Physics 570

Explanation for FRW Maple Worksheets

We begin with the Friedman-Robertson-Walker metric for an assumed homogeneous and isotropic universe, with

$$\mathbf{g} = R^2(t)\left\{\frac{(dr)^2}{1-\eta\, r^2} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\right\} - dt^2 \;=\; R^2(t)\left\{d\chi^2 + r^2\, d\Omega^2 - d\eta^2\right\} \;.$$

We follow the (standard) reduction of the Einstein field equations for a source determined by a perfect fluid in two parts, namely reasonably-slow-moving matter and radiation, so that $\rho = \rho_m + \rho_r$ and $P = P_r = \rho_r/3$, which gives us the following equations:

$$\rho_m = \left(\frac{R_0}{R}\right)^3 \rho_{m0} \;, \qquad \rho_r = \left(\frac{R_0}{R}\right)^4 \rho_{r0} \;,$$

$$H^2 \equiv \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3}(\rho_m + \rho_r) - \frac{k}{R^2} - \tfrac{1}{3}\Lambda \equiv \frac{A}{R^3} + \frac{B^2}{R^4} - \frac{k}{R^2} - \tfrac{1}{3}\Lambda \;.$$

We will work through two separate cases, namely $k = +1$, a closed, finite, spherically-curved universe, and $k = -1$, an open, infinite, hyperbolically-curved universe. For non-zero values of the all these parameters, this "work" has to be done numerically, or via elliptic functions, which we will abjure here. I will choose some sets of reasonably-realistic values for H_0, R_0, ρ_{m0}, ρ_{r0}, and Λ, and make some comments about how they have been chosen, but will not worry too much about their values since I would accept almost any non-zero set of values that you chose to use for your integrations. We begin with the usual (big-bang) boundary condition, namely that $R(0) = 0$, and we do the numerical integration with a parameter, η, so that we actually integrate the following system of equations:

$$\left(\frac{dR}{d\eta}\right)^2 = AR + B^2 - kR^2 - \frac{\Lambda}{3}R^4 \;, \qquad \frac{dt}{d\eta} = R \;,$$

$$A \equiv \frac{\kappa}{3}\rho_{m0}{R_0}^3 \;, B^2 = \frac{\kappa}{3}\rho_{r0}{R_0}^4 \;.$$

To begin the questions about the selection of constants, we note that the Hubble constant, H_0, is just such that its square is the value of the left-hand-side of this equation as measured

now. Experimental criteria leave us with rather good reasons to believe that H_0 lies somewhere between 50 and 90 km/sec/Mpc, which, when changed to units of just Mpc^{-1}, and squared, gives us a value somewhere between

$$1/(6000 \text{ Mpc})^2 \leq H_0^2 \leq 1/(3333 \text{ Mpc})^2 \ .$$

We therefore need to ensure that the major term in our equation, namely $\frac{A}{R^3}$, which when measured now is just $8\pi\rho_{m_0}/3$, be at least a little larger than a number somewhere in this range. When measured in terms of the more usual g/cm^3, this puts ρ_{m_0} between 5 and 15 $\times 10^{-30}$ g/cm^3. We also decide to choose the quantity R_0 in this range, i.e., between 3333 Mpc and 6000 Mpc, although this is simply the easiest approach; there are surely more sophisticated ways to make the terms cancel here and there, and have different values for R_0. The term containing B^2, when evaluated now, is of the form $8\pi\rho_{r_0}/3$; knowing that ρ_{r_0} is on the order of a 1000-th of ρ_{m_0}, we can choose this term around 10^{-6} of the term containing A; therefore now it will not be of too much interest even though its behavior will dominate at very early times.

For the $k = 1$ case I will pick the following paramters:

$$k = +1 \ , \quad \begin{cases} \rho_{m_0} = 2 \times 10^{-29} \text{g/cm}^3 = 1.41 \times 10^{-8} \text{ Mpc}^{-2} \ , \\ \rho_{r_0} = 10^{-32} \text{g/cm}^3 \ , \quad R_0 = 4500 \text{Mpc} \quad , \end{cases}$$

which implies, using the conversion factor, 1 g/cm^3 = 7.05 $\times 10^{20}$ Mpc^{-2}, that

$$A = \frac{8\pi}{3}\rho_{m_0}R_0^3 = 10,772 \text{ Mpc} \ , \quad B^2 = \frac{8\pi}{3}\rho_{r_0}R_0^4 = 24,237 \text{ Mpc}^2 \ , \ \Lambda/3 = (1/8,226 \text{ Mpc})^2 \ ,$$

