

Summary of Formulae for the Robertson-Walker Metric

1. A (local) **diffeomorphism** is a mapping of some (open) neighborhood of a manifold M into itself that is invertible and both it and its inverse are differentiable.
2. A (local) **isometry** is a (local) diffeomorphism that also preserves the metric structure of the manifold within that neighborhood.
3. A space is **homogeneous** in some neighborhood if any point, $P \in U \subseteq M$ can be mapped by an isometry into any other point, $Q \in U \subseteq M$.
4. A space is **isotropic** in some neighborhood, $U \subseteq M$, if at every point $P \in U$, there exists an isometry that leaves the point P fixed, but which maps any basis vector in \mathcal{T}_P into any other such basis vector.

Note: Isotropy implies the existence of the maximal number of independent isometries possible.

Note: **Isotropy about every point implies homogeneity**

5. A **spatially homogeneous spacetime** is one which is *foliated by* a one-parameter family, Σ_t , of spacelike hypersurfaces, each one of which is homogeneous.
6. A **spatially isotropic spacetime** is one which contains a congruence of timelike curves, with tangent \tilde{u} , i.e., (co-moving) observers, that fill the spacetime and are such that for each point $P \in M$ the spacelike-directed curves orthogonal to \tilde{u}_P , that correspond locally to directions to other such observers, fill a (locally) spacelike surface that is an isotropic 3-surface.

Presuming that this isotropicity is defined everywhere, then these spacelike surfaces are in fact just the spacelike hypersurfaces Σ_t discussed above. By adjusting their origins and scales, this congruence of observers possesses a uniform proper time, which we refer to as “cosmic time,” and we use the variable t to refer to it.

7. The 3 metric(s) discovered, separately, by Friedmann, with Lemaitre involved, and again by Robertson and Walker describe all possible 4-dimensional spacetimes which are spatially

isotropic (and homogeneous). They therefore require the existence, as above, of a foliation by spacelike hypersurfaces $\{\Sigma_t \mid t = -\infty \dots +\infty\}$. On these surfaces, we denote the 3-dimensional, spatial metric there by $\sigma \equiv \sigma_{ij} dx^i dx^j$, and note that there are only 3 possible such metrics, i.e., which are everywhere homogeneous and isotropic, all of which of course have constant curvature:

$$\begin{aligned} \text{positive curvature, a 3-sphere:} & \quad \sigma = d\psi^2 + \sin^2 \psi (d\theta^2 + \sin^2 \theta d\varphi^2) , \\ \text{zero curvature, flat Euclidean space:} & \quad \sigma = d\psi^2 + \psi^2 (d\theta^2 + \sin^2 \theta d\varphi^2) , \\ \text{negative curvature, a 3-hyperboloid:} & \quad \sigma = d\psi^2 + \sinh^2 \psi (d\theta^2 + \sin^2 \theta d\varphi^2) . \end{aligned}$$

It is of course more usual to use the symbol r instead of ψ when the space is flat; nonetheless, this generates a uniformity in the appearance which has some value. Actually, let's do that, but let's do it somewhat more generically: We define a symbol r for each of these three allowed spaces:

$$d\psi \equiv \begin{cases} \frac{dr}{\sqrt{1-r^2}} , & \text{for the 3-sphere,} \\ dr , & \text{for flat 3-space,} \\ \frac{dr}{\sqrt{1+r^2}} , & \text{for the 3-hyperboloid,} \end{cases} \implies r \equiv f(\psi) = \begin{cases} \sin \psi , & \text{for the 3-sphere,} \\ \psi , & \text{for flat 3-space,} \\ \sinh \psi , & \text{for the 3-hyperboloid.} \end{cases}$$

We may then insert the one or the other of these metrics into the notion above, of foliating the spacetime by homogeneous, isotropic 3-surfaces, and having normal, timelike geodesics for them. We use the symbol t for the associated ‘‘cosmic time,’’ i.e., the affine variable along those timelike geodesics. As well, we suppose the possibility that there is a time-dependent scaling, with dimensions of length, that would multiply each 3-metric, which allow us to write the most general such metric in the following form:

$$\mathbf{g} = -dt^2 + R^2(t) \sigma = -dt^2 + R^2(t) \left\{ \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right\} , \quad k = +1, 0, -1 . \quad (1)$$

It is, however, also useful to introduce a different time variable, often referred to as ‘‘arc-time,’’ which demonstrates explicitly that our metric is conformally equivalent to a simpler one. We define $d\eta \equiv dt/R(t)$, and may write

$$\mathbf{g} = R^2(t) \{ -d\eta^2 + d\psi^2 + f^2(\psi) d\Omega^2 \} . \quad (1a)$$

We define an orthonormal tetrad for the Robertson-Walker metric in the obvious way,

$$\varpi^r \equiv R d\psi = R \frac{dr}{\sqrt{1-kr^2}}, \quad \varpi^\theta \equiv R r d\theta, \quad \varpi^\varphi \equiv R r \sin\theta d\varphi, \quad \varpi^t = dt. \quad (2)$$

