{36}(3) - 2\tfrac{1}{24}(0) + \tfrac{1}{16}(4) = \frac{1}{3} \ , \\ \Longrightarrow \quad & ((\langle \alpha_j, \alpha_k \rangle)) = \frac{1}{6} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \ . \end{aligned} \tag{2.16}$$

III. Generation of the Roots

A. General Properties of Two Arbitrary Roots

Now let $\alpha, \beta \in \mathcal{H}^*$ be 2 roots. Then we easily have

$$\begin{aligned}[h,[x_\alpha,x_\beta]] = [x_\alpha,[h,x_\beta]] - [x_\beta,[h,x_\alpha]] &= \beta(h)[x_\alpha,x_\beta] - \alpha(h)[x_\beta,x_\alpha] \\ &= \{\alpha(h)+\beta(h)\}[x_\alpha,x_\beta]\ .\end{aligned} \tag{3.1}$$

We can conclude several useful pieces of information from this result.

1.) The first conclusion is that the algebra in question is *graded*, which means the following. We recall that $\mathcal{G}_\alpha$ is the linear span of all those eigenvectors that correspond to the root α, i.e., to a specific set of eigenvalues for an arbitrary element of $\mathcal{H}$. A graded algebra is one satisfying the following requirement where $\{\alpha,\beta\}$ are members of some set closed under a definition of addition:

$$[\mathcal{G}_\alpha\,,\mathcal{G}_\beta] \subseteq \mathcal{G}_{\alpha+\beta}\ . \tag{3.2}$$

That our algebra is graded under the set $\boldsymbol{\Sigma}$ follows from our result.

2.) The detailed information given in Eq. (3.1) allows a division into three distinct, non-overlapping categories, for our given pair of roots, α and β, remembering that 0 is **not** a root, i.e., not an element of the set $\boldsymbol{\Sigma}$:

a. if $\alpha+\beta \in \boldsymbol{\Sigma}$, then $[x_\alpha,x_\beta] \propto x_{\alpha+\beta}$, (3.3)

b. or if $\alpha+\beta = 0$, then $[x_\alpha,x_\beta] = [x_\alpha,x_{-\alpha}] \in \mathcal{G}_0 = \mathcal{H}$, (3.4)

c. or, otherwise, $[x_\alpha,x_\beta] = 0$. (3.5)

The first of these statements is equivalent to the idea that the eigenvectors x_α, under the action of the commutator, act as raising operators, raising the eigenvalue β up to $\beta+\alpha$, whenever $\beta+\alpha \in \boldsymbol{\Sigma}$.

To understand a little more about the content of these first two options, we now use the associativity of the Killing form, rewriting the value of $(h,[x_\alpha,x_\beta])$ in two different ways:

$$\begin{aligned}(h,[x_\alpha,x_\beta]) &= ([h,x_\alpha],x_\beta) = \alpha(h)(x_\alpha,x_\beta)\,, \\ (h,[x_\alpha,x_\beta]) &= ([h,x_\alpha],x_\beta) = -([x_\alpha,h],x_\beta) = -(x_\alpha,[h,x_\beta]) = -\beta(h)(x_\alpha,x_\beta)\,, \\ &\Longrightarrow \quad \{\alpha(h)+\beta(h)\}(x_\alpha,x_\beta) = 0\ .\end{aligned} \tag{3.6}$$

Therefore, for the case when $0 \neq \alpha + \beta \in \boldsymbol{\Sigma}$, we see that the two associated vector spaces are perpendicular, i.e.,

$$\mathcal{G}_\alpha \perp \mathcal{G}_\beta \; . \tag{3.7a}$$

On the other hand, in the case that $\alpha + \beta = 0$, i.e., when $\beta = -\alpha$, we have

$$(h, [x_\alpha, x_{-\alpha}]) = \alpha(h)(x_\alpha, x_{-\alpha}) = (x_\alpha, x_{-\alpha})(t_\alpha, h) \; , \tag{3.7b}$$

so that the linearity in h will allow us to write the conclusion above about this commutator, i.e., $[x_\alpha, x_{-\alpha}]$ in a somewhat more certain fashion:

$$[x_\alpha, x_{-\alpha}] = (x_\alpha, x_{-\alpha})\, t_\alpha \in \mathcal{H} \; . \tag{3.7c}$$

B. Now we ask which (integer) Multiples of a Root Exist?

First, we show that all non-zero roots come in pairs, α and $-\alpha$. To see this, presume that it is in fact not true, i.e., that there exists (at least) one root $\alpha \in \boldsymbol{\Sigma}$ such that $-\alpha \notin \boldsymbol{\Sigma}$, i.e., $\mathcal{G}_{-\alpha} = 0$. It then follows that for all $\beta \in \boldsymbol{\Sigma}$, $(\mathcal{G}_\alpha, \mathcal{G}_\beta) = 0$, since it is never true that $\alpha + \beta = 0$. This means that there are vectors in the space for which the Killing form is singular, contrary to Cartan's Criterion for semi-simple Lie algebras. The result must then be that our presumption is wrong, i.e., it is true that all non-zero roots come in ± pairs.

Then we want to show that this is really the only allowed, non-trivial multiple of a root; i.e., the only allowed multiples of a given root, α are just multiplication by $0, \pm 1$. As a (useful) byproduct, we will also show that each root has exactly one eigenvector associated with it, i.e., that the multiplicity of each non-zero eigenvalue is just $+1$, or, in other words, that the dimension of the (vector) subspace $\mathcal{G}_\alpha$ is just $+1$, for all $\alpha \in \boldsymbol{\Sigma}$. We begin by noting that Eq. (3.7c), above, shows that we can always write t_α in the form of the commutator of two elements of $\mathcal{G}$. However, the trace of a commutator of two matrices is always zero, which tells us that the trace of the adjoint representation of any t_α must be zero, i.e., $\mathrm{tr}\ \mathrm{ad}(t_\alpha) = 0$.

