Lie Derivatives and (Conformal) Killing Vectors

A conformal Killing vector, $\underset{\sim}{\xi}$, is a vector field on a manifold that satisfies either of the following equivalent equations:

$$\underset{\underset{\sim}{\xi}}{\mathcal{L}} g_{\mu\nu} = \xi_{(\mu;\nu)} = 2\chi\, g_{\mu\nu} \; , \tag{0.1}$$

for some scalar field χ. The physical understanding is that when the metric is dragged along some congruence of curves then it remains itself **modulo some scale factor**, χ, which may, perhaps, vary from place to place on the manifold. In the case that χ is zero, then we refer to this as a *true* Killing vector, and, clearly the metric is left completely invariant as it is dragged along. An obvious example is dragging a spherically symmetric metric along a path of constant latitude on a sphere. On the other hand, when the quantity χ is only constant, the Killing vector is said to be *homothetic*, and the metric is being changed by a (constant) scale factor as it moves along. This too is quite interesting from first principles; a simple example is a dilatation, where objects are, for instance, enlarged but otherwise unchanged as you proceed in some direction—toward (spacelike) infinity, perhaps. Lastly, when χ is actually a function on the manifold, then this is a true conformal Killing vector. The meaning of these is not quite as obvious, but will be shown below to be related to certain invariances not of the metric but, rather, of the curvature. [Killing vectors are named for a Norwegian mathematician named W. Killing, who first described these notions in 1892.]

In order to understand this definition we must first back up quite a bit, to understand the Lie derivative, with respect to some vector field, used in the equation above. As well, we eventually want to proceed forward from here, because there are integrability conditions, on the connection and curvature, before they may be satisfied by the manifold, to permit the existence of such a vector field. We want to describe all these things here in some detail.

1. Maps between manifolds, and their pullbacks and pushforwards

On some manifold, $\mathcal{M}$, we first choose a point $P \in U \subseteq \mathcal{M}$ and a map, ϕ, from U to some other manifold, $\mathcal{N}$, which might well just be a different region of our original manifold

$\mathcal{M}$. In particular, let $Q \equiv \phi(P) \in W \subseteq \mathcal{N}$. Since there are various tensor spaces attached to the point P, it is reasonable to suppose that this map has a method of correlating tensors over $P \in \mathcal{M}$ to tensors over $Q \equiv \phi(P) \in \mathcal{N}$. To explain how this happens we begin with functions, and define the pullback of functions from $\mathcal{N}$ to $\mathcal{M}$.

Let f be a function defined over the neighborhood $W \subseteq \mathcal{N}$, i.e., $f : W \subseteq \mathcal{N} \to \mathbb{R}$. Then we may define an associated function $\phi^* f : U \subseteq \mathcal{M} \to \mathbb{R}$, the *pullback of f via ϕ* as the following:

$$\forall S \in U \subseteq \mathcal{M}, \quad (\phi^* f)(S) \equiv f(\phi(S)) \ . \tag{1.1}$$

Since individual coordinates are just functions, this allows me to use a coordinate system near $Q \in \mathcal{N}$ to define a coordinate system near $P \in \mathcal{M}$. Let $\{y^\mu\}_1^n$ be a coordinate system defined over $W \subseteq \mathcal{N}$; then $\{x^\mu \equiv \phi^* y^\mu\}_1^n$ is a proper coordinate system defined over $U \subseteq \mathcal{M}$.

Now, since tangent vectors act as operators on functions, this allows us to define the *pushforward* of tangent vector fields:

$$\forall\, \widetilde{v} \in \boldsymbol{\mathcal{T}}_{|U}\ , \quad (\phi_* \widetilde{v}) \in \boldsymbol{\mathcal{T}}_{|W}\ , \quad \text{where } \forall f \in \boldsymbol{\mathcal{F}}_{|W} \quad (\phi_* \widetilde{v})(f) \equiv \widetilde{v}(\phi^* f) \ . \tag{1.2}$$

To see that there actually is something to this definition, let us calculate the relationship between the components of the two vector fields, relative to their appropriate coordinate systems, $\{y^\mu\}$ and $\{x^\mu\}$, as defined above, noting that in coordinates, we may think of f as a function of the coordinates y^μ, while $\phi^* f$ is a function of the coordinates x^μ:

$$\begin{aligned}(\phi_* \widetilde{v})(f) = (\phi_* \widetilde{v})^\mu \frac{\partial}{\partial y^\mu}(f) = \widetilde{v}(\phi^* f) = (\widetilde{v})^\nu \frac{\partial}{\partial x^\nu}(\phi^* f) = (\widetilde{v})^\nu \frac{\partial y^\mu}{\partial x^\nu}\frac{\partial}{\partial y^\mu}(f) \\ \Longrightarrow (\phi_* \widetilde{v})^\mu = v^\nu \, \frac{\partial y^\mu}{\partial x^\nu} \ .\end{aligned} \tag{1.3}$$

