Notes for the standard central, single mass metric in Kruskal coordinates

I. Relation to Schwarzschild coordinates

One originally relates the Kruskal coordinates to the Schwarzschild coordinates in the following way:

$$\left.\begin{aligned} u &= \sqrt{r/2m-1}\, e^{r/4m}\ \cosh(t/4m)\ , \\ v &= \sqrt{r/2m-1}\, e^{r/4m}\ \sinh(t/4m)\ , \end{aligned}\right\} ,\quad r \geq 2m\ ,$$
$$\left.\begin{aligned} u &= \sqrt{1-r/2m}\, e^{r/4m}\ \sinh(t/4m)\ , \\ v &= \sqrt{1-r/2m}\, e^{r/4m}\ \cosh(t/4m)\ , \end{aligned}\right\} ,\quad r \leq 2m\ . \tag{1.1}$$

As it turns out, however, once this has been done, the manifold is incomplete, and one must append an additional copy of it. We refer to the first two portions of it, above, as *Quadrant I*, when $r \geq 2m$, and *Quadrant II*, when $r \geq 2m$. The complete extension of the manifold requires another copy of these quadrants, for which we have the same equations as above, but with u and v replaced by $-u$ and $-v$. We then refer to those sections as *Quadrant III*, when $r \geq 2m$ but $u < 0$, and also *Quadrant IV*, when $r \leq 2m$ and $v < 0$. [We will consider inserting here a graph of this (well-known) situation.] Our earth lives always in Quadrant I.

It is then straightforward to resolve these equations for (a function of) r and for t:

$$u^2 - v^2 = (r/2m - 1)\, e^{r/2m} = (1 - 2m/r)\left(\frac{4m}{f}\right)^2 , \qquad t = 4m \begin{cases} \tanh^{-1}(v/u)\,, & I\ \&\ III\ , \\ \tanh^{-1}(u/v)\,, & II\ \&\ IV\ . \end{cases} \tag{1.2}$$

The curves of constant r are hyperbolae in the (u, v) coordinate space, where they are considered as represented as a standard spacetime diagram, with u as spatial and v as timelike.

The Schwarzschild metric then has the following equivalent forms, in the two coordinate systems:

$$\mathbf{g} = \frac{dr^2}{1-2m/r} + r^2\, d\Omega^2 - (1-2m/r)\, dt^2 = \frac{32m^3}{r}\, e^{-r/2m}\, (du^2 - dv^2) + r^2\, d\Omega^2\ , \tag{1.3}$$

where it is convenient to give the coefficient of du^2 a name. We define

$$f(r) \equiv \frac{4m}{\sqrt{r/2m}}\, e^{-r/4m} = 4m\sqrt{\frac{e^{-r/2m}}{r/2m}}\ , \tag{1.4}$$

so that the u, v-part of the metric is now just $f^2(du^2 - dv^2)$.

It is then straightforward to see that

$$du = \tfrac{1}{4m}\left\{ \frac{r/2m}{\sqrt{r/2m-1}}\, e^{r/4m}\,\cosh(t/4m)\,dr + \sqrt{r/2m-1}\,e^{r/4m}\,\sinh(t/4m)\,dt \right\} ,$$
$$dv = \tfrac{1}{4m}\left\{ \frac{r/2m}{\sqrt{r/2m-1}}\, e^{r/4m}\,\sinh(t/4m)\,dr + \sqrt{r/2m-1}\,e^{r/4m}\,\cosh(t/4m)\,dt \right\} , \tag{1.5a}$$

from which we may quickly verify that the two forms of the metric are equivalent. On the other hand, we may invert the equations, to determine dr and dt:

$$dr = \frac{f}{4m}\left(u\,\underset{\sim}{\omega}^{\hat{u}} - v\,\underset{\sim}{\omega}^{\hat{v}}\right) , \qquad dt = \frac{f}{4m}\left(\frac{r/2m}{r/2m-1}\right)\left(-\,v\,\underset{\sim}{\omega}^{\hat{u}} + u\,\underset{\sim}{\omega}^{\hat{v}}\right) . \tag{1.5b}$$