Now we pick a particular root α, and consider all its possible, positive integer multiples, i.e., $2\alpha = \alpha + \alpha$, $3\alpha = \alpha + \alpha + \alpha$, etc., which we may denote by $\{k\alpha \mid k = 1, 2, 3, \ldots\}$, and let

$\bigoplus_k \mathcal{G}_{k\alpha}$ be the subspace spanned by the set of those. Not being too sure what is the dimension of any of these spaces, we denote the dimension, i.e., the multiplicity of the eigenvalue, of each one by the symbol d_k, for $k = 1, 2, 3, \ldots$. Lastly, we know that there is at least one eigenvector $x_{-\alpha}$, which may or may not also have a multiplicity greater than 1. Therefore, pick some particular element from that space, which we call y; i.e., pick some eigenvector such that $[h_\alpha, y] = -\alpha(h_\alpha)\, y$, and also such that $[x_\alpha, y] \propto h_\alpha \in \mathcal{H} \equiv \mathcal{G}_0$. Then, we may look at the linear span of all these vectors, i.e., $V = \langle y, h_\alpha, x_\alpha, x_{k\alpha} \mid k = 2, 3, 4, \ldots \rangle$.

Note that $V \subseteq \mathcal{G}$ is preserved when it is acted upon by each, separately, of $\mathrm{ad}(y)$ and $\mathrm{ad}(x_\alpha)$, since they simply permute the various pieces around. This invariance allows us to create a basis for $\mathcal{G}$ which contains first a basis for V and then a basis from the rest of $\mathcal{G}$, only; this causes each of the matrices $\mathrm{ad}(y)$ and $\mathrm{ad}(x_\alpha)$ to have a block diagonal form with a zero below the upper block, which will then therefore also be true of the product. From this we infer that the restrictions of $\mathrm{ad}(y)$ and $\mathrm{ad}(x_\alpha)$ to V determine completely the restriction of $\mathrm{ad}(h_\alpha)$ to V; therefore the trace of the restriction of $\mathrm{ad}(h_\alpha)$ to V is still zero. However, since the chosen basis for this subspace, V, is a basis of the eigenvectors of h_α, the matrix $\mathrm{ad}(h_\alpha)$ is in diagonal form, so that the trace is very easy to calculate:

$$0 = \mathrm{tr}\,\{\mathrm{ad}(h_\alpha)\Big|_V\} = \alpha(h_\alpha)\,(-1 + 0 + d_1 + d_2 + d_3 + \ldots)\ . \tag{3.10}$$

However, we know that $d_1 \geq +1$, since we began with the assumption that $\alpha \in \boldsymbol{\Sigma}$. As well, we are quite sure that dimensions are never negative numbers, and are always integers. Therefore, the only allowed solution to the problem is that

$$d_1 = 1; \quad 0 = d_2 = d_3 = d_4 = \ldots\ , \tag{3.11}$$

which is the same as saying that only ± 1 and 0 are allowed multiples of any root, $\alpha \in \boldsymbol{\Sigma}$, and **also** that the multiplicity of α is exactly 1. By giving, then, the same argument, but beginning with $-\alpha$ as the given root, we could show that the multiplicity of $-\alpha$ is also $+1$. This completes our desired proof, namely that the only allowed multiples of α as a root are

± 1, and, trivially, zero, and that each of $+\alpha$ and $-\alpha$ have exactly one and only one associated eigenvector.

C. Strings of Roots generated by Two Non-Zero Roots

On the other hand, now let $\alpha, \beta \in \boldsymbol{\Sigma}$ be two arbitrary roots, but such that $\alpha \neq \beta \neq -\alpha$. It may well be, then, that $\alpha + \beta$ could be a root, and, then, perhaps, $\alpha + 2\beta$, etc.. Similarly, it might be that $\alpha - \beta$ could be a root, and, then, perhaps, $\alpha - 2\beta$, etc. A sequence like this can be characterized in general in the form

$$\alpha + p\beta, \ldots, \alpha + \beta, \alpha, \alpha - \beta, \ldots, \alpha - m\beta \; . \tag{3.12}$$

where, of course, m and p are integers. Such a sequence is called **the $\boldsymbol{\beta}$-string containing $\boldsymbol{\alpha}$**. The integers in question must be finite since there are only finitely many roots, so that any such string cannot continue indefinitely! The highest element in the string, i.e., the one with the value p, we will call the "highest root" in the string; it is convenient to give it a name for the moment, so set $\gamma \equiv \alpha + p\,\beta$. We now want to show that a knowledge of α and β is sufficient to characterize this (joint) string. We begin by first thinking about Eq. (3.3) which tells us that the result of the action on x_α by x_β is proportional to $x_{\alpha+\beta}$, assumed non-zero. However, as these various x_{α_i} are only defined as eigenvectors of $\mathrm{ad}(h)$, their normalization is as yet undetermined. Therefore, we are allowed to normalize them as desired. Following Cahn, p. 34, we may begin with $x_\gamma \equiv x_{\alpha+p\beta}$ and use the action of $\mathrm{ad}(x_{-\beta})$ to create each of the "lower" elements of the string, normalizing as we go by taking the so-created eigenvector to be correctly normalized:

$$\mathrm{ad}(x_{-\beta})^j x_\gamma \equiv x_{\gamma - j\beta} \; . \tag{3.13a}$$