where the numerical value of Λ has been created as follows. Firstly, I think about it as if it corresponded to some (negative) density of matter within space-time, preferably not as much as the actual (positive) matter, itself, for instance, we choose it to correspond to a density of $\rho_\Lambda = 5 \times 10^{-30}$ g/cm^3, which implies $\frac{\Lambda}{3} = (\frac{4\pi}{3}) 5 \times 10^{-30}$ g/cm^3 $= 1.478 \times 10^{-8}$ Mpc^{-2} $= (1/8,226 \text{ Mpc})^2$. It is worthwhile, however, to point out that the restrictions that

$(H_0)^2$ should be positive, and, actually, lie within the range specified already, namely between $1/(6000 \text{ Mpc})^2$ and $1/(3333 \text{ Mpc})^2$, puts constraints on $\Lambda/3$; for the case that $k = +1$, this says that it should not exced either $1/(2,192 \text{ Mpc})^2$ or $1/(6,725 \text{ Mpc})^2$.

For constants for the case when k is negative, I want to use as many of the same input numbers as possible, just to simplify my own "workload." Therefore I will use the same values for A and B^2. For Λ, I will simply change the sign, since I also want to use this as an example of a universe which does **not** re-collapse on itself. (The same sign of Λ would in fact insist that it re-collapse.) When all this is done, the value of H_0 is rather high, i.e., 117 km/sec per Mpc. Since this is noticeably over 100, I change the value of R_0 to adjust that, changing it to $R_0 = 6500 Mpc$, which will bring H_0 into a range more in agreement with data, although it does make the density of the universe rather lower, since I have maintained the same value for A. Therefore the values chosen for this case are

$$k = -1\,, \qquad \begin{cases} A = \dfrac{8\pi}{3}\rho_{m_0} R_0^3 = 10,772 \text{ Mpc}\,, \quad B^2 = \dfrac{8\pi}{3}\rho_{r_0} R_0^4 = 24,237 \text{ Mpc}^2\,, \\ \qquad\qquad R_0 = 6500 \text{ Mpc}\,, \quad \Lambda/3 = -(1/8,226 \text{ Mpc})^2\,. \end{cases}$$

Therefore, we may now proceed to determine the current values of H_0 in the two models:

$$(H_0)^2 = \frac{10,772}{4,500^3} + \frac{24,237}{4,500^4} - \frac{1}{4,500^2} - \frac{1}{8,226^2} = 5.411 \times 10^{-8} \ \text{ Mpc}^{-2} = \left[69.8 \frac{\text{km/sec}}{\text{Mpc}}\right]^2, \quad k = +1\,,$$

$$(H_0)^2 = \frac{10,772}{6,500^3} + \frac{24,237}{6,500^4} + \frac{1}{6,500^2} + \frac{1}{8,226^2} = 7.7685 \times 10^{-8} \ \text{ Mpc}^{-2} = \left[83.56 \frac{\text{km/sec}}{\text{Mpc}}\right]^2, \quad k = -1\,,$$

At this point one should go to the Maple worksheets, posted, where I worked out all the details of these two cases, one of which has finite lifetime, and one does not. Also on those worksheets are shown the graphs of $R = R(\eta)$, $t = t(\eta)$, and $R = R(t)$, along with the Maple command language to generate them. In addition, the answers to all the questions asked in the exam question are worked out there, for each case. As a useful summary, I remind you of them below:

i. the current size of the universe, in Megaparsecs,

ii. the current age of the universe, in billions of years,

iii. the rate of increase of the size of the universe, at the current epoch, in Megaparsecs per billion years,

iv. the current value of the deceleration parameter, q_0,

v. the maximum size the universe will attain, in Megaparsecs,

vi. the time that that maximum size will be attained, i.e., the half-period of the universe,

vii. the density of the universe at the time of maximum size,

viii. the age of the universe at the time when the radiation and matter densities were equal,

ix. the metric distance of a source that would be observed today on Earth with a redshift of $Z = 5.0$, and also the time that the light in question had been travelling,

x. the distance to the particle horizon, i.e., what is the metric distance from Earth to the place from which the light from the first instants of time would just now be reaching us, assuming that our telescopes were powerful enough to see it, AND also assuming that that early light was not lost in the mixing happening continually before the final recombination time.