A possibly useful/interesting comment is that $u_{,t} = v$ and $v_{,t} = u$, so that

$$\partial_t = v\partial_u + u\partial_v = f(v\widetilde{e}_{\hat{u}} + u\widetilde{e}_{\hat{v}}) . \tag{1.5c}$$

Since ∂_t is a Killing vector, this suggests that this is the form of the Killing vector in these coordinates, so that the following should be a constant of the motion:

$$-A = u_t = \mathbf{g}(\partial_t, \widetilde{u}) = f(v\,u_{\hat{u}} + u\,u_{\hat{v}}) = f(v\,u^{\hat{u}} - u\,u^{\hat{v}}) . \tag{1.5d}$$

Since it is difficult to resolve Eqs. (1.2) for r explicitly—such explication would involve Lambert's function, as described, for instance, by Hille—we use the given form of the equation to determine $r_{,u}$ and $r_{,v}$, which will be needed. Differentiating that equation by ∂_u, or by ∂_v, and then dividing appropriately we may determine

$$r_{,u} = (2m)^2\frac{2u}{r}\,e^{-r/2m} = \frac{u}{4m}\,f^2(r) , \quad r_{,v} = -(2m)^2\frac{2v}{r}\,e^{-r/2m} = -\frac{v}{4m}\,f^2(r) . \tag{1.6}$$

It is also useful to go ahead and explicitly determine derivatives of our function f:

$$f' \equiv \frac{df}{dr} = -\frac{1+r/2m}{(r/2m)^{3/2}}\,e^{-r/4m} = -\frac{1+r/2m}{r/2m}\,\frac{f}{4m} = -\frac{1+r/2m}{2r}\,f . \tag{1.7}$$

2. Curvature and geodesics in these coordinates

We now define an orthonormal basis in our Kruskal coordinates:

$$\begin{aligned} &\underset{\sim}{\omega}^{\hat{u}} \equiv f\,du , \quad \underset{\sim}{\omega}^{\hat{v}} \equiv f\,dv , \quad \underset{\sim}{\omega}^{\hat{\theta}} \equiv r\,d\theta , \quad \underset{\sim}{\omega}^{\hat{\varphi}} \equiv r\,\sin\theta\,d\varphi , \\ &\widetilde{e}_{\hat{u}} = \tfrac{1}{f}\,\partial_u , \quad \widetilde{e}_{\hat{v}} = \tfrac{1}{f}\,\partial_v , \quad \widetilde{e}_{\hat{\theta}} = \tfrac{1}{r}\,\partial_\theta , \quad \widetilde{e}_{\hat{\varphi}} = \tfrac{1}{r\sin\theta}\,\partial_\varphi . \end{aligned} \tag{2.1}$$

One may then ask GRTensorII to calculate the connections and curvatures:

$$\begin{aligned}\Gamma_{\hat u\hat\theta\hat\theta} &= -\frac{u}{r}\frac{f}{4m} = \Gamma_{\hat u\hat\varphi\hat\varphi}\ ,\\ \Gamma_{\hat v\hat\theta\hat\theta} &= +\frac{v}{r}\frac{f}{4m} = \Gamma_{\hat v\hat\varphi\hat\varphi}\ ,\\ \Gamma_{\hat u\hat v\hat u} = +\frac{v}{2r}(1+r/2m)\frac{f}{4m}\ ,\ \Gamma_{\hat u\hat v\hat v} &= -\frac{u}{2r}(1+r/2m)\frac{f}{4m}\ ,\quad \Gamma_{\hat\theta\hat\varphi\hat\varphi} = -\frac{\cot\theta}{r}\ .\end{aligned} \tag{2.2}$$

As well we determine the components of the Riemann tensor, where one needs to replace $u^2 - v^2$ by its equal $(r/2m-1)\,e^{r/2m}$ several times:

$$\begin{aligned}R_{\hat u\hat\theta\hat u\hat\theta} = -\frac{m}{r^3} = R_{\hat u\hat\varphi\hat u\hat\varphi} &= -R_{\hat v\hat\theta\hat v\hat\theta} = -R_{\hat v\hat\varphi\hat v\hat\varphi}\ ,\\ R_{\hat\theta\hat\varphi\hat\theta\hat\varphi} = 2\frac{m}{r^3} &= -R_{\hat u\hat v\hat u\hat v}\ .\end{aligned} \tag{2.3}$$