Of course, by our definitions of p and m, this process continues until we have a total of $p+m+1$ elements in the string. Having created them with this normalization, we can not then ignore the normalization a second time; therefore, when considering the action of $\mathrm{ad}(x_\beta)$ on these objects, we define normalization coefficients $N_{\delta,\beta}$ so that

$$\mathrm{ad}(x_\beta) x_\delta = N_{\delta,\beta} x_{\delta+\beta} \; , \tag{3.13b}$$

for any root δ that exists in this string. We now seek a recursion relation for these coefficients $N_{\gamma-k\beta,\beta}$, as follows. However, notational simplicities will make the succeeding calculations easier to follow; therefore, we adopt the notation N_k as a nice shorthand for $N_{\gamma-k\beta,\beta}$ and also the symbol x_{-k} as a shorthand for $x_{\gamma-k\beta}$. We may now give the calculation that creates the desired recursion relation:

$$\begin{aligned} N_k\, x_{-k+1} &= \mathrm{ad}(x_\beta)\, x_{-k} = \mathrm{ad}(x_\beta)\, \mathrm{ad}(x_{-\beta}) x_{-k+1} = \{\mathrm{ad}(x_{-\beta})\, \mathrm{ad}(x_\beta) + (x_\beta, x_{-\beta})\, \mathrm{ad}(t_\beta)\}\, x_{-k+1} \\ &= \{\mathrm{ad}(x_{-\beta})\, N_{k-1} \mathrm{ad}(x_{-k+2}) + (x_\beta, x_{-\beta})\, [\gamma(t_\beta) - (k-1)\beta(t_\beta)] x_{-k+1}\} \\ &= \{N_{k-1} + (x_\beta, x_{-\beta})\ [\langle\gamma,\beta\rangle - (k-1)\langle\beta,\beta\rangle]\}\ x_{-k+1}\ , \end{aligned} \tag{3.14}$$

which gives us the value of N_k in terms of N_{k-1} and other quantities, i.e., a genuine 1-term recursion relation. However, we also know that $N_0 \equiv 0$, since γ is by definition the highest root in this string, and the action of $\mathrm{ad}(x_\beta)$ on it must therefore give zero. This provides the reference point needed to determine a solution to the equation. By writing down various next levels of the equation, for example

$$\begin{gathered} k = 0: \quad N_0 = 0\ , \\ k = 1: \quad N_1 = (x_\beta, x_{-\beta})\, \langle\gamma,\beta\rangle\ , \\ k = 2: \quad N_2 = N_1 + (x_\beta, x_{-\beta})\ [\langle\gamma,\beta\rangle - \langle\beta,\beta\rangle] = (x_\beta, x_{-\beta})\ [2\langle\gamma,\beta\rangle - \langle\beta, beta\rangle]\ , \\ k = 3: \quad N_3 = N_2 + (x_\beta, x_{-\beta})\ [\langle\gamma,\beta\rangle - 2\langle\beta,\beta\rangle] = (x_\beta, x_{-\beta})\ [3\langle\gamma,\beta\rangle - 3\langle\beta,\beta\rangle]\ , \\ k = 4: \quad N_4 = N_3 + (x_\beta, x_{-\beta})\ [\langle\gamma,\beta\rangle - 3\langle\beta,\beta\rangle] = (x_\beta, x_{-\beta})\ [4\langle\gamma,\beta\rangle - 6\langle\beta,\beta\rangle]\ , \\ \ldots \\ N_k = (x_\beta, x_{-\beta})\ \left[k\langle\gamma,\beta\rangle - \tfrac{1}{2}k(k-1)\langle\beta,\beta\rangle\right]\ . \end{gathered} \tag{3.15}$$

At this point we add the "extra" boundary condition to this problem that allows us to actually determine the constants themselves. We have already inserted the fact that the string does not go "upward" past p; now we must also tell the recursion relation that the string does go "downward" past m, i.e., the entire string has only $p+m+1$ members:

$$N_{p+m+1}\, x_{p+m} = \mathrm{ad}(x_\beta)\, \mathrm{ad}(x_{-\beta})\, x_{p+m} = 0\ , \tag{3.16}$$

so that our recursion relation, unusually, has a boundary condition at the other end as well, so that its values are explicitly determined. Inserting the value $k = p + m + 1$ into Eq. (3.15), we may write

$$0 = N_{p+m+1} = (p + m + 1)(x_\beta, x_{-\beta})\ \left\{\langle\gamma, \beta\rangle - \tfrac{1}{2}(p + m)\langle\beta, \beta\rangle\right\}\ . \tag{3.17}$$

Since the minimum allowed values for p and m are each zero, we now have the very useful equation

$$p + m = 2\frac{\langle\gamma, \beta\rangle}{\langle\beta, \beta\rangle} = 2\frac{\langle\alpha + p\beta, \beta\rangle}{\langle\beta, \beta\rangle}\ , \tag{3.18}$$

where the last equality comes by recalling the definition of the root γ. From that last equality as well, we can recover an even more useful equation:

$$m - p = 2\frac{\langle\alpha, \beta\rangle}{\langle\beta, \beta\rangle}\ , \tag{3.19}$$

which tells us that this interesting ratio of Killing forms of two distinct roots is in fact an integer! It is this equation that will guide us in many of our (successful) attempts to uncover the (rather unexpected) nature of the lattice of roots that lies in $\mathcal{H}^*$. We will describe these properties in the following sequence of "lemmas" concerning them. It may also be convenient to define a "re-normalized" root, $\check{\beta}$, defined so that

$$\check{\beta} \equiv 2\frac{\beta}{\langle\beta, \beta\rangle} \qquad \Longrightarrow \qquad m - p = \langle\alpha, \check{\beta}\rangle \equiv 2\frac{\langle\alpha, \beta\rangle}{\langle\beta, \beta\rangle}\ . \tag{3.19$'$}$$

1.) A string of roots can contain no more than 4 roots.