Once we have a way to do this for tangent vectors, we may easily perform the reverse sort of mapping for 1-forms, remembering that 1-forms map tangent vectors to real numbers: the *pullback* of a 1-form, $\underset{\sim}{\alpha} \in \Lambda^1_{|W}$ is denoted by $\phi^* \underset{\sim}{\alpha} \in \Lambda^1_{|U}$:

$$\forall\, \widetilde{v} \in \mathcal{T}_{|U}\ , \quad (\phi^* \underset{\sim}{\alpha})(\widetilde{v}) \equiv \underset{\sim}{\alpha}(\phi_* \widetilde{v}) \quad \Longrightarrow \quad (\phi^* \underset{\sim}{\alpha})^\nu = \alpha^\mu \, \frac{\partial x^\nu}{\partial y^\mu} \ . \tag{1.4}$$

It is perhaps useful to give a summary of the various other maps, just described above, induced by our original mapping between manifolds:

$$
\begin{aligned}
&\phi : \mathcal{M} \ \to \ \mathcal{N} \ , \\
&\phi^* : \boldsymbol{\mathfrak{F}}\Big|_{W\subseteq\mathcal{N}} \ \to \ \boldsymbol{\mathfrak{F}}\Big|_{U\subseteq\mathcal{M}} \ , \text{ via} \qquad \phi^* f \equiv f\circ\phi \ , \\
&\qquad\qquad \text{Important Ex. } \ \phi^* y^\mu = x^\mu \ ; \ \ \text{i.e., coord's on } \mathcal{N} \text{ pulled back to coord's on } \mathcal{M}. \\
&\phi_* : \boldsymbol{\mathfrak{T}}\Big|_{U\subseteq\mathcal{M}} \ \to \ \boldsymbol{\mathfrak{T}}\Big|_{W\subseteq\mathcal{N}} \ , \text{ via} \qquad (\phi_*\widetilde{v})(f) \equiv \widetilde{v}(\phi^* f) \quad \Longrightarrow \quad (\phi_*\widetilde{v})^\mu = v^\nu \, \frac{\partial y^\mu}{\partial x^\nu} \ , \\
&\phi^* : \Lambda\Big|_{W\subseteq\mathcal{N}} \ \to \ \Lambda\Big|_{U\subseteq\mathcal{M}} \ , \text{ via} \qquad (\phi^*\underset{\sim}{\alpha})(\widetilde{v}) \equiv \underset{\sim}{\alpha}(\phi_*\widetilde{v}) \quad \Longrightarrow \quad (\phi^*\underset{\sim}{\alpha})^\nu = \alpha^\mu \, \frac{\partial x^\nu}{\partial y^\mu} \ .
\end{aligned}
\tag{1.5}
$$

2. Definitions for the Lie derivative of tensor fields

The Lie derivative is a particular method to determine how vector fields are changing in vector spaces over nearby given points of interest. It is different from the covariant derivative, as it relies on the behavior of a vector field defined in the neighborhood of the point, rather than the existence of a connection there.

Following the discussion in §1, we now suppose that the manifold $\mathcal{N}$ is indeed the same as $\mathcal{M}$, and that there is a given congruence of curves connecting $U \subseteq \mathcal{M}$ and $W \subseteq \mathcal{N}$. These curves all have the same tangent vector field, ξ, but of course begin at different points in the neighborhood of P. We take a parameter λ along the curves, and may then describe the curve that begins at R near $P \in U$ via the formalism $\Gamma(\lambda) = e^{\lambda\widetilde{\xi}}R$. For any particular, fixed value of λ, the set of these curves, through points in U, may be thought of as constituting a particular example of the map ϕ between manifolds, as described above, mapping, for instance P into Q; therefore, it is reasonable to denote this mapping by the symbol ϕ_λ, since there is a different such map for each value of λ. In particular, ϕ_0 is then the identity map, and $\phi_{-\lambda} = (\phi_\lambda)^{-1}$, i.e., the inverse map to ϕ_λ exists and is just $e^{-\lambda\widetilde{\xi}}$.

As an important aside, this construction gives us **two modes of thinking about a set of curves defined over some neighborhood**, either as

i.) a congruence of curves, i.e., maps, $\Gamma_P(\lambda)$, from $\lambda \in \mathbb{R}$ to the manifold, parameterized by the set of (nearby) points on the manifold at which they begin, or ii.) a family of maps, ϕ_λ, from the manifold to itself, parameterized by the real number λ. As these are maps from the manifold to itself that contain the identity map—for $\lambda = 0$—and maps near the identity—for small values of λ—these are often/usually referred to as *a family of flows* of the manifold into itself.

