

General Lorentz Transformations, Acting on Some Important Physical Quantities

We are interested in transforming measurements made in a reference frame \mathcal{O}' into measurements of the same quantities as made in a reference frame \mathcal{O} , where the reference frame \mathcal{O} measures \mathcal{O}' to be moving with constant velocity \vec{v} , so that $\gamma \equiv \gamma_v \equiv 1/\sqrt{1 - (v/c)^2}$, where \vec{v} is a 3-vector in a totally arbitrary direction. It is therefore immediately useful to divide all other vector quantities into their parts which are parallel and perpendicular to \vec{v} ; for instance, for the location vector \vec{r} we write

$$\vec{r} = \vec{r}_{\parallel} + \vec{r}_{\perp} = (\hat{v} \cdot \vec{r})\hat{v} + \hat{v} \times (\vec{r} \times \hat{v}) .$$

The appropriate Lorentz transformation equations for the location vector are then

$$\begin{aligned} \vec{r}_{\parallel} &= \gamma[\vec{r}'_{\parallel} + t'\vec{v}] , & \vec{r}_{\perp} &= \vec{r}'_{\perp} , \\ t &= \gamma[t' + \vec{v} \cdot \vec{r}'/c^2] , \\ \text{or } \vec{r} &= \vec{r}' + (\gamma - 1)(\vec{r}' \cdot \hat{v})\hat{v} + \gamma t'\vec{v} . \end{aligned}$$

Since the transformations mix together \vec{r} and t , it seems profitable to create a 4-dimensional vector, usually referred to simply as a “4-vector,” which then transforms between the two frames via a so-called “*boost matrix*”, denoted by $\Lambda(\vec{v})$. We denote the location-and-time 4-vector by \tilde{r} , and write the earlier transformations in the following way, with a 4×4 matrix performing the transformation, where we must note that the superscript T means the transpose of the matrix. It is somewhat unfortunate, however, that the matrix is quite messy for the general case; therefore, often the parallel and perpendicular decompositions above are much simpler to use, although of course this is not easy when there are more than two reference frames to consider:

$$\begin{aligned} \begin{pmatrix} \vec{r} \\ ct \end{pmatrix} &\equiv \tilde{r} = \Lambda(\vec{v})\tilde{r}' = \Lambda(\vec{v}) \begin{pmatrix} \vec{r}' \\ ct' \end{pmatrix} , \\ \Lambda(\vec{v}) &= \begin{pmatrix} I_3 + (\gamma - 1)\hat{v}\hat{v}^T & \gamma\vec{v}/c \\ \gamma\vec{v}^T/c & \gamma \end{pmatrix} \\ &= \begin{pmatrix} 1 + (\gamma - 1)(\hat{v})_x(\hat{v})_x & (\gamma - 1)(\hat{v})_x(\hat{v})_y & (\gamma - 1)(\hat{v})_x(\hat{v})_z & \gamma v_x/c \\ (\gamma - 1)(\hat{v})_x(\hat{v})_y & 1 + (\gamma - 1)(\hat{v})_y(\hat{v})_y & (\gamma - 1)(\hat{v})_y(\hat{v})_z & \gamma v_y/c \\ (\gamma - 1)(\hat{v})_x(\hat{v})_z & (\gamma - 1)(\hat{v})_y(\hat{v})_z & 1 + (\gamma - 1)(\hat{v})_z(\hat{v})_z & \gamma v_z/c \\ \gamma v_x/c & \gamma v_y/c & \gamma v_z/c & \gamma \end{pmatrix} \end{aligned}$$

Now that we have this structure, however, we may create other 4-vectors, all of which transform in the same way, and may also use those transformations to determine the relation to the transformations of the ordinary 3-vector quantities. I will outline those below; however, I also want to note here that there are other, important, physical quantities that are not part of 4-vectors, but, rather, something more complicated. The most immediate ones are the electromagnetic fields, which, in (4-dimensional) spacetime belong as the 6 components of a second-rank, antisymmetric tensor. For easy reference I will also describe them in these notes, after the discussion of 4-vectors and their associated 3-vectors.

