Stereographic Projection of the Sphere: continued

II. Vector Potentials for Magnetic Monopoles

Now let's use these constructions to consider the magnetic field of a magnetic monopole, were one to exist! It is reasonably obvious what the magnetic field ought to look like, even though no one has ever found such an object. Using the usual freshman-physics sort of approach, where we are looking at the field at some arbitrary point $\vec{r}$, due to a monopole where we take the origin.

$$\vec{B}(\vec{r}) = \frac{g}{r^2}\hat{r} \; . \tag{2.1a}$$

However, to consider all this from the current point of view we want to write out the 1-form $\underset{\sim}{B}$, in 3-dimensional space, in spherical coordinates, which would have the same content:

$$\underset{\sim}{B} = \frac{g}{r^2}\, dr \; . \tag{2.1b}$$

However, it is also true that the magnetic field is a polar vector rather than an axial vector; this is of course evidenced by the fact that (a) it always finds itself in equations involved in a cross product, and (b) when one executes a parity transformation the magnetic field is left invariant, even though most usual vectors—axial vectors—change sign, such as displacement or velocity. Therefore, the correct way to deal with this 1-form is to find the associated (dual) 2-form which incorporates its true physics. In order to determine this we need to use the (Hodge) dual of dr—in 3 spatial dimensions only—which we work out by the usual formulae, using

$$\begin{aligned} &*dr = r^2 \sin\theta\, d\theta \wedge d\varphi \; , \\ \Longrightarrow \quad &*\underset{\sim}{B} = \; + g \sin\theta\, d\theta \wedge d\varphi \; . \end{aligned} \tag{2.2}$$

The point of going this way, of course, is that we all know that that particular one of Maxwell's Equations that is $\nabla \cdot \vec{B} = 0$ allows us to write $\vec{B}$ in terms of a vector potential, i.e., there exists some 3-vector $\vec{A}$ such that $\vec{B} = \nabla \times \vec{A}$. When all this is written in terms of the magnetic 2-form it fits much more easily into the context of Poincaré's Lemma. We recall that Poincaré's

Lemma tells us that, at least in a sufficiently nice neighborhood of some point, any closed (differential) form may be written as the exterior derivative of some form of one lower order:

$$\text{for } *\underset{\sim}{B} \text{ such that } d*\underset{\sim}{B}=0\,,\quad \text{there exists a form, } \underset{\sim}{A},\quad \text{such that } *\underset{\sim}{B}=d\underset{\sim}{A}\ . \tag{2.3}$$

We apply this to our particular $*\underset{\sim}{B}$, we easily see that it is indeed closed:

$$d*\underset{\sim}{B}=d\{g\sin\theta\,d\theta\wedge d\varphi\}=0\ . \tag{2.4a}$$

Therefore, there should be some $\underset{\sim}{A}$ the exterior derivative of which would give us $*\underset{\sim}{B}$. It is reasonably easy to see how to obtain such a 1-form, although it also turns out that it's not particularly unique:

$$\underset{\sim}{A}=\mp g(1\pm\cos\theta)\,d\varphi\quad\Longrightarrow\quad d\underset{\sim}{A}=+g\sin\theta\,d\theta\wedge d\varphi=*\underset{\sim}{B}\ . \tag{2.4b}$$

Still, the fact that it is not unique is not too much trouble, since we already know that there should be gauge freedom in vector potentials, where one can add, to the usual 3-vector formulation, a gradient of some scalar. In our particular approach based on differential forms, it should be obvious that two different 1-forms that differ by a total derivative of a scalar function would give the same exterior derivative. In symbols one says this in the following way, using the fact that the second exterior derivative is always zero:

$$\underset{\sim}{A}=\underset{\sim}{A}'+df\quad\Longrightarrow\quad d\underset{\sim}{A}=d\underset{\sim}{A}'+ddf=d\underset{\sim}{A}'\ . \tag{2.5}$$

Then, for our particular case, we may consider the two different forms of $\underset{\sim}{A}$ as given above as an $\underset{\sim}{A}$ and a $\underset{\sim}{A}'$, and consider the gauge function, f that relates them:

$$\underset{\sim}{A}-\underset{\sim}{A}'=\Big[g(1-\cos\theta)\,d\varphi\Big]-\Big[-g(1+\cos\theta)\,d\varphi\Big]=2g\,d\varphi=d(2g\varphi)\equiv df\ . \tag{2.6}$$

Unfortunately, the quantity φ is NOT a continuous function on the sphere, as already discussed. In particular notice that if, for instance, we pick some circle on the sphere of constant θ, and integrate this 1-form $d\varphi$ all the way around it, continuity alone would require that this integral give zero; i.e., $\int_a^a df = f(a) - f(a) = 0$. Unfortunately the integral of df above gives a non-zero result. As well, the integral of either version of $\underset{\sim}{A}$ would also give a non-zero result! This tells us that these scalars and 1-forms are NOT properly defined.