The use of these maps, generated by our family of flows, or congruence of curves, allows us to consider the behavior of whatever type of tensor we want between, for instance, from the tensor spaces over $P \in \mathcal{M}$ to those over $Q \in \mathcal{N}$; in the language above of course, such a relationship would be referred to as a "push-forward." Since these maps always have inverses, at least locally, if the tensor is of type $[p, q]$, then we would, technically, be pulling back its covariant components from $\mathfrak{T}_Q$, using ϕ_λ^{-1}, and pushing forward its contravariant components to $\mathfrak{T}_Q$, using ϕ_λ itself. As we are therefore using the behavior both of $(\phi_\lambda)_*$ and $(\phi_{-\lambda})^*$, this generates a minor notational problem for a single symbol to use as the action on $\mathfrak{T}_P$. We will not make too much "noise" about being rigorous about this, and simply use the symbol $(\phi_\lambda)_*$ as appropriate for this push-forward action on T, and $(\phi_\lambda)^*$ is we want to begin with elements from $\mathfrak{T}_Q$. Therefore, as an example, and in somewhat more mathematical detail, we consider the particular, easily-generalized, case where $\mathbf{T}$ is a type [1,1] tensor over P, then we may define its pushforward value, $(\phi_\lambda)_*\mathbf{T}$, a tensor over Q, via the following. We want to understand a tensor by its (linear) actions on its appropriate domains; therefore, we first choose $\underset{\sim}{\alpha} \in \Lambda^1_{|Q}$ and $\widetilde{v} \in \mathfrak{T}_{|Q}$, appropriate domain elements for our desired pushforward, and then pullback the 1-form along the flows, while pushing forward the vector field along the inverse flows:

$$\begin{gathered}
\underset{\sim}{\alpha} \in \Lambda^1\Big|_Q \;\Rightarrow\; (\phi_\lambda)^*\underset{\sim}{\alpha} \in \Lambda^1\Big|_P \,, \quad \widetilde{v} \in \mathfrak{T}\Big|_Q \;\Rightarrow\; (\phi_{-\lambda})_*\widetilde{v} \in \mathfrak{T}\Big|_P \,, \\
\{(\phi_\lambda)_*(\mathbf{T})\}(\underset{\sim}{\alpha}, \widetilde{v}) \equiv \mathbf{T}[(\phi_\lambda)^*\underset{\sim}{\alpha}, (\phi_\lambda^{-1})_*\,\widetilde{v}] \quad \Longrightarrow \quad [(\phi_\lambda)_*\mathbf{T}]^\mu{}_\alpha = \left(\frac{\partial y^\mu}{\partial x^\nu}\right)\left(\frac{\partial x^\beta}{\partial y^\alpha}\right)\mathbf{T}^\nu{}_\beta \;.
\end{gathered} \tag{2.1}$$

Likewise, if $\mathbf{B}$ is a type [1,1] tensor over Q, then we may define its pullback $(\phi_\lambda)^*\mathbf{B}$, a tensor over P via the following

$$\begin{gathered}
\underset{\sim}{\beta} \in \Lambda^1\Big|_P \;\Rightarrow\; (\phi_{-\lambda})^*\underset{\sim}{\beta} \in \Lambda^1\Big|_Q \quad \widetilde{w} \in \mathfrak{T}\Big|_P \;\Rightarrow\; (\phi_\lambda)_*\widetilde{w} \in \mathfrak{T}\Big|_Q \,, \\
[(\phi_\lambda)^*\mathbf{B})(\underset{\sim}{\beta}, \widetilde{w}) \equiv \mathbf{B}[(\phi_\lambda^{-1})^*\underset{\sim}{\beta}, (\phi_\lambda)_*\widetilde{w}] \quad \Longrightarrow \quad [(\phi_\lambda)^*\mathbf{B}]^\nu{}_\beta = \left(\frac{\partial x^\nu}{\partial y^\mu}\right)\left(\frac{\partial y^\alpha}{\partial x^\beta}\right)\mathbf{B}^\mu{}_\alpha \;.
\end{gathered} \tag{2.2}$$

Now if we also suppose that our tensor is defined over an entire neighborhood including both P and Q, and the curve with tangent vector $\widetilde{\xi}$ joining them, we may suppose that we take the tensor $\mathbf{B}$, say, in the vector space over point $Q = \phi_\lambda(P)$, and pull it back to the tensor space over the point P, using $(\phi_\lambda)^*$. Will it be the same thing as the original tensor $\mathbf{B}$ in the tensor space over the point P. The answer is "probably not"! However, as both tensors lie in the same vector space we may certainly ask for the difference, which surely depends on λ:

$$\Delta_\lambda \mathbf{T}_{|_P} \equiv (\phi_\lambda)^* \left(\mathbf{T}_{|_{\phi_\lambda(P)}} \right) - \mathbf{T}_{|_P} \ . \tag{2.3a}$$

We may then ask, for the given curve $\widetilde{\xi}$, what is the first-order change of this sort, which we will call the Lie derivative in the direction $\widetilde{\xi}$. More precisely, we define the following tensor in the (appropriate) tensor space at P:

$$\mathop{\mathcal{L}}_{\widetilde{\xi}} \mathbf{T} \Big|_P \equiv \lim_{\lambda \to 0} \frac{1}{\lambda} \left\{ [(\phi_\lambda)^* (\mathbf{T}_{|_{\phi_\lambda(P)}})]_{|_P} - \mathbf{T}_{|_P} \right\} \ . \tag{2.3b}$$

We shall see below that this is indeed a tensor of the same type as $\mathbf{T}$.