Proof: Suppose that it contains 5! In that event, we can surely label them so that both m and p have the value 2, i.e., we can name them $\{\alpha + 2\beta, \alpha + \beta, \alpha, \alpha - \beta, \alpha - 2\beta\}$. Now, of course β is also a root; however, $2\beta = (\alpha + 2\beta) - \alpha$ is not a root; likewise while $\beta + \alpha$ is a root $2(\beta + \alpha) = (\alpha + 2\beta) + \alpha$ is not a root. Therefore the root $\eta \equiv \alpha + 2\beta$ is in an ***α*-string** of roots with only one element in it, since either subtracting or adding α to it results in something which is not a root. This implies that for this string, the corresponding values of m and p are both zero. Likewise, $\zeta \equiv \alpha - 2\beta$ has the same property, i.e., $-2\beta = (\alpha - 2\beta) - \alpha$ is not a root, and $2(\alpha - \beta) = (\alpha - 2\beta) + \alpha$ is not a root, so that the corresponding m and p are again zero:

$$\begin{aligned} 0 = m_\eta - p_\eta &= 2\frac{\langle \eta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \ , \\ 0 = m_\zeta - p_\zeta &= 2\frac{\langle \zeta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \ . \end{aligned} \tag{3.20}$$

Adding these two equalities, we immediately see that it must be that $\langle \alpha, \alpha \rangle = 0$. However, since the scalar product on $\mathcal{H}^*$ is positive-definite, the only conclusion we can draw is that $\alpha = 0$, so that the root does not in fact have 5 elements.

The conclusion to be drawn from the above is that the length of such a string can be not more than 4. We will in fact see that it can be 4, by the consideration, later on, of specific cases where this holds—actually in the second rank algebra, $\mathbf{G}_2$, which we will look at in some detail. Therefore, we may go ahead and take this to be the case for now.

We then consider all allowed options, to divine associated allowed values of m and p.

i. if the string has 4 elements, we write it in the form $\{\alpha + 2\beta, \alpha + \beta, \alpha, \alpha - \beta\}$; then $m - p$ may only be ± 3 or ± 1, depending on where we begin in that string;
ii. if the string has only three elements, then either $m - p$ is ± 2 or 0;
iii. if the string has only two elements, then $m - p = \pm 1$, and
iv. if the string has just the single element, then $m - p = 0$.

2.) We can now study the properties of the cosine of the angle between α and β, as given in Eq. (2.5), by re-thinking the derivation of Eq. (3.19) above, as it would be if applied to the converse notion, i.e., **an $\boldsymbol{\alpha}$-string containing $\boldsymbol{\beta}$**. Labelling the m and p from the first construction by a subscript α, and these second ones by a subscript β, analogously to that used in the previous derivation, we have

$$\begin{gathered} m_\alpha - p_\alpha = 2\frac{\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle} \ , \qquad m_\beta - p_\beta = 2\frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \ , \\ \Longrightarrow \quad \cos^2\theta(\alpha, \beta) = \frac{\langle \alpha, \beta \rangle \, \langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle \, \langle \beta, \beta \rangle} = \frac{1}{4}(m_\alpha - p_\alpha)(m_\beta - p_\beta) \ . \end{gathered} \tag{3.21}$$

> This tells us that these angles at which two roots meet are incredibly circumscribed; in all cases, the square of the cosine of that angle must be a fourth of some integer, i.e., it can only take on the values 0, $\frac{1}{4}$, $\frac{2}{4} = \frac{1}{2}$, or $\frac{3}{4}$.

That these angles are so very strongly circumscribed tells us a great deal about the possible sorts of lattices that can be allowed for the root space of a simple Lie algebra, which will become clearer yet when we determine a reasonable basis for the vector space, $\mathcal{H}^*$.

IV. Simple Roots

A. Positive and Simple Roots

As is plausible, among a set of eigenvalues, there is no particular natural ordering of the roots. Nonetheless, we shall see that the structure of the root space will allow itself to be discovered with almost any choice of an ordering. Since we know that every root, α, has its counterpart $-\alpha$, it is natural to attempt to choose the "positive" roots. Unfortunately, it is not immediately clear which ones these are; happily it doesn't seem to matter much. One simply goes ahead and makes a choice. The method used by Cahn is "as good as any other," and goes like this. First choose a basis for $\mathcal{H}^*$, and call it $\{\alpha^i\}_1^r$. Then, of course, any root, ρ, may be written as a linear combination of this basis, say $\rho = \sum c_i \alpha^i$. We arbitrarily declare that ρ is a positive root provided the first non-zero c_i is a positive number, where the c_i's are ordered as expected, namely $c_1, c_2, \ldots$. Otherwise, the root in question is negative. Since the roots in fact come in pairs, it is obvious that all roots are either positive or negative, and that the numbers of these are equal.

Having made a choice of an ordered basis for the root space, then we easily determine the so-induced division of all the roots, i.e., the eigenvalues of $\mathrm{ad}(h)$, into the positive roots, and the negative roots. We may now define a ***simple root*** as a positive root which cannot be written as the sum of two positive roots, and refer to this subset by the symbol Π, and label the simple roots by some labelling set I; i.e., $\Pi \equiv \{\alpha_i \mid i \in I\}$.

The simple roots have several quite interesting, and useful, properties:

a. the difference of two simple roots cannot be a root at all.

Proof: Let $\alpha, \beta \in \Pi$, and suppose that $\alpha - \beta \in \boldsymbol{\Sigma}$. Then either $\alpha - \beta$ or $\beta - \alpha$ is a positive root. Therefore, either $(\alpha - \beta) + \beta = \alpha$ or $(\beta - \alpha) + \alpha = \beta$ is the sum of two positive roots. However, as α and β are both simple roots, this is impossible. The only conclusion available is that the difference is in fact not a root.

b. for any two simple roots, $\alpha, \beta \in \Pi$, we must have $\langle \alpha, \beta \rangle \leq 0$.

Proof: Since $\beta - \alpha$ is not a root, the value of m for the **$\boldsymbol{\alpha}$-string containing $\boldsymbol{\beta}$** is 0, so that $m - p$ is negative or zero. However, our equation for $m - p$ then tells us that $\langle \beta, \alpha \rangle \leq 0$, as desired.

c. the set of simple roots is linearly independent.