We may then write down the form of a timelike tangent vector, $\widetilde{u}$, normalized to have length $-1 = \widetilde{u}^2$:

$$\begin{aligned}\widetilde{u} = \frac{d}{d\tau} &= \frac{du}{d\tau}\,\partial_u + \frac{dv}{d\tau}\,\partial_v + \frac{d\theta}{d\tau}\,\partial_\theta + \frac{d\varphi}{d\tau}\,\partial_\varphi = u^{\hat u}\,\widetilde{e}_{\hat u} + u^{\hat v}\,\widetilde{e}_{\hat v} + u^{\hat\theta}\,\widetilde{e}_{\hat\theta} + u^{\hat\varphi}\,\widetilde{e}_{\hat\varphi}\ ,\\ \Longrightarrow\quad u^{\hat u} &= f\,\frac{du}{d\tau}\ ,\quad u^{\hat v} = f\,\frac{dv}{d\tau}\ ,\quad u^{\hat\theta} = r\,\frac{d\theta}{d\tau}\ ,\quad u^{\hat\varphi} = r\sin\theta\,\frac{d\varphi}{d\tau}\ .\end{aligned} \tag{2.4}$$

The basic geodesic equations then say that

$$\begin{aligned}\frac{du^{\hat u}}{d\tau} &= -\Gamma_{\hat u\hat\theta\hat\theta}(u^{\hat\theta})^2 - \Gamma_{\hat u\hat\varphi\hat\varphi}(u^{\hat\varphi})^2 - \Gamma_{\hat u\hat v\hat u}u^{\hat v}u^{\hat u} - \Gamma_{\hat u\hat v\hat v}(u^{\hat v})^2\\ &= \frac{u}{r}\frac{f}{4m}[(u^{\hat\theta})^2 + (u^{\hat\varphi})^2] - \frac{vu^{\hat u} - uu^{\hat v}}{2r}(1+r/2m)\frac{f}{4m}u^{\hat v}\ ,\\ \frac{du^{\hat v}}{d\tau} &= +\Gamma_{\hat v\hat\theta\hat\theta}(u^{\hat\theta})^2 + \Gamma_{\hat v\hat\varphi\hat\varphi}(u^{\hat\varphi})^2 + \Gamma_{\hat v\hat u\hat u}(u^{\hat u})^2 + \Gamma_{\hat v\hat u\hat v}u^{\hat u}u^{\hat v}\\ &= +\frac{v}{r}\frac{f}{4m}[(u^{\hat\theta})^2 + (u^{\hat\varphi})^2] - \frac{vu^{\hat u} - uu^{\hat v}}{2r}(1+r/2m)\frac{f}{4m}u^{\hat u}\ ,\\ \frac{du^{\hat\theta}}{d\tau} &= \frac{\cot\theta}{r}(u^{\hat\varphi})^2 - \frac{uu^{\hat u} - vu^{\hat v}}{r}\frac{f}{4m}u^{\hat\theta}\ ,\\ \frac{du^{\hat\varphi}}{d\tau} &= -\frac{\cot\theta}{r}u^{\hat\theta}u^{\hat\varphi} - \frac{uu^{\hat u} - vu^{\hat v}}{r}\frac{f}{4m}u^{\hat\varphi}\ .\end{aligned} \tag{2.5}$$

We may check that the normalization of the 4-velocity is indeed preserved, i.e., multiplying each equation for $du^\alpha/d\tau$ by u_α and summing, we may check that the sum adds up to just zero, as desired for $d(-1)/d\tau$.

Looking at the last two of these equations, we easily see that there do exist orbits that remain in the equatorial plane if they begin there; i.e., we consider $\theta_0 = \pi/2$ and

$u^{\hat\theta} = 0$, which tells us that $du^{\hat\theta}/d\tau = 0$. All single worldlines may be considered of this type, as we may orient the equatorial plane so that it contains that worldline. On the other hand, there are, in particular, radial orbits in that plane: if we set $u^{\hat\varphi} = 0$, then we have $du^{\hat\varphi}/d\tau = 0$.