To evaluate this geometrically-based idea in terms of components, we first take the coordinates of P as x^μ, and then recall that $Q = \phi_\lambda(P)$ lies along the path with tangent vector $\widetilde{\xi}$, so that

$$x^\mu(Q) = x^\mu(P) + \lambda\, \xi^\mu(P) + O(\lambda^2) \ . \tag{2.4a}$$

This allows us to think of the tensor fields in terms of a functional dependence on the coordinates of the points at which they exist, so that we can represent $\mathbf{T}_P$ as $\mathbf{T}[x(P)]$. Since we are comparing two different vectors in the same vector space, with respect to the same basis set, it is simplest if we look at the problem in terms of its components. Therefore, let us suppose, again, that $\mathbf{T}$ is a type [1,1] tensor, so that with respect to whatever basis is being used its components are $T^\nu{}_\beta$. We then which gives us

$$T^\nu{}_\beta[x(Q)] = T^\nu{}_\beta(x + \lambda\xi) = T^\nu{}_\beta(x) + \lambda\, \widetilde{\xi}(T^\nu{}_\beta)(x) + O(\lambda^2) \ , \tag{2.4b}$$

However, as well we have to effect the tensor transformations—push forwards or pull backs—described above in Eqs. (1.5), which requires that we know the relation between the coordinates $\{y^\mu\}$ and $\{x^\nu\}$ at the same point, so that we may act on these components with either $\partial y^\mu/\partial x^\nu$, for the contravariant components, or $\partial x^\nu/\partial y^\mu$, for the covariant components. It is reasonable to want this relationship at the point P, since that is where we are eventually calculating the Lie derivative; therefore, in an analogous way to the expression of $x^\nu(Q)$ via its value, and that of its first derivative, at P, using Taylor's Theorem, we may begin with the coordinates $y^\mu(Q)$ and use Taylor's Theorem to express their values, to lowest order in λ, at the point P, remembering that we go from Q to P along the inverse (or reverse) of our curve, so that Taylor's Theorem tells us

$$y^\mu(P) = y^\mu(Q) - \lambda\,\xi^\mu(Q) + O(\lambda^2)\ . \tag{2.4c}$$

However, next we use the fact that the coordinates $\{y^\mu\}$ are those coordinates near Q that are pulled back by the map induced by the curve to give the coordinates $\{x^\nu\}$w near P; in particular we have the following:

$$x^\nu(P) = \{(\phi_\lambda)^*(y^\nu)\}(P) = y^\nu\left(\phi_\lambda(P)\right) = y^\nu(Q)\ . \tag{2.4d}$$

Inserting this into the previous equation we have

$$y^\mu(P) = x^\mu(P) - \lambda\,\xi^\mu(Q) + O(\lambda^2)\ . \tag{2.4e}$$

However, next we relate the values of our vector field, $\widetilde{\xi}$, at these two (nearby) points, again using Taylor's Theorem, going from P to Q:

$$\xi^\mu(Q) = \xi^\mu(P) + \lambda\,\widetilde{\xi}\,(\xi^\mu)\,(P) + O(\lambda^2)\ . \tag{2.4f}$$

Inserting this into the previous equation then gives us the relationship that we need, i.e., the relationship between the two coordinate sets, both as they have values in the near neighborhood of the point P, where we are dropping additional terms of $O(\lambda^2)$:

$$
\begin{aligned}
y^\mu(P) = \ & x^\mu(P) - \lambda\,\xi^\mu(P) + O(\lambda^2)\,,\\
\Longrightarrow \quad & \frac{\partial y^\mu}{\partial x^\nu}\big|_P = \delta^\mu_\nu\big|_P - \lambda\,\frac{\partial \xi^\mu}{\partial x^\nu}\big|_P + O(\lambda^2)\,,\\
& \text{and the inverse relation,}\quad \frac{\partial x^\nu}{\partial y^\mu}\big|_P = \delta^\nu_\mu\big|_P + \lambda\,\xi^\nu_{,\mu}\big|_P + O(\lambda^2)\,.
\end{aligned}
\tag{2.4g}
$$

Putting all this together gives us the following (complicated-appearing) expression as the component evaluation of Eq. (2.3a), where all quantities concerned are evaluated at the initial point P:

$$
\Delta_\lambda T^\mu{}_\nu = \left\{T^\alpha{}_\beta + \lambda\,\widetilde{\xi}(T^\alpha{}_\beta)\right\}\{\delta^\mu_\alpha - \lambda\,\xi^\mu{}_{,\alpha})\}\,\{\delta^\beta_\nu + \lambda\,\xi^\beta_{,\nu}\} + O(\lambda^2) - T^\mu{}_\nu\,. \tag{2.4h}
$$

As the zero-order terms (in λ) cancel, it is safe to divide by λ; then taking the limit as $\lambda \to 0$ eliminates the higher-order terms and gives us the desired form for the Lie derivative:

$$
\mathcal{L}_{\widetilde{\xi}}\, T^\mu{}_\nu = \xi^\eta \partial_\eta\, T^\mu{}_\nu - \xi^\mu_{,\eta}\, T^\eta{}_\nu + \xi^\eta_{,\nu}\, T^\mu{}_\eta\,. \tag{2.5}
$$