A very important quantity involved with their definitions is the so-called proper time, τ , which is a scalar quantity. We know that any two points on the world line of some observer are “time-like separated.” Therefore, for any two such points, A and B , we define the difference of their proper time by

$$(\Delta\tau)_{AB}^2 \equiv (\Delta t)_{AB}^2 - (\Delta\vec{r})_{AB}^2/c^2 ,$$

where the sign is chosen so that $\Delta\tau_{AB} \equiv \tau_B - \tau_A$ is positive when B is to the future of A , along the worldline in question. It should be noted that this is just the time shown on the “wrist-watch” of the observer whose worldline this is. Obviously, this quantity is an invariant one, having the same value no matter which observer happens to measure it, i.e., happens to be making measurements on this particular worldline’s behavior. As well, we see that one may label uniquely points on a given worldline by their value of τ , relative to some chosen origin and some chosen scale of units, i.e., seconds, years, etc. Therefore we may take derivatives along that worldline with respect to the proper time:

1. The velocity 4-vector is given by

$$\frac{d}{d\tau}\tilde{r} \equiv \tilde{u} = \gamma_u \begin{pmatrix} \vec{u} \\ c \end{pmatrix} , \quad \vec{u} \equiv \frac{d}{dt}\vec{r} , \quad \frac{dt}{d\tau} = \gamma_u .$$

2. For a particle of mass m , moving with velocity \vec{u} , the energy-momentum 4-vector is given by

$$m\tilde{u} \equiv \tilde{p} \equiv \begin{pmatrix} \vec{p} \\ E/c \end{pmatrix} = m\gamma_u \begin{pmatrix} \vec{u} \\ c \end{pmatrix} .$$

3. The acceleration 4-vector is then given by

$$\tilde{a} \equiv \frac{d}{d\tau}\tilde{u} = \frac{d^2}{d\tau^2}\tilde{r} = \gamma_u^2 \begin{pmatrix} \vec{a} + \gamma_u^2(\vec{u} \cdot \vec{a})\vec{u}/c^2 \\ \gamma_u^2(\vec{u} \cdot \vec{a})/c \end{pmatrix} , \quad \vec{a} \equiv \frac{d}{dt}\vec{u} = \frac{d^2}{dt^2}\vec{r} .$$

4. The 4-vector force is defined so that (the appropriate generalization of) Newton's Second Law is still true, where we also recall that $dE/dt = \vec{F} \cdot \vec{u}$:

$$\tilde{K} \equiv \frac{d}{d\tau} \tilde{p}, \quad \tilde{K} = \gamma_u \left(\frac{\vec{F}}{\frac{1}{c} dE/dt} \right),$$

Using these definitions, and the fact that each of them is a 4-vector and therefore transforms very simply by multiplication by $\Lambda(\vec{v})$, we may work out the Lorentz transformations of the associated 3-vectors, which are, in general, as expected, not very nice, except for the 3-momentum and energy/c, which transform exactly the same way as does the 3-location and c(time):

1. The 3-velocity, \vec{u} , and its associated function γ_u :

$$\vec{u}_{\parallel} = \frac{\vec{u}'_{\parallel} + \vec{v}}{1 + \vec{v} \cdot \vec{u}'/c^2}, \quad \vec{u}_{\perp} = \frac{\gamma_v^{-1} \vec{u}'_{\perp}}{1 + \vec{v} \cdot \vec{u}'/c^2},$$

$$\text{or } \vec{u} = \frac{\gamma_v^{-1} \vec{u}' + (1 - \gamma_v^{-1})(\hat{v} \cdot \vec{u}')\hat{v} + \vec{v}}{1 + \vec{v} \cdot \vec{u}'/c^2},$$

$$\text{and } \frac{1}{\sqrt{1 - (u/c)^2}} \equiv \gamma_u = \gamma_v(1 + \vec{v} \cdot \vec{u}'/c^2)\gamma_{u'} = \gamma_v \frac{(1 + \vec{v} \cdot \vec{u}'/c^2)}{\sqrt{1 - (u'/c)^2}}, .$$

2. The 3-momentum, \vec{p} and the energy E :

$$\vec{p}_{\parallel} = \gamma_v(\vec{p}'_{\parallel} + E'\vec{v}/c), \quad \vec{p}_{\perp} = \vec{p}'_{\perp}, \quad E = \gamma_v(E' + \vec{v} \cdot \vec{p}'),$$

$$\text{or } \vec{p} = \vec{p}' + (\gamma_v - 1)(\vec{p}' \cdot \hat{v})\hat{v} + \gamma_v E'\vec{v}/c^2.$$

3. The 3-acceleration, \vec{a} :

$$\vec{a}_{\parallel} = \frac{\gamma_v^{-3}}{(1 + \vec{v} \cdot \vec{u}'/c^2)^3} \vec{a}'_{\parallel}, \quad \vec{a}_{\perp} = \frac{\gamma_v^{-2}}{(1 + \vec{v} \cdot \vec{u}'/c^2)^3} \{ \vec{a}'_{\perp} + \vec{v} \times (\vec{a}' \times \vec{u}')/c^2 \}$$

$$= \frac{1 - (v/c)^2}{(1 + \vec{v} \cdot \vec{u}'/c^2)^3} \left\{ \vec{a}'_{\perp} - \frac{\vec{v} \cdot \vec{a}'/c}{1 + \vec{v} \cdot \vec{u}'/c^2} \frac{\vec{u}'_{\perp}}{c} \right\}$$

$$\text{or } \vec{a} = \frac{\gamma^{-3}}{(1 + \vec{v} \cdot \vec{u}'/c^2)^3} \{ \vec{a}'_{\parallel} + \gamma \vec{a}'_{\perp} + \gamma \vec{v} \times (\vec{a}' \times \vec{u}')/c^2 \} .$$