On the other hand in case you think that the problems here are just because we are looking at in terms of these differential forms, instead of in terms of the “real” 3-vectors that one “should” use, let us go back and look at it that way. In this approach we need a 3-vector $\vec{A}$ such that

$$\vec{B} = \frac{g}{r^2}\hat{r} = \nabla \times \vec{A} \quad \Longrightarrow \quad \vec{A} = \pm\frac{g}{r}\frac{\sin\theta}{1 \pm \cos\theta}\hat{\varphi}\ . \tag{2.7a}$$

It is clear that each of these two versions for the vector potential has a very serious difficulty at either the North pole, where $\cos\theta = 1$, or the South pole, where $\cos\theta = -1$. In this form of the vector potentials, one can again ask about the gauge function relating the two distinct choices:

$$\vec{A} - \vec{A}' = \frac{g\sin\theta}{r}\left\{\frac{1}{1+\cos\theta} - \frac{-1}{1-\sin\theta}\right\}\hat{\varphi} = \frac{2g}{r\sin\theta}\hat{\varphi} = \nabla(2g\varphi)\ . \tag{2.7b}$$

As before there is this serious problem that φ is not a well-defined, or even everywhere continuous, function.

Having discussed the fact that all this does not really make quite good sense, I will point out that this bothered Dirac a lot, and he concluded—using the 3-vector approach described just above—that one had to insist on restrictions on physical quantities which would negate the problems here. In particular he chose that version of the vector potential which was well-behaved at the North pole, so that it had a problem at the South pole. He therefore insisted that the monopole had to have some sort of “string” attached to it that extended to infinity along the South pole, which required him to insist that integrals of flux around the South pole were integer multiples of 2π—since they occurred in the argument of an exponential.

On the other hand, let us try to see more what he might ought to have thought by using our properly defined N-stereographic and/or S-stereographic coordinates on the sphere to re-write all of this discussion in terms of proper quantities. We begin by considering the N-stereographic coordinates, and differentiating the mapping in Eqs. (1.1):

$$\begin{aligned} dx_1 &= d\frac{\xi/a}{1-\zeta/a} &&= -\frac{\cos\varphi\, d\theta + \sin\theta\, \sin\varphi\, d\varphi}{1-\cos\theta}\ , \\ dy_2 &= d\frac{\eta/a}{1-\zeta/a} &&= -\frac{\sin\varphi\, d\theta - \sin\theta\, \cos\varphi\, d\varphi}{1-\cos\theta}\ , \\ \Longrightarrow \quad dx_1 \wedge dy_2 &= &&-\frac{\sin\theta}{(1-\cos\theta)^2} d\theta \wedge d\varphi\ . \end{aligned} \tag{2.8a}$$

On the other hand, it is easy to determine that

$$1 + x_1^2 + y_1^2 = 1 + \left(\frac{\sin\theta}{1-\cos\theta}\right)^2 = \frac{2}{1-\cos\theta}\ . \tag{2.8b}$$

Therefore we may determine that the presentation of the magnetic 2-form in these (smooth, i.e., approved) N-pole coordinates is given by

$$g \sin\theta\, d\theta \wedge d\varphi = *\underset{\sim}{B} = -4g\frac{dx_1 \wedge dy_1}{[1 + x_1^2 + y_1^2]^2}\ , \tag{2.9a}$$

where we can see that the expression is well-behaved, differentiable, etc., in all of U_1, i.e., any open region that excludes the N pole. It is also still obvious that this 2-form is closed. (Actually all 2-forms on a 2-dimensional manifold are closed, of course.) One can go through the same calculation in U_2, which gives us the presentation in those coordinates:

$$*\underset{\sim}{B} = +4g\frac{dx_2 \wedge dy_2}{[1 + x_2^2 + y_2^2]^2}\ , \tag{2.9b}$$

which is again valid in any open region that avoids the S pole. On the other hand, more important is the fact that if we restrict ourselves to the region of overlap of their coordinate charts, these two are equivalent; i.e., in $U_1 \cap U_2$, we may use the transition functions to change the expression of one into the other coordinates, which in fact gives us the other expression. That is, since $U_1 \cup U_2$ is the entire sphere, we may use the two expressions in the two coordinate patches to define a *globally well-defined* 2-form everywhere on the sphere.