Proof: Linear dependence would say that there are two different linear combinations of the simple roots that are equal, i.e., there exist sets $\{a_i \mid i \in I\}$ and $\{b_j \mid j \in I\}$ such that

$$\sum_{i \in I} a_i \alpha_i = \sum_{j \in I} b_j \alpha_j \ ,$$

where we move terms from one side of the equation to the other as necessary in order to ensure that the same root does not occur on both sides of the equation, **and** so that all the coefficients $\{a_i\}$ and $\{b_j\}$ are positive.

Assuming that this equation is true, we multiply both sides of it by the left-hand side and take the scalar product, which gives us

$$\left\langle \sum_I a_i \alpha_i, \sum_I a_j \alpha_j \right\rangle = \left\langle \sum_I a_i \alpha_i, \sum_I b_j \alpha_j \right\rangle = \sum_{i \in I} \sum_{j \in I} a_i b_j \langle \alpha_i, \alpha_j \rangle \ .$$

Now, since the expression on the left is a perfect square it is surely positive; however, the expression on the right involves a sum over positive numbers multiplied by different terms of the form $< \alpha_i, \alpha_j >$, which is always negative, since $i \neq j$. This is clearly impossible; therefore, our assumption was incorrect, and it is indeed true that the simple roots are linearly independent.

d. any positive root may be written as a linear combination of simple roots, **with positive coefficients**.

V. The Cartan matrix for a semi-simple Lie algebra

The content of Eq. (3.19) is that a certain ratio is always an integer. We create an $r \times r$ matrix with integer entries, called *the Cartan matrix for* $\mathcal{G}$, by using this fact with the set of simple roots, Π, labelling them as $\{\alpha_j\}_1^r$:

$$A_{ij} \equiv 2\frac{\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} \ . \tag{5.1}$$

This matrix has all 2's on its diagonal, and, for the off-diagonal elements it has the property that A_{ij} is proportional to A_{ji}, which ensures that if $A_{ij} = 0$ then $A_{ji} = 0$, as well as arranging it so that it can be made symmetric by multiplication by a diagonal matrix. (This last statement is often stated by simply saying that *the matrix is symmetrizable.* From our results concerning the angles between simple roots, its off-diagonal elements will never be positive, and can only have values selected from the set $\{0, -1, -2, -3\}$. Moreover, since our equations tell us that the product $A_{ij}A_{{}_{JI}} < 4$, it follows that if, for instance, $A_{ij} = -2$ or $A_{ij} = -3$, then it must be that $A_{ji} = -1$, since their product must be less than 4.

If the Lie algebra is semi-simple, it will be the direct sum of the Cartan matrix for each of the simple parts which make it up. All the information needed to re-create the structure of a simple Lie algebra is contained within the Cartan matrix, so that this is an important way to visualize things. The standard approach is to **choose a normalization** for the basis of each $\mathcal{G}_{\pm 1}$, i.e., for the eigenvectors of the adjoint representation, labelling them by $\{e_i\}_1^r$ for $\mathcal{G}_1$ and $\{f_j\}_1^r$ for $\mathcal{G}_{-1}$, which satisfies the following (normalized) commutation relations:

$$[e_i, f_j] = \delta_{ij} h_i \ , \quad [h_i, h_j] = 0 \ , \quad [h_i, e_j] = A_{ij} e_j \ , \quad [h_i, f_j] = -A_{ij} f_j \ , \tag{5.2a}$$

the existence and consistency of which was first proved by Chevalley. One may then prove that, for finite- or affine-Lie algebras, the *free (Lie) algebra* on these generators is still constrained by the Serre conditions:

$$(\mathrm{ad}\, e_i)^{-A_{ij}} e_j \neq 0 \ , \ (\mathrm{ad}\, e_i)^{1-A_{ij}} e_j = 0 \ ; \qquad (\mathrm{ad}\, f_i)^{1-A_{ij}} f_j = 0 \ , (\mathrm{ad}\, f_i)^{-A_{ij}} f_j \neq 0 \ . \tag{5.2b}$$

(DO NOTE that there are NO sums on any of the indices.) The Serre conditions are simply a simple, compact way of re-stating the conclusions we have already reached above, concerning the lengths of various strings, etc.

Let us try to visualize the associated lattice a little bit:

1. First notice that

   if the matrix is symmetric, then the simple roots will all have the same length.

   The proof is straightforward, since

$$A_{ij} \equiv 2\frac{\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} \ ; \quad A_{ji} \equiv 2\frac{\langle \alpha_j, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} \ .$$

   As $\langle \alpha_j, \alpha_i \rangle = \langle \alpha_i, \alpha_i i \rangle$, equivalence of the matrix entries requires that the denominators are equal.

2. Of course, even if it is only true that $A_{ij} = A_{ji}$, for some values of i and j, then those (associated) roots are of equal length.

3. Next we calculate the cosine of the angle between two simple roots, α_i and α_j, given by our earlier formula, Eq. (3.21), but now using the entries in the Cartan matrix:

$$\cos^2\theta(\alpha_i, \alpha_j) = \frac{\langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle \langle \alpha_j, \alpha_j \rangle} = \frac{1}{4} A_{ij} A_{ji} \quad \Longrightarrow \quad \cos\theta(\alpha_i, \alpha_j) = -\frac{1}{2}\sqrt{A_{ij} A_{ji}} \ ,$$

   where the minus sign has been inserted because we know that $A_{ij} \leq 0$, for $i \neq j$. This of course says that the angle between two simple roots will always be obtuse, or right, i.e., $90° \leq \theta < 180°$. For instance, the Cartan matrix for $\boldsymbol{sl}(3)$ is 2×2, symmetric, and with $A_{12} = -1$. This tells us that the two simple roots are the same length, and the angle between them is such that $\cos\theta = -1/2$, i.e., the angle is 120°.