Some relevant first examples are

$$
\begin{aligned}
&\text{for}\ \ f \in \mathfrak{F}, \quad \mathcal{L}_{\widetilde{\xi}}\, f = \xi^\eta \partial_\eta f = \widetilde{\xi}(f)\,,\\
&\text{for}\ \ \widetilde{v} \in \mathfrak{T}, \quad \mathcal{L}_{\widetilde{\xi}}\, \widetilde{v} = \widetilde{e}_\mu\left\{\xi^\nu\, v^\mu_{,\nu} - \xi^\mu_{,\nu} v^\nu\right\} = \widetilde{e}_\mu\{\widetilde{\xi}(v^\mu) - \widetilde{v}(\xi^\mu)\} \quad = \quad [\widetilde{\xi}, \widetilde{v}]\,,\\
&\text{for}\ \ \underset{\sim}{\alpha} \in \Lambda^1\,, \quad \mathcal{L}_{\widetilde{\xi}}\, \underset{\sim}{\alpha} = \underset{\sim}{\omega}^\nu\left\{\xi^\mu \alpha_{\nu,\mu} + \xi^\mu_{,\nu}\alpha_\mu\right\} \quad = \quad d\underset{\sim}{\alpha}(\widetilde{\xi}, \,\cdot\,) + d[\underset{\sim}{\alpha}(\widetilde{\xi})]\,,
\end{aligned}
\tag{2.6}
$$

where the equalities at the end of the two last lines are important ways to think about the Lie derivatives of tangent vectors and 1-forms, respectively. As well, the $\cdot$ in the argument of $d\underset{\sim}{\alpha}$ simply indicates that it is waiting for yet one more vector on which to operate; i.e., that term is a 1-form, as it should be.

I now want to change the above equation so that it involves only covariant derivatives. Therefore, I will first agree that all calculations in this set of notes will be done in terms of an ordinary coordinate-basis for tangent vectors, and 1-forms, which will simplify the presentation of the algebra. However, when we acquire the final results they will only use covariant derivatives, in such a way that those results are basis independent. We therefore now show that the derivatives above could be covariant derivatives, without any change in the equation, remembering that in the (current) coordinate basis the connection is symmetric in its last two indices. This gives me

$$\begin{aligned}\underset{\tilde{\xi}}{\mathcal{L}} A^\mu{}_\nu &= \left(A^\mu{}_{\nu;\lambda}\xi^\lambda - \Gamma^\mu{}_{\rho\lambda}A^\rho{}_\nu\xi^\lambda + \Gamma^\rho{}_{\nu\lambda}A^\mu{}_\rho\xi^\lambda\right) - \xi^\mu_{,\lambda}A^\lambda{}_\nu + \xi^\lambda_{,\nu}A^\mu{}_\lambda \\ &= A^\mu{}_{\nu;\lambda}\xi^\lambda - \xi^\mu{}_{;\lambda}A^\lambda{}_\nu + \xi^\lambda{}_{;\nu}A^\mu{}_\lambda \ .\end{aligned} \tag{2.7}$$

From this calculation we see that for a tensor of arbitrary type, there should be a term which is simply the covariant derivative, in the ξ-direction, of the components, and then a term involving the covariant derivative of ξ, for each of the covariant components, in turn, with a plus sign, and a term for each of the contravariant components, in turn, with a minus sign. **We take the above as a general, algebraic description of the Lie derivative of an arbitrary tensor.**

As it is a very important special case, we can write down the Lie derivative of the metric:

$$\underset{\tilde{\xi}}{\mathcal{L}} g_{\mu\nu} = \xi^\eta\, g_{\mu\nu;\eta} + \xi^\eta_{;\mu}\, g_{\eta\nu} + \xi^\eta_{;\nu}\, g_{\mu\eta} = \xi_{nu;\mu} + \xi_{\mu;\nu} = \xi_{(\mu;\nu)} \ , \tag{2.8}$$

where we have used the fact that the covariant derivative of the metric is identically zero.

We will consider some other important tensors; however, at the moment let us instead consider the Lie derivative of the components of the connection, since they have a different transformation law, and therefore we would expect also a different form for the Lie derivative:

$$\begin{aligned}\phi^*_\lambda\Gamma^\mu{}_{\nu\lambda}{}_{|_P} &= X^\mu{}_\tau(X^{-1})^\eta{}_\nu(X^{-1})^\sigma{}_\lambda\Gamma^\tau{}_{\eta\sigma}{}_{|_P} + X^\mu{}_\tau(X^{-1})^\tau{}_{\nu,\lambda}{}_{|_P} \ , \\ \Longrightarrow \ \underset{\tilde{\xi}}{\mathcal{L}}\Gamma^\mu{}_{\nu\lambda} &= \xi^\eta\,\Gamma^\mu{}_{\nu\lambda,\eta} - \xi^\mu_{,\tau}\Gamma^\tau{}_{\nu\lambda} + \xi^\tau_{,\nu}\Gamma^\mu{}_{\tau\lambda} + \xi^\tau_{,\lambda}\Gamma^\mu{}_{\nu\tau} + (\xi^\mu_{,\nu})_{,\lambda} \ .\end{aligned} \tag{2.9a}$$