Radial Orbits:

The normalization condition, already considered, now says that

$$(u^{\hat v})^2 - (u^{\hat u})^2 = +1 = f^2[(v')^2 - (u')^2] \,. \tag{2.6a}$$

The conservation law, because of the timelike Killing vector, from Eq. (1.5d), says that

$$f(u\,u^{\hat v} - v\,u^{\hat u}) = A = f^2[u\,v' - v\,u'] \,. \tag{2.6b}$$

To give a more direct proof that this energy-like quantity is truly conserved, we re-write it in the form

$$\begin{aligned}\frac{d}{d\tau}A = \frac{d}{d\tau}\left[f u\,u^{\hat v} - f v\,u^{\hat u}\right] &= f u\frac{du^{\hat v}}{d\tau} - f v\frac{du^{\hat u}}{d\tau} + f\,u^{\hat v}u' - f\,u^{\hat u}v' + (u\,u^{\hat v} - v\,u^{\hat u})\frac{df}{d\tau}\\ &= -(v\,u^{\hat u} - u\,u^{\hat v})(1 + r/2m)\frac{f^2}{8mr}(u\,u^{\hat u} - v\,u^{\hat v})\\ &\quad - (u\,u^{\hat v} - v\,u^{\hat u})(1 + r/2m)\frac{f^2}{8mr}(u\,u^{\hat u} - v\,u^{\hat v}) \;=\; 0\,,\end{aligned} \tag{2.7}$$

where we have used Eqs. (1.7) and (1.5b) to calculate $df/d\tau$, and the two central terms, among the 6 terms in the first line, have cancelled since $fu' = u^{\hat u}$ and $fv' = u^{\hat v}$.

It is perhaps also interesting to re-write the equations in a more coordinate-based point of view. We may introduce a name for the extra part of the equations not involving f, but involving r directly, and first re-write the equations already given:

$$\left.\begin{aligned}\frac{d}{d\tau}u^{\hat u} &= -\,\mathcal{A}f^3(vu' - uv')v'\,,\\ \frac{d}{d\tau}u^{\hat v} &= -\,\mathcal{A}f^3(vu' - uv')u'\,,\\ \frac{d}{d\tau}f &= -\,\mathcal{A}f^3(uu' - vv')\,,\end{aligned}\right\} \qquad \mathcal{A} \equiv \frac{1 + r/2m}{2mr}\,. \tag{2.8}$$

But now using the fact that $u^{\hat u} \equiv fu'$ and $u^{\hat v} \equiv fv'$, we may obtain the following:

$$\begin{aligned}u'' &= \mathcal{A}f^2\{u[(u')^2 + (v')^2] - 2vu'v'\}\,,\\ v'' &= -\,\mathcal{A}f^2\{v[(u')^2 + (v')^2] - 2uu'v'\}\,.\end{aligned} \tag{2.9}$$

Next we may note that the factor may be re-written in many ways:

$$\mathcal{A}f^2 = \frac{1 + r/2m}{r/2m}4\frac{e^{-r/2m}}{r/2m} = \left(1 - \frac{1}{\rho^2}\right)\frac{4}{u^2 - v^2}\,, \qquad \rho \equiv r/2m\,. \tag{2.10}$$

Given the two conserved quantities displayed in Eqs. (2.6), and noticing, from the definition of $f = f(r)$, that as $r \to 0$, $1/f^2$ vanishes like r, it follows that $(v')^2$ approaches $(u')^2$, which is the same as saying that the speed of light is being approached. If we also notice that $v' > 0$ while $u' < 0$, then we may say that v' approaches $-u'$. This then also allows us to see that $u + v$ approaches zero like r.

As an aside I note that the radial equations in the $\{r, t\}$ coordinates are given as follows:

$$\left(\frac{dr}{d\tau}\right)^2 = A^2 - 1 + \frac{2m}{r} \ , \qquad \frac{dt}{d\tau} = \frac{A}{1 - 2m/r} \ , \tag{2.11}$$

and have (parametric) solution, for the case where $A \leq 1$:

$$r = R\cos^2(\eta/2) = \tfrac{1}{2}R(1 + \cos\eta) \ , \qquad \tau = \tfrac{1}{2}R\sqrt{\tfrac{R}{2m}}(\eta + \sin\eta) \ ,$$

where R is the maximum value of r.