As a detailed example, let us look at **the Cartan matrix for $\boldsymbol{sl}(4)$**, which is given by

$$A \Longrightarrow \begin{pmatrix} +2 & -1 & 0 \\ -1 & +2 & -1 \\ 0 & -1 & +2 \end{pmatrix} \ . \tag{5.3a}$$

We now show how we may use this matrix to create all the details of the algebra. To begin, we note that the algebra has rank 3, and the basic details from the matrix may be written as

$$\begin{aligned}
&[h_1,e_1]=2e_1\ ,\ [h_1,e_2]=-e_2\ ,\ [h_1,e_3]=0\ ,\\
&[h_2,e_1]=-e_1\ ,\ [h_2,e_2]=2e_2\ ,\ [h_2,e_3]=-e_3\ ,\\
&[h_3,e_1]=0\,,\ [h_3,e_2]=-e_2\ ,\ [h_3,e_3]=2e_3\ ,
\end{aligned}\tag{5.3b}$$

where the 3 triplets of generators, $\{h_i\}_1^3$, $\{e_i\}_1^3$, $\{f_i\}_1^3$, are the Chevalley generators for $\boldsymbol{sl}(4)$. Then the Serre conditions say that all the following commutators vanish:

$$[e_1,[e_1,e_2]]\,,\ [e_1,e_3]\,,\ [e_2,[e_2,e_1]]\,,\ [e_2,[e_2,e_3]]\,,\ [e_3,e_1]\,,\ [e_3,[e_3,e_2]]\ ,\tag{5.3c}$$

along with the analogous ones for the f_j's. In addition, these last conditions tell us of the (expected) existence of some additional non-zero commutators, to which we now give names:

$$[e_1,e_2]\equiv e_{12}\neq 0\ ,\quad [e_2,e_3]\equiv e_{23}\neq 0\ ;\quad [f_1,f_2]\equiv f_{12}\neq 0\ ,\quad [f_2,f_3]\equiv f_{23}\neq 0\ ,\tag{5.3d}$$

along with the re-statement of the remainder of the Serre conditions (above) as

$$[e_1,e_{12}]=0=[e_2,e_{12}]\,,\quad [e_2,e_{23}]=0=[e_3,e_{23}]\ ,\tag{5.3e}$$

again along with the analogous equations for the f_j's. None of this tells us that the commutators $[e_1,e_{23}]$ and $[e_3,e_{12}]$ should vanish, so we suppose that they do not; however, we can use the Jacobi relation to show that they are (closely) related:

$$e_{123}\equiv[e_1,e_{23}]=[e_1,[e_2,e_3]]=-[e_2,[e_3,e_1]]-[e_3,[e_1,e_2]]=0-[e_3,e_{12}]\ ,\tag{5.3f}$$

with, again, one last time, the analogous statement for the f_k's. To continue, we must look at higher-order commutators, using the Jacobi relation. We begin with

$$\begin{aligned}
&[e_1,e_{123}]=-[e_1,[e_3,e_{12}]]=+[e_1,[e_{12},e_3]]=-[e_{12},[e_3,e_1]]-[e_3,[e_1,e_{12}]]=-0-0=0\ ,\\
&[e_3,e_{123}]=[e_3,[e_1,e_{23}]]=-[e_1,[e_{23},e_3]]-[e_{23},[e_3,e_1]]=-0-0=0\ .
\end{aligned}\tag{5.3g}$$

However the remaining one requires a somewhat more delicate approach:

$$\begin{aligned}[e_2, e_{123}] = [e_2, [e_1, e_{23}]] = -[e_2, [e_{23}, e_1]] = [e_{23}, [e_1, e_2]] + [e_1, [e_2, e_{23}]] = [e_{23}, e_{12}] \\ = -[e_{12}, [e_2, e_3]] = +[e_2, [e_3, e_{12}]] + [e_3, [e_{12}, e_2]] = -[e_2, e_{123}] + 0 \ .\end{aligned} \tag{5.3h}$$

This tells us that the desired commutator equals its own negative, and therefore must in fact vanish; however, as well, along the way it told us that another interesting commutator vanishes, namely $[e_{12}, e_{23}] = 0$. Therefore, we have shown that there are no new, non-zero commutators among the positive grades, and also among the negative grades, by analogy. It is of course desirable to know about the commutators between the two signs. However, that is surely not going to describe new non-zero elements. Counting, we see that we now have 15 entries, the appropriate number for traceless 4×4 matrices. Notice that $\{e_1, e_2, e_3\}$ span $\mathcal{G}_1$ while $\{e_{12}, e_{23}\}$ are in $\mathcal{G}_2$, with only e_{123} a member of $\mathcal{G}_3$, those higher yet being all zero.

In this normalization the simple roots are given directly as the rows (or, possibly, columns, depending on one's conventions) of the Cartan matrix. Therefore, for $\boldsymbol{sl}(4)$ we already have the choice of $\{h_i\}_1^3$ as a basis for its Cartan subalgebra, and the roots may be presented:

$$\alpha_1 \Longrightarrow (2, -1, 0) \ , \quad \alpha_2 \Longrightarrow (-1, 2, -1) \ , \quad \alpha_3 \Longrightarrow (0, -1, 2) \ . \tag{5.3i}$$

Notice that the matrix is symmetric, so that all 3 simple roots have the same length. As well the three of them span a 3-dimensional lattice, with angle of 120° between α_2 and both α_1 and α_3, while α_1 and α_3, themselves, are at 90°. For instance, we may set up a standard Cartesian basis set so that α_2 is along the $\hat{z}$-axis. Then, choose a plane that makes an angle of 120° with $\hat{z}$, and pick a pair of vectors in that plane that are perpendicular. A (non-symmetric) choice, normalized to the same length as α_2, namely 1, is $2\alpha_1 = \sqrt{3}\hat{x} - \hat{z}$ and $2\alpha_3 = -\hat{z} - \hat{x}/\sqrt{3}+$

We then recall that $[e_1, e_2]$, $[e_2, e_3]$, and $[e_1, [e_2, e_3]]$ are non-zero, and therefore should correspond to the roots:

$$\alpha_1 + \alpha_2 \Longrightarrow (1, 1, -1) \ , \quad \alpha_2 + \alpha_3 \Longrightarrow (-1, 1, 1) \ , \quad \alpha_1 + \alpha_2 + \alpha_3 \Longrightarrow (1, 0, 1) \ . \tag{5.3j}$$

As well, of course, the negative of each of these 6 roots is also a root, for a total of 12 non-zero roots, and therefore a total of $12 + 3 = 15 \;=\; 4^2 - 1$ linearly independent (basis) elements of the Lie algebra $\boldsymbol{sl}(4)$.