Now we rewrite this second derivative in terms of a second covariant derivative plus whatever extra terms are needed, and insert it into the calculation above:

$$\xi^\mu{}_{;\nu\lambda} = \xi^\mu{}_{;\nu,\lambda} + \Gamma^\mu{}_{\sigma\lambda}\xi^\sigma{}_{;\nu} - \Gamma^\sigma{}_{\nu\lambda}\xi^\mu{}_{;\sigma} = \xi^\mu_{,\nu\lambda} + \Gamma^\mu{}_{\sigma\nu,\lambda}\xi^\sigma + \Gamma^\mu{}_{\sigma\nu}\xi^\sigma_{,\lambda} + \Gamma^\mu{}_{\sigma\lambda}\xi^\sigma{}_{;\nu} - \Gamma^\sigma{}_{\nu\lambda}\xi^\mu{}_{;\sigma} \tag{2.9b}$$

This gives us two pairs of terms which are simply the difference of the covariant derivative and the ordinary derivative, which we replace by the form involving the connection, which results in the following quite interesting result:

$$\mathop{\mathcal{L}}_{\widetilde{\xi}}\Gamma^\mu{}_{\nu\lambda} = \xi^\mu{}_{;\nu\lambda} + 2\xi^\eta\left\{\Gamma^\mu{}_{\nu[\lambda,\eta]} + \Gamma^\mu{}_{\tau[\eta}\Gamma^\tau{}_{\nu\lambda]}\right\} = \xi^\mu{}_{;\nu\lambda} - \xi^\eta R^\mu{}_{\nu\lambda\eta}\ . \tag{2.10}$$

Since the curvature tensor has arisen, it is reasonable that we should also ask about its Lie derivative. Following the rules above for arbitrary tensors, we may immediately write that down:

$$\mathop{\mathcal{L}}_{\widetilde{\xi}}R_{\mu\nu\lambda\eta} = \xi^\tau R_{\mu\nu\lambda\eta;\tau} + 2\xi^\tau{}_{;[\mu}R_{\tau\nu]\lambda\eta} + 2\xi^\tau{}_{;[\lambda}R_{\mu\nu\tau\eta]}\ , \tag{2.11}$$

so that Eqs. (2.8-11) give us the form for the Lie derivative of all the quantities of interest for our next discussion point, namely the discovery and importance of Killing vectors. [Recall that the function of the 2's in the recent equations are simply to cancel the "built-in" 1/2 in the definition of the commutator brackets for indices.]

3. Implications from the existence of a Killing Vector

The existence of a Killing vector tells us immediately about the symmetries of the metric. If, for instance, $\widetilde{K} = \partial/\partial q$ is a Killing vector for some manifold, then it is always possible to arrange a coordinate system, with q as one of the coordinates, so that the components of the metric with respect to that coordinate basis do not depend on q. It should be noted that this can be done for any collection of Killing vectors **provided** that they commute with one another; otherwise, one must choose whichever one is desired. Therefore, for instance, in the static, spherically-symmetric metric, we may choose coordinates so that $g_{\mu\nu}$ are independent of φ and t, but may not also eliminate the dependence on θ.

Another very useful feature of Killing vectors is that they may be used to determine "constants of the motion," i.e., quantities that will be constant along any given geodesic. To see this, consider a Killing vector $\widetilde{K}$ and a 4-velocity $\widetilde{u}$, tangent to some geodesic motion. Then it is claimed that the scalar quantity $\widetilde{K}\cdot\widetilde{u} \equiv \mathbf{g}(\widetilde{K},\widetilde{u})$ is constant along that motion:

$$\widetilde{u}(\widetilde{K}\cdot\widetilde{u}) = u^\mu\nabla_\mu(K^\eta g_{\eta\lambda}u^\lambda) = K^\eta\, u^\mu\nabla_\mu u_\eta + u^\lambda u^\mu\nabla_\mu K_\lambda = u^\lambda u^\mu K_{(\lambda;\mu)} = 0\ , \tag{3.1}$$

where the first term vanished because $\widetilde{u}$ is a geodesic, and the summation into two copies of u^λ in the second term indicated that only the symmetric part of those coefficients contributed.

4. Further conditions for the existence of a (conformal) Killing Vector

The symmetries that allow the existence of a Killing vector also put constraints on the connections and curvatures of the manifold; or the values of the connections and curvatures put additional constraints on the existence of a Killing vector, as one chooses to proceed. We now proceed, therefore, to resolve the complete set of such constraints, which we think of as "integrability conditions" for the existence of a Killing vector.

The curvature tensor of a Riemannian manifold measures the lack of commutativity of covariant derivatives, and also satisfies the first and second Bianchi identities; therefore, beginning with our Killing vector, we may write down the following equations:

$$\begin{aligned}
\xi_{\mu;[\nu\lambda]} &= +R^\eta{}_{\mu\nu\lambda}\xi_\eta \iff \xi^\mu{}_{;[\nu\lambda]} = -R^\mu{}_{\eta\nu\lambda}\xi^\eta \,, \\
R^\eta{}_{\mu\nu\lambda} &+ R^\eta{}_{\nu\lambda\mu} + R^\eta{}_{\lambda\mu\nu} = 0 \,, \\
R_{\sigma\lambda\nu\mu;\eta} &+ R_{\eta\sigma\nu\mu;\lambda} + R_{\lambda\eta\nu\mu;\sigma} = 0 \,.
\end{aligned} \tag{4.1}$$