4. The 3-force, and its associated quantity $\dot{E} = dE/dt = \vec{F} \cdot \vec{u}$, when it is acting on an object with velocity \vec{u} :

$$\vec{F}_{\parallel} = \frac{\vec{F}'_{\parallel} + (\vec{u}' \cdot \vec{F}')\vec{v}/c^2}{1 + \vec{v} \cdot \vec{u}'/c^2}, \quad \vec{F}_{\perp} = \frac{\gamma^{-1} \vec{F}'_{\perp}}{1 + \vec{v} \cdot \vec{u}'/c^2},$$

$$\text{or } \vec{F} = \frac{\gamma^{-1} \vec{F}' + (1 - \gamma^{-1})(\hat{v} \cdot \vec{F}')\hat{v} + (\vec{u}' \cdot \vec{F}')\vec{v}/c^2}{1 + \vec{v} \cdot \vec{u}'/c^2},$$

and a related set of equations, showing the true relationships between the 3-force and the 3-acceleration when objects are moving quite fast:

$$\begin{aligned} \vec{F} &= \gamma_u m [\vec{a} + \gamma_u^2 (\vec{u} \cdot \vec{a}) \vec{u} / c^2] , \\ \text{or } \vec{F}_{\parallel \vec{u}} &= \gamma_u^3 m \vec{a}_{\parallel \vec{u}} , \quad \vec{F}_{\perp \vec{u}} = \gamma_u m \vec{a}_{\perp \vec{u}} . \end{aligned}$$

We now want to say a little bit about the electromagnetic fields. They properly belong as components of a particular second-rank, antisymmetric tensor, usually referred to as the Faraday. Choosing the simplest, Cartesian basis for our 3-vectors, the Faraday is presented as the following matrix:

$$F^{\mu\nu} = \begin{pmatrix} 0 & B^z & -B^y & -E^x/c \\ -B^z & 0 & B^x & -E^y/c \\ B^y & -B^x & 0 & -E^z/c \\ E^x/c & E^y/c & E^z/c & 0 \end{pmatrix} .$$

Since we know that a 4-vector transforms via the Lorentz boost matrix, as described earlier, via $\tilde{r} = \Lambda(\vec{v}) \tilde{r}'$, we may surmise, or believe, that this 2-index object should transform as

$$F^{\mu\nu} = \Lambda(\vec{v})^\mu{}_\alpha F'^{\alpha\beta} \Lambda(\vec{v})^\nu{}_\beta \quad \iff \quad F = \Lambda(\vec{v}) F' \Lambda(\vec{v})^T ,$$

where the second equality is simply the same as the first one, but written in terms of square matrices, using the usual rules for matrix arithmetic.

As before, such a set of transformation equations is perhaps easier to understand if written out in terms of those components parallel and perpendicular to the transformation velocity:

$$\left. \begin{aligned} \vec{E}_{\parallel} &= \vec{E}'_{\parallel} , \\ \vec{E}_{\perp} &= \gamma_v \{ \vec{E}'_{\perp} - \vec{v} \times \vec{B}' \} , \end{aligned} \right\} \quad \left\{ \begin{aligned} \vec{B}_{\parallel} &= \vec{B}'_{\parallel} , \\ \vec{B}_{\perp} &= \gamma_v \{ \vec{B}'_{\perp} + \vec{v} \times \vec{E}' / c^2 \} , \end{aligned} \right.$$

Notice that these equations are really rather different from those for a 4-vector, especially in the sense that for the 3-vector portion of a 4-vector it is that part perpendicular to \vec{v} which is unchanged, while for the 3-vectors \vec{E} and \vec{B} , it is their portion parallel to the velocity which is unchanged.

Again, as a useful reference, there is an second way to present the elements of the electromagnetic field, in terms of a different, second-rank, antisymmetric tensor, referred to as *the dual of the Faraday*, denoted by $*F$, and also sometimes named the Maxwell tensor. It is presented, in the same way as above by the following matrix:

$$(*F)^{\mu\nu} = \begin{pmatrix} 0 & -E^z/c & +E^y/c & -B^x \\ +E^z/c & 0 & -E^x/c & -B^y \\ -E^y/c & +E^x/c & 0 & -B^z \\ +B^x & +B^y & +B^z & 0 \end{pmatrix} .$$