Lastly we present the multiplication table for $\boldsymbol{sl}(4)$, which also was generated simply by the use of the knowledge of the generators encoded in the Cartan matrix, along with consistent use of the Jacobi identity:

	h_1	h_2	h_3	e_1	f_1	e_2	f_2	e_3	f_3	e_{12}	f_{12}	e_{23}	f_{23}	e_{123}	f_{123}
h_1	0	0	0	$2e_1$	$-2f_1$	$-e_2$	f_2	0	0	e_{12}	$-f_{12}$	$-e_{23}$	f_{23}	e_{123}	$-f_{123}$
h_2		0	0	$-e_1$	f_1	$2e_2$	$-2f_2$	$-e_3$	f_3	e_{12}	$-f_{12}$	e_{23}	$-f_{23}$	0	0
h_3			0	0	0	$-e_2$	f_2	$2e_3$	$-2f_3$	$-e_{12}$	f_{12}	e_{23}	$-f_{23}$	e_{123}	$-f_{123}$
e_1				0	h_1	e_{12}	0	0	0	0	f_2	e_{123}	0	0	f_{23}
f_1					0	0	f_{12}	0	0	e_2	0	0	f_{123}	e_{23}	0
e_2						0	h_2	e_{23}	0	0	$-f_1$	0	f_3	0	0
f_2							0	0	f_{23}	$-e_1$	0	e_3	0	0	0
e_3								0	h_3	$-e_{123}$	0	0	$-f_2$	0	$-f_{12}$
f_3									0	0	$-f_{123}$	$-e_2$	0	$-e_{12}$	0
e_{12}										0	$-h_1 - h_2$	0	0	0	$-f_3$
f_{12}											0	0	0	$-e_3$	0
e_{23}												0	$-h_2 - h_3$	0	f_1
f_{23}													0	e_1	0
e_{123}														0	$h_1 + h_2 + h_3$
f_{123}															0

In some sense the previous construction is “cheating,” since we claim that we already know the total number of independent basis elements of the algebra. If, in fact, we did not know this we need some slightly more algorithmic method to ensure that we truly have, at some level, determined all the roots. Of course, since every root has its negative, it is sufficient if we can ensure that we have determined all the positive roots. Therefore, let us proceed a bit more toward the creation of such an algorithm, as is described, for instance, in Chapter 7 of

Cahn's book. We begin by choosing an arbitrary simple root, α_i, and recalling Eq. (3.19) for the "length" of the α_i string containing an arbitrary positive root β, which is of course such that there exist non-negative integers $\{k_j \mid j \in I\}$ to write β as a sum of positive roots:

$$\beta = \sum_{j\in I} k_j \alpha_j \quad \Longrightarrow \quad m_{_\beta} - p_{_\beta} = 2\frac{<\beta, \alpha_i>}{<\alpha_i, \alpha_i>} = 2\sum_j k_j \frac{<\alpha_j, \alpha_i>}{<\alpha_i, \alpha_i>} = \sum_j k_k A_{ji} \;. \tag{5.3k}$$

Since we know that the p-th element in the string is a root if and only if p is a positive integer, this gives us the algorithm that says that

$$\beta + p\alpha_i \;\in\; \boldsymbol{\Sigma} \quad \Leftrightarrow \quad p = m - \sum_j k_k A_{ji} > 0 \;.$$

To understand this algorithm a little better, we could use it to go back through the calculation above to show that it tells us when to stop; however, it is true that the situation with it is, perhaps, a little bit too straightforward. A perhaps clearer example is to follow through with this algorithm for an algebra with a non-symmetric Cartan matrix; therefore, we now demonstrate its use in detail using the matrix for the algebra $\boldsymbol{G}_2$. It is also an interesting example since its dimension is not a priori known. Therefore we note that the Cartan matrix is

$$\begin{gathered} \boldsymbol{G}_2 : \qquad \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} \\ \Longrightarrow \quad \alpha_1(h) \Longrightarrow \quad [\,2 \quad -3\,]\,, \quad \alpha_2(h) \Longrightarrow \quad [\,-1 \quad 2\,]\;. \end{gathered} \tag{5.3l}$$

We then consider the α_2-string containing α_1, which of course has $m_1 - p_1 = -3$. However, as both these roots are simple, we know that their difference is not a root, so that $m_1 = 0$, so that $p_1 = 3$, telling us that the entire string indeed has length 4, i.e., it is given by

$$\alpha_1 \,,\; \alpha_1 + \alpha_2 \,,\; \alpha_1 + 2\alpha_2 \,,\; \alpha_1 + 3\alpha_2 \;.$$

We may of course also consider the α_1-string containing α_2, which again has $m_2 = 0$, so that $-1 = A_{21} = m_2 - p_2 = 0 - p_2$, which tells us that $p_2 = 1$, so that there is only one element in that string, namely $\alpha_2 + \alpha_1$, which we already knew was a root.