We may insert into this identity the requirements for a conformal Killing vector. Putting the first two of Eqs. (4.1) together, and then inserting Eqs. (0.1), we have

$$\begin{aligned}
&\xi_{\mu;\nu\lambda} - \xi_{\mu;\lambda\nu} + \xi_{\nu;\lambda\mu} - \xi_{\nu;\mu\lambda} + \xi_{\lambda;\mu\nu} - \xi_{\lambda;\nu\mu} = 0 \,, \\
\Longrightarrow \quad &\xi_{\mu;\nu\lambda} + \xi_{\nu;\lambda\nu} + \xi_{\lambda;\mu\nu} = \chi_{,\lambda}g_{\mu\nu} + \chi_{,\nu}g_{\mu\lambda} + \chi_{,\mu}g_{\lambda\nu} \,, \\
\Longrightarrow \quad \xi_{\mu;\nu\lambda} &= -\xi_{\nu;\lambda\mu} - \xi_{\lambda;\mu\nu} + \chi_{;\lambda}g_{\mu\nu} + \chi_{;\nu}g_{\mu\lambda} + \chi_{;\mu}g_{\lambda\nu} \\
&= \xi_{\lambda;[\nu\mu]} + [-\chi_{,\mu}g_{\nu\lambda} + \chi_{,\nu}g_{\lambda\mu} + \chi_{,\lambda}g_{\mu\nu}] \\
&= R^\eta{}_{\lambda\nu\mu}\xi_\eta + [-\chi_{,\mu}g_{\nu\lambda} + \chi_{,\nu}g_{\lambda\mu} + \chi_{,\lambda}g_{\mu\nu}] \,,
\end{aligned} \tag{4.2}$$

which may be thought of as the first integrability condition for our original system. However, using Eq. (2.10), the definition of the Lie derivative for the connection, and multiplying by $g^{\mu\eta}$ to raise the first index, we may rewrite this requirement in the following more geometric form:

$$\mathop{\mathcal{L}}_{\xi}\Gamma^\mu{}_{\nu\lambda} = g^{\mu\eta}[-\chi_{,\eta}g_{\nu\lambda} + \chi_{,\nu}g_{\lambda\eta} + \chi_{,\lambda}g_{\eta\nu}] \,. \tag{4.3}$$

We now inquire as to whether these integrability conditions have integrability conditions of their own. We do this by considering again the form in Eqs. (4.1) the application of the relation between commutators of covariant derivatives and curvature, for the third covariant derivatives of our Killing vector:

$$\xi_{\mu;\nu[\lambda\eta]} = R^{\sigma}{}_{\mu\lambda\eta}\xi_{\sigma;\nu} + R^{\sigma}{}_{\nu\lambda\eta}\xi_{\mu;\sigma} \ . \tag{4.4}$$

We may, however, also calculate the left-hand side of this equation by taking the covariant derivative of our first integrability equations, Eq. (4.2), which gives:

$$\xi_{\mu;\nu[\lambda\eta]} = R^{\sigma}{}_{[\lambda\nu\mu}\xi_{\sigma;\eta]} + R^{\sigma}{}_{[\lambda\nu\mu;\eta]}\xi_{\sigma} + [-\chi_{;\mu[\eta}g_{\nu\lambda]} + \chi_{;\nu[\eta}g_{\lambda]\mu} + \chi_{;[\lambda\eta]}g_{\mu\nu}] \ . \tag{4.5}$$

Noting that the very last term above vanishes, because χ is a scalar, we may equate the two expressions for the commutator of the derivatives and obtain the following:

$$\begin{aligned}\xi^{\sigma}(R_{\sigma\lambda\nu\mu;\eta} - R_{\sigma\eta\nu\mu;\lambda}) + \xi^{\sigma}{}_{;\eta}R_{\sigma\lambda\nu\mu} - \xi^{\sigma}{}_{;\lambda}R_{\sigma\eta\nu\mu} + \xi^{\sigma}{}_{;\nu}R_{\sigma\mu\nu\lambda} \\ + \xi_{\mu;\sigma}R^{\sigma}{}_{\nu\eta\lambda} + \chi_{;\nu\eta}g_{\lambda\nu} - \chi_{;\nu\lambda}g_{\eta\mu} - \chi_{;\mu\eta}g_{\nu\lambda} + \chi_{;\mu\lambda}g_{\nu\eta} = 0 \ .\end{aligned} \tag{4.6}$$

If we now intervene with the second Bianchi identity, and use the definition of the Lie derivative of the curvature tensor, Eq. (8), we may rewrite the last equation in the following form:

$$\underset{\xi}{\mathcal{L}} R_{\eta\lambda\nu\mu} = 2\chi R_{\eta\lambda\nu\mu} - \chi_{;\eta[\nu}g_{\lambda\mu]} - \chi_{;\lambda[\mu}g_{\eta\nu]} \ . \tag{4.7}$$