Since $*\underset{\sim}{B}$ is globally defined, and also closed, we may look for a 1-form which is its potential. While it may not be completely obvious to you the N-coordinate answer is

$$\underset{\sim}{\rho} \equiv -\frac{g}{2}\left\{\frac{x_1\,dy_1 - y_1\,dx_1}{1 + x_1^2 + y_1^2}\right\} \ , \tag{2.10}$$

which is well-defined in all of U_1. One could have derived this result by, again, performing a transformation from the poorly-defined one given above in terms of θ and φ; after all, that one is acceptable in a region that includes neither pole. However, I will prove that it is true by simply taking its exterior derivative, to obtain our desired 2-form:

$$\begin{aligned} d\underset{\sim}{\rho} = & -\frac{g}{2}\left\{\frac{2dx_1\wedge dy_1}{1 + x_1^2 + y_1^2} - 2(x_1dx_1 + y_1dy_1)\wedge\frac{x_1dy_1 - y_1dx_1}{[1 + x_1^2 + y_1^2]^2}\right\} \\ & = -g\frac{dx_1\wedge dy_1}{[1 + x_1^2 + y_1^2]^2} = *\underset{\sim}{B}\Big|_{U_1} \ . \end{aligned} \tag{2.11}$$

On the other hand, if we were to work in the S-coordinates, the appropriate 1-form is

$$\begin{aligned} \underset{\sim}{\sigma} \equiv & \ +\frac{g}{2}\frac{x_2dy_2 - y_2dx_2}{1 + x_2^2 + y_2^2} \ , \\ \Rightarrow \quad d\underset{\sim}{\sigma} = & \ +g\frac{dx_2\wedge dy_2}{[1 + x_2^2 + y_2^2]^2} = *\underset{\sim}{B}\Big|_{U_2} \ . \end{aligned} \tag{2.12}$$

Recall that the magnetic field 2-form has a different coordinate expression in the two different coordinate charts, U_1 and U_2, but, in the overlap zone of those two charts—the entire sphere except for small neighborhoods of the two poles—the two expressions are equivalent. On the other hand, this is not true for the two different vector potential 1-forms shown just above. To see this we transform $\underset{\sim}{\rho}$ into the S-coordinates, again well-defined in the region of overlap. A little algebra gives us

$$\begin{aligned} -\frac{g}{2}\left\{\frac{x_1\,dy_1 - y_1\,dx_1}{1 + x_1^2 + y_1^2}\right\} & = \underset{\sim}{\rho} = -\left\{\frac{g/2}{(x_2/2)^2 + (y_2/2)^2}\left(\frac{x_2\,dy_2 - y_2\,dx_2}{1 + x_2^2 + y_2^2}\right)\right\} \\ & = \underset{\sim}{\sigma} - \frac{g}{2}\frac{x_2dy_2 - y_2dx_2}{x_2^2 + y_2^2} \equiv \underset{\sim}{\sigma} + \underset{\sim}{\tau} \ . \end{aligned} \tag{2.13}$$

It is clear that the 1-form τ is not zero; moreover, it becomes singular at **both** the N- and the S-poles. Nonetheless, it is closed as we see by computing its exterior derivative:

$$d\tau = -g\frac{dx_2\wedge dy_2}{x_2^2 + y_2^2} + g\frac{(x_2dx_2 + y_2dy_2)\wedge(x_2dy_2 - y_2dx_2)}{(x_2^2 + y_2^2)^2} = 0 \ . \tag{2.14}$$

Therefore, as we already knew, $d\rho$ and $d\sigma$ are the same, wherever they are both defined. The conclusion is the following:

> **The 2-form $*\underset{\approx}{B}$ is defined over the entire sphere, albeit of course that its coordinate presentation is different depending on which coordinate chart one uses, and that there is no single coordinate chart valid over the entire sphere. Moreover, it is everywhere closed, so that in any small, arc-connected (or star-shaped) neighborhood there is a potential 1-form, i.e., one such that its exterior derivative is the original 2-form.**
>
> **However, there does NOT exist a single 1-form which has the property that it constitutes a potential, as defined above. In different portions of the sphere the 1-form in question is not equivalent to all the other 1-forms appropriate in other coordinate charts. That such a 2-form exists is a consequence of the topology of the sphere on which things are being defined.**

I lastly simply point out that the 1-forms ρ and σ are indeed the two forms shown earlier for $\vec{A}$ and $\vec{A}'$, in Eqs. (2.7a,b) and their difference, τ is exactly the same as the difference, proportional to $d\varphi$ shown in Eq. (2.7b).