Nonetheless, we can continue because we have a new root, indeed 3 new ones. We display below each of their presentations:

$$(\alpha_1+\alpha_2)(h)=\alpha_1(h)+\alpha_2(h)=[\,1\quad -1\,]\ ,$$

$$(\alpha_1+2\alpha_2)(h)=\alpha_1(h)+2\alpha_2(h)=[\,0\quad +1\,]\ ,$$

$$(\alpha_1+3\alpha_2)(h)=\alpha_1(h)+3\alpha_2(h)=[\,-1\quad +3\,]\ ,$$

We then look at the values of m and p for each of them in turn, for both the options of adding additional multiples of α_2 or α_1. We first think about just adding more multiples of α_2 to each of them, labelling those as m_{A2}, m_{B2} and m_{C2}, respectively, knowing, or at least hoping, that the algorithm will tell us what we have already been told before, namely that this is the entire string:

$$m_{A2}-p_{A2}=-1\ ,\quad m_{A2}=1\ \Rightarrow\ p_{A2}=2\ ,$$

$$m_{B2}-p_{B2}=+1\ ,\quad m_{B2}=2\ \Rightarrow\ p_{B2}=1\ ,$$

$$m_{C2}-p_{C2}=+3\ ,\quad m_{C2}=3\ \Rightarrow\ p_{C2}=0\ .$$

All this is as was expected, consistently. Now, however, let us consider adding to each of them a multiple of α_1:

$$m_{A1}-p_{A1}=+1\ ,\quad m_{A1}=1\ \Rightarrow\ p_{A1}=0\ ,$$

$$m_{B1}-p_{B1}=0\ ,\quad m_{B1}=0\ \Rightarrow\ p_{B1}=0\ ,$$

$$m_{C1}-p_{C1}=-1\ ,\quad m_{C1}=0\ \Rightarrow\ p_{C1}=1\ .$$

We now explain these last three lines in a little more detail. For the first line, we are first determining m_{A1}, i.e., for what maximum value of m_{A1} is $(\alpha_1+\alpha_2)-m_{A1}\alpha_1$ a root? We know the answer is +1 because α_2 is a root and $\alpha_2-\alpha_1$ is **not** a root. Therefore we can determine p_{A1}, which turns out to be zero, so that there are no more elements in the string. At the next line we are asking about the value of m_{B2}, i.e., for the maximum value such that

$(\alpha_1 + 2\alpha_2) - m_{B1}\alpha_1$ is still a root. This time the answer must be that $m_{B1} = 0$, for if we were to subtract even one multiple of α_1, we would obtain just $2\alpha_2$ which we know is **not** a root. Therefore, again we may determine p_{B1} which again is zero, so that there are no extra elements in this string; i.e., it is a string of length 1. Lastly, however, we come to the question as to the string terminating with $(\alpha_1 + 3\alpha_2) - m_{C1}\alpha_1$. As usual we can now see easily that $m_{C1} = 0$, since $3\alpha_2$ is surely not a root. However, this tells us that $p_{C1} = 1$, meaning that we are allowed to add one more α_1 to this string, and still have a root; i.e., we have now found a new root,

$$(\alpha_1 + 3\alpha_2) + \alpha_1 = 2\alpha_1 + 3\alpha_2 \ \in \ \boldsymbol{\Sigma} \ .$$

In principle we must now determine its structure, starting with

$$(2\alpha_1 + 3\alpha_2)(h) = [\,1 \quad 0\,] \implies \begin{cases} m_{D1} - p_{D1} = 1 \ , \\ m_{D2} - p_{D2} = 0 \ . \end{cases}$$

We already know that $m_{D1} = 1$ so that $p_{D1} = 0$, so that we cannot add a multiple of α_1. Also we know that we cannot subtract any α_2 since if we did so we would have $2(\alpha_1 + \alpha_2)$, which would be twice an actual root, which is not allowed; therefore, we have $m_{D2} = 0$, which tells us that $p_{D2} = 0$. Therefore, we may not add any more simple roots to this one, and we are finished. However, we could probably have argued this one slightly more easily by noting that adding one more α_1 would have caused it to be three times the root $\alpha_1 + \alpha_2$, while adding one more α_2 would have caused it to be twice the root $\alpha_1 + 2\alpha_2$; therefore, this is easily the end of the line!

VI. Dynkin diagrams associated to a given Cartan matrix

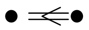
Dynkin diagrams are a simple, 2-dimensional way of recording all the information in an arbitrary Cartan matrix in terms of some very simple, graphical approach; the idea is very useful both in remembering the information as well as in the construction of proofs that this or that diagram is, or is not, allowed.

To create a Dynkin diagram, we arrange r copies of $\bullet$, referring to them as vertices. Sometimes it is valuable to label them, for instance by 1 for the root α_1, by 2 for the root α_2, and eventually, for instance, by $r-1$, for the root α_{r-1}. For each pair of vertices, α_i and α_j, connect them by a number of straight lines given by the maximum of $|A_{ij}|$ or $|A_{ji}|$. Then, if $|A_{ij}|$ is greater than 1, put an arrow on the multi-line connecting the vertices that points toward α_i.

In a different set of notes we will give the Dynkin diagrams for a greater many known, simple Lie algebras. On the other hand, right now we will simply give a few examples, noting first that the Dynkin diagram for the algebra $\boldsymbol{sl}(4)$ is simply •—•—•

$$\boldsymbol{A}_2 = \boldsymbol{sl}(3): \quad \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \text{ that corresponds to}$$

$$\boldsymbol{C}_2 = \boldsymbol{sp}(4): \quad \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} \text{ that corresponds to}$$



$$\boldsymbol{G}_2: \quad \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \text{ that corresponds to}$$

$$\boldsymbol{B}_3 = \boldsymbol{so}(7): \quad \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix} \text{ that corresponds to}$$

$$\boldsymbol{D}_4 = \boldsymbol{so}(8): \quad \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix} \text{ that corresponds to}$$