5. Specialization to Homothetic Killing Vectors

A homothetic symmetry requires that χ be constant, possibly zero of course, in which case it would actually be a *true* Killing vector. We therefore now specialize to the constant case. Specialization of the equations above gives us

$$\begin{aligned}\underset{\widetilde{\xi}}{\mathcal{L}} g_{\mu\nu} &= 2\chi\, g_{\mu\nu} \ , \\ \underset{\widetilde{\xi}}{\mathcal{L}}\Gamma^{\mu}{}_{\nu\lambda} &= 0 = \underset{\widetilde{\xi}}{\mathcal{L}} R^{\mu}{}_{\nu\lambda\eta} \ .\end{aligned} \tag{5.1}$$

The first requirement on the second line is simply a repeat of Eqs. (4.3) in the current situation; however, it is not immediately obvious that the relation on the curvature given in that line is consistent with the previous Eq. (4.7). In order to show that consistency we consider the following sequence of calculations. First, we find the Lie derivative of the inverse metric:

$$
\begin{aligned}
0 = \underset{\widetilde{\xi}}{\mathcal{L}}\delta^\mu_\lambda &= \underset{\widetilde{\xi}}{\mathcal{L}}\left(g^{\mu\nu}g_{\nu\lambda}\right) = \left(\underset{\widetilde{\xi}}{\mathcal{L}}g^{\mu\nu}\right)g_{\nu\lambda} + g^{\mu\nu}\underset{\widetilde{\xi}}{\mathcal{L}}g_{\nu\lambda} \\
&= \left(\underset{\widetilde{\xi}}{\mathcal{L}}g^{\mu\nu}\right)g_{\nu\lambda} + g^{\mu\nu}2\chi\, g_{\nu\lambda} = \left(\underset{\widetilde{\xi}}{\mathcal{L}}g^{\mu\nu}\right)g_{\nu\lambda} + 2\chi\delta^\mu_\lambda \;, \\
&\Longrightarrow \underset{\widetilde{\xi}}{\mathcal{L}}g^{\mu\nu} = -2\chi\, g^{\mu\nu} \;.
\end{aligned}
\tag{5.2}
$$

We may then use that to raise the index on the Lie derivative for the curvature:

$$
\begin{aligned}
\underset{\widetilde{\xi}}{\mathcal{L}}R^\mu{}_{\nu\lambda\eta} &= \underset{\widetilde{\xi}}{\mathcal{L}}\left(g^{\mu\sigma}R_{\sigma\nu\lambda\eta}\right) = \left\{\underset{\widetilde{\xi}}{\mathcal{L}}g^{\mu\sigma}\right\}R_{\sigma\nu\lambda\eta} + g^{\mu\sigma}\left\{\underset{\widetilde{\xi}}{\mathcal{L}}R_{\sigma\nu\lambda\eta}\right\} \\
&= -2\chi R^\mu{}_{\nu\lambda\eta} + 2\chi\, g^{\mu\sigma}R_{\sigma\nu\lambda\eta} = 0 \;.
\end{aligned}
\tag{5.3}
$$

Following that line of calculations, it is also then worthwhile to consider various pieces of the curvature tensor:

$$
\begin{aligned}
\underset{\widetilde{\xi}}{\mathcal{L}}\mathcal{R}_{\nu\eta} &= \underset{\widetilde{\xi}}{\mathcal{L}}R^\mu{}_{\nu\mu\eta} = 0 \;, \\
\underset{\widetilde{\xi}}{\mathcal{L}}\mathcal{R} &= \underset{\widetilde{\xi}}{\mathcal{L}}\left(g^{\nu\eta}\mathcal{R}_{\nu\eta}\right) = -2\chi\, g^{\nu\eta}\mathcal{R}_{\nu\eta} = -2\chi\,\mathcal{R} \;, \\
\text{and also}\quad \underset{\widetilde{\xi}}{\mathcal{L}}\Gamma_{\mu\nu\eta} &= \underset{\widetilde{\xi}}{\mathcal{L}}\left(g_{\mu\sigma}\Gamma^\sigma{}_{\nu\eta}\right) = 2\chi\,\Gamma_{\mu\nu\eta} \;.
\end{aligned}
\tag{5.4}
$$

6. Specialization to Pure Killing Vectors

Here I simply want to summarize the complete set of integrability conditions to find **pure** Killing vectors on a given metric manifold with metric-compatible connection, which means that we re-write the equations above in the special case when the proportionality factor, χ, vanishes (again the 2's simply cancel out the 1/2's buit into the defn of the brackets):

$$
\begin{aligned}
0 &= \underset{\widetilde{\xi}}{\mathcal{L}}\, g_{\mu\nu} = \xi_{(\mu;\nu)} \;, \\
0 &= \underset{\widetilde{\xi}}{\mathcal{L}}\,\underset{\sim}{\Gamma}^\mu{}_{\nu\lambda} = \xi^\mu{}_{;\nu\lambda} - \xi^\eta\, R^\mu{}_{\nu\lambda\eta} \;, \\
0 &= \underset{\widetilde{\xi}}{\mathcal{L}}\, R_{\mu\nu\lambda\eta} = \xi^\tau R_{\mu\nu\lambda\eta;\tau} - 2\xi^\tau{}_{;[\mu}R_{\nu]\tau\lambda\eta} - 2\xi^\tau{}_{;[\lambda}R_{\eta]\tau\mu\nu} \;.
\end{aligned}
\tag{6.1}
$$