

Weight Spaces for Representations of Simple Lie Algebras

I. Introduction

Let \mathcal{G} be a semi-simple Lie algebra, V a (finite-dimensional) vector space, of dimension n , $L(V)$ the space of linear operators on V , and $\rho : \mathcal{G} \rightarrow L(V)$ a representation of \mathcal{G} . We simplify the notation so that for the standard Chevalley basis, $\{h_i, e_i, f_i\}$, we denote their representations by $\{H_i, E_i, F_i\}$; i.e., $H_i \equiv \rho(h_i)$, etc. As all the H_i commute one with another, we may diagonalize them simultaneously, so that they possess a common set of eigenvectors, $\{\Phi^a\}$, which will span V . We specify the eigenvalues by $H_i \Phi^A = M_{ai} \Phi^A$, where there is no sum on the index a . [We use the capital index A to remind us of this. The convention is that A and a should have the same values, at every time that they take on specific values, but that no sum on all possible values should be generated, since one of them is a capital letter.]

We then define a generic element of the Cartan subalgebra $\mathcal{H} \subset \mathcal{G}$ by $h \equiv c^i h_i$, and its representation by $H \equiv c^i H_i$. This allows us to introduce the notion of *weights* for the (given) representation as linear functionals over the span of the H_i 's via

$$H \Phi^A = M_a(H) \Phi^A = c^i H_i \Phi^A = c^i (M_{ai} \Phi^A) .$$

As the H_i are a representation of the h_i , it follows that the M_a are also linear functionals over the Cartan subalgebra, i.e.,

$$\forall a = 1, \dots, n, \quad M_a \in \mathcal{H}^* \implies M_a = \mu_a^i \alpha_i ,$$

since the simple roots, $\{\alpha_i\}$, form a (vector-space) basis for \mathcal{H}^* . (We will later show that that the μ_a^i have the form of integers divided by the determinant, \mathcal{A} , of the Cartan matrix, A_{ij} .)

The straight-forward calculation

$$H \{E_\alpha \Phi^A\} = E_\alpha \{H \Phi^A\} + \alpha(h) \{E_\alpha \Phi^A\} = (M_a + \alpha)(h) \{E_\alpha \Phi^A\}$$

shows us that if M_a is a weight then $M_a + \alpha$ is another weight, so long as they are such that $E_\alpha \Phi^A \neq 0$. Therefore, by the same line of reasoning as was given for weights, one finds that there is an α -string of weights through M_a , from $M_a + p\alpha$ down to $M_a - m\alpha$, such that

for the α_i -string of weights through \mathbf{M}_a ,

$$m_i - p_i = 2 \frac{\langle M_a, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} .$$

While we have written the weights down in terms of their coefficients with respect to the simple roots as a basis for \mathcal{H}^* , the equation describing the lengths of strings of weights through a given one motivates a different choice of basis, referred to as a *reciprocal basis* for \mathcal{H}^* , namely $\{\lambda^j\}_1^n$ which are such that

$$2 \frac{\langle \lambda^j, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} = \delta_i^j .$$

To find the relationship between a given weight's components with respect to one and the other basis, we insert the expansion of M_a in terms of the root-basis, $M_a \equiv \mu_a^i \alpha_i$, and also its expansion in terms of the reciprocal basis, $M_a \equiv \nu_{ai} \lambda^i$, into the preceding expression for the differences of the integers m_i and p_i for the appropriate α_i -string of weights through \mathbf{M}_a :

$$\begin{aligned} 2 \frac{\langle M_a, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} &= 2 \mu_a^j \frac{\langle \alpha_j, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} = \mu_a^j A_{ji} , \quad \text{and} \quad 2 \frac{\langle M_a, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} = 2 \nu_{aj} \frac{\langle \lambda^j, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} = \nu_{aj} \delta_i^j = \nu_{ai} , \\ &\implies \mu_a^j A_{ji} = \nu_{ai} \Leftrightarrow \mu_a^j = \nu_{ai} (A^{-1})^{ij} . \end{aligned}$$

Since the entries in the Cartan matrix are all integers, the entries in the inverse matrix must be simply integers divided by the determinant of that matrix, so that, as stated previously, the coefficients μ_a^j have the form of integers divided by that determinant. On the other hand, the components ν_{ai} are always integers, so that, for a given representation, the set of all associated weights forms an integral lattice within \mathcal{H}^* , with basis elements given by the reciprocal basis, $\{\lambda^i\}_1^n$.

One may now define a *dominant weight* as one for which all the lattice coefficients $\nu_{ai} \geq 0$, and the *highest weight*, Λ as that dominant weight which has the largest coefficients, in the

order already chosen by the ordering of the simple roots. The simple criterion to decide on a highest weight is that $\forall i = 1, \dots, r, \Lambda + \alpha_i$ is **not a root**, i.e., that $p_i = 0$ for every simple root. For any representation, the weight vector space for the highest weight is 1-dimensional. Moreover, the highest weight determines uniquely a given irreducible representation, determined simply by letting all possible matrices in the representation act upon it as many times as possible. The algebra created, for a given representation, by considering all possible sums and (ordinary matrix) products of all the associated matrices is called the *enveloping algebra for that representation*.

The **fundamental representations** are those whose highest weight corresponds to a weight vector from the reciprocal basis; they are irreducible. All other representations may then be described by giving their (non-negative, integer) components with respect to the fundamental representations. Given a highest weight in this form, one may then determine all the other (associated) weights for that representation by creating strings, via the lowering operators, as described above. The only difficulty with this method is that it still leaves open the question of the dimensionality of the various associated weight spaces; to resolve this question we will eventually need to determine a dimensionality formula, famous ones being due to Freudenthal and to Weyl.

Let us try working through some details for the algebra $\mathbf{A}_2 = \mathbf{su}(3)$.