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MONOPOLES AND FIBER BUNDLES

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Magnetic Monopole and Need to Introduce Sections

The magnetic monopole is the magnetic charge. While the idea of magnetic monopoles must have been discussed in classical physics early in the history of electricity and magnetism, modern discussions date back to 1931 in the important paper of Dirac[1] in which he pointed out that magnetic monopoles in quantum mechanics exhibit some extra and subtle features. In particular, with the existence of a magnetic monopole of strength g, electric charges and magnetic charges must necessarily be quantized, in quantum mechanics. We shall give a new derivation of this result in a few minutes.

If one wants to describe the wave function of an electron in the field of a magnetic monopole, it is necessary to find the vector potential $\vec{A}$ around the monopole. Dirac chose a vector potential which has a string of singularities. The necessity of such a string of singularities is obvious if we prove the following theorem[2].

Theorem. Consider a magnetic monopole of strength $g \neq 0$ at the origin and consider a sphere of radius R around the origin. There does not exist a vector potential $\vec{A}$ for the monopole magnetic field which is singularity free on the sphere. This theorem can be easily proved in the following way. If there were a singularity free $\vec{A}$ we consider the loop integral

$$\oint A_\mu dx^\mu$$

around a parallel on the sphere as indicated in Figure 1. By

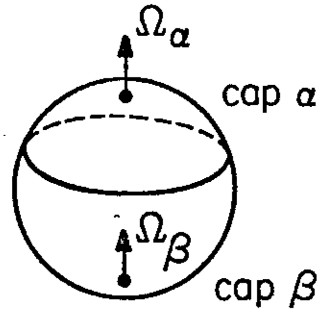

FIGURE 1. A sphere of radius R with a magnetic monopole at its center. The parallel divides the sphere into two caps α and β.

Stoke's theorem this loop integral is equal to the total magnetic flux through the cap alpha:

$$\oint A_\mu dx^\mu = \Omega_\alpha . \tag{1}$$

Similarily we can apply Stoke's theorem to cap β obtaining

$$\oint A_\mu dx^\mu = \Omega_\beta .. \tag{2}$$

Here Ω_α and Ω_β are the total upward magnetic flux through the caps α and β, both of which are bordered by the parallel. Subtracting these two equations we obtain

$$0 = \Omega_A - \Omega_B , \tag{3}$$

which is equal to the total flux out of the sphere, which in turn is equal to $4\pi g \neq 0$. We have thus reached a contradiction.

Having proved this theorem, we observe that R is arbitrary. Thus one concludes that there must be a string of singularities or strings of singularities in the vector potential to describe the monopole field. Yet we know that the magnetic field around the monopole is singularity free. This suggests that the string of singularities is not a real physical difficulty. Indeed the situation is reminiscent of the problem that one faces when one wants to find a parametrization of the surface of the globe. The coordinate system that we usually use, the latitude and the longitude, is not singularity free. It has singularities at the north pole and at the south pole. Yet the surface of the globe is evidently without singularities. We deal with this situation usually in something like the way illustrated in figure 2. We consider a rubber sheet with nicely defined coordinates and stretch and wrap it down onto the globe so that it covers more than the northern hemisphere. Similarly,

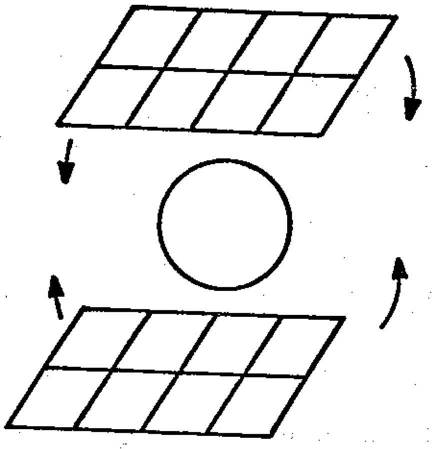

FIGURE 2. Method of parametrizing the globe.

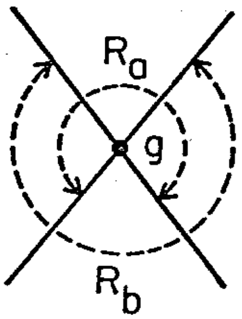

FIGURE 3. Division of space outside of monopole g into overlapping regions R_a and R_b.

we consider another rubber sheet with nicely defined coordinates and stretch and wrap it upwards so that it covers more than the southern hemisphere. We now have a double system of coordinates to describe the points on the globe. The description is analytic in the domain covered by each sheet, if we had done no violence in the stretching and wrapping. In the overlapping region covered by both sheets, one has two coordinate systems which are transformable into each other by an analytic non-vanishing Jacobian. This double coordinate system is an entirely satisfactory way to parametrize the globe.

Following this idea we shall now try to exorcise the string of singularities in the monopole problem by dividing space into two regions. We shall call the points outside of the origin, above the lower cone in figure 3, region R_a. Similarly, we shall call the points outside of the origin, under the upper cone, R_b.

The union of these two regions gives all points outside of the origin. In R_a we shall choose a vector potential for which there is only one non-vanishing component of A, the azimuthal component:

$$(A_r)_a = (A_\theta)_a = 0, \quad (A_\phi)_a = \frac{g}{r \sin\theta}\,(1 - \cos\theta), \tag{4}$$

It is important to notice that this vector potential has no singularities anywhere in R_a. Similarly in R_b we choose the vector potential

$$(A_r)_b = (A_\theta)_a = 0, \quad (A_\phi)_b = \frac{-g}{r \sin\theta}\,(1 - \cos\theta), \tag{5}$$

which has no singularities in R_b. It is simple to prove that the curl of either of these two potentials give correctly the magnetic field of the monopole.

In the region of overlap, since both of the two sets of vector potentials share the same curl, the difference between them must be curlless and therefore must be a gradient. Indeed a simple calculation shows

$$(A_\mu)_a - (A_\mu)_b = \partial_\mu \alpha \text{ , where } \alpha = 2g\phi \tag{6}$$

where ϕ is the azimuthal angle. The Schrodinger equation for an electron in the monopole field is thus

$$\frac{1}{2m}\,(p - eA_a)^2 \psi_a + V\psi_a = E\psi_a, \qquad \text{in } R_a,$$

$$\frac{1}{2m}\,(p - eA_b)^2 \psi_b + V\psi_b = E\psi_b, \qquad \text{in } R_b,$$

where ψ_a and ψ_b are respectively the wave functions in the two regions. The fact that the two vector potentials in these two equations are different by a gradient tells us, by the well known gauge principle, that ψ_a and ψ_b are related by a phase factor transformation

$$\psi_a = S\psi_b, \quad S = \exp\,(ie\alpha), \tag{7}$$

or

$$\psi_a = [\exp\,(2iq\phi)]\psi_b, \quad q = eg. \tag{8}$$

Around the equator which is entirely in R_a, ψ_a is single valued. Similarly, since the equator is also entirely in R_b, ψ_b is single valued around the equator. Therefore, S must return to its original value when one goes around the equator. That implies Dirac's quantization condition:

$$2q = \text{integer}. \tag{9}$$