Then we may immediately calculate the connection 1-forms,

$$\begin{aligned} \mathfrak{L}_{r\theta} &= -\frac{\sqrt{1-kr^2}}{rR} \varpi^\theta, & \mathfrak{L}_{r\varphi} &= -\frac{\sqrt{1-kr^2}}{rR} \varpi^\varphi, & \mathfrak{L}_{\theta\varphi} &= -\frac{\cot\theta}{rR} \varpi^\varphi, \\ \mathfrak{L}_{rt} &= \frac{\dot{R}}{R} \varpi^r, & \mathfrak{L}_{\theta t} &= \frac{\dot{R}}{R} \varpi^\theta, & \mathfrak{L}_{\varphi t} &= \frac{\dot{R}}{R} \varpi^\varphi, \end{aligned} \quad (3)$$

and the curvature 2-forms,

$$\begin{aligned} \mathfrak{Q}_{r\theta} &= \frac{\dot{R}^2 + k}{R^2} \varpi^r \wedge \varpi^\theta, & \mathfrak{Q}_{r\varphi} &= \frac{\dot{R}^2 + k}{R^2} \varpi^r \wedge \varpi^\varphi, & \mathfrak{Q}_{\theta\varphi} &= \frac{\dot{R}^2 + k}{R^2} \varpi^\theta \wedge \varpi^\varphi, \\ \mathfrak{Q}_{rt} &= -\frac{\ddot{R}}{R} \varpi^r \wedge \varpi^t, & \mathfrak{Q}_{\theta t} &= -\frac{\ddot{R}}{R} \varpi^\theta \wedge \varpi^t, & \mathfrak{Q}_{\varphi t} &= -\frac{\ddot{R}}{R} \varpi^\varphi \wedge \varpi^t, \end{aligned} \quad (4)$$

or one can describe the curvatures by the very simple forms:

$$R_{r\theta r\theta} = R_{r\varphi r\varphi} = R_{\theta\varphi\theta\varphi} = \frac{\dot{R}^2 + k}{R^2}, \quad R_{rt rt} = R_{\theta t \theta t} = R_{\varphi t \varphi t} = -\frac{\ddot{R}}{R}. \quad (5)$$

From this one calculates immediately the conformal and Einstein parts of the curvature:

$$C_{\mu\nu\lambda\eta} = 0, \quad (6)$$

$$\mathcal{G}_{rr} = \mathcal{G}_{\theta\theta} = \mathcal{G}_{\varphi\varphi} = -\left\{ 2\frac{\ddot{R}}{R} + \frac{\dot{R}^2 + k}{R^2} \right\}, \quad \mathcal{G}_{tt} = 3\frac{\dot{R}^2 + k}{R^2}, \quad \mathcal{G}_{\mu\nu} = 0, \quad \mu \neq \nu$$

Given that we want to solve Einstein's equations, it is worth noting that the structure we have is that in this frame,

- a.) Einstein's tensor is diagonal,
- b.) all 3 of the diagonal, spatial components of Einstein's tensor are equal, and
- c.) the diagonal, temporal component is different.

These are exactly the distinguishing characteristics of the stress-energy tensor for a perfect fluid, characterized by its pressure, P , and its rest-energy density, ρ ; therefore, if we wanted to set Einstein's tensor equal to some stress-energy tensor for some sort of matter, it would

actually have to correspond to that for a perfect fluid. The two Einstein equations, then, setting $\mathcal{G}_{\mu\nu} = 8\pi T_{\mu\nu} + \Lambda g_{\mu\nu}$, are given by

$$\begin{aligned} 8\pi P + \Lambda &= - \left(2\frac{\ddot{R}}{R} + \frac{\dot{R}^2 + k}{R^2} \right) \\ 8\pi \rho - \Lambda &= 3\frac{\dot{R}^2 + k}{R^2}, \end{aligned} \tag{7}$$

so that we could “identify” those parts of the curvature that act like pressure, and like energy density.

These equations may be re-written in various useful ways. One approach is

$$\begin{aligned} \frac{\ddot{R}}{R} &= -\frac{4\pi}{3}(\rho + 3P) - \frac{\Lambda}{3}, \\ H^2 \equiv \left(\frac{\dot{R}}{R}\right)^2 &= \frac{8\pi}{3}\rho - \frac{k}{R^2} - \frac{\Lambda}{3}. \end{aligned} \tag{7a}$$

An approach that has **recently** become common is to claim that one should re-define Λ and k so that they look like densities. For instance, first considering Λ , we may define $\rho_\Lambda \equiv -\Lambda/8\pi = -P_\Lambda$. (As shown below, in Eq. (8), if $\rho + P = 0$, then ρ is a constant, consistent with the fact that Λ is a constant.) This construction not only takes the (Friedmann) equation for the Hubble parameter, H , and puts it in a form “easier to remember,” but also does the same thing for the acceleration equation:

$$\begin{aligned} \frac{\ddot{R}}{R} &= -\frac{4\pi}{3}[(\rho_{\text{tr}} + 3P_{\text{tr}}) + (\rho_\Lambda + 3P_\Lambda)], \\ \left(\frac{\dot{R}}{R}\right)^2 &= \frac{8\pi}{3}(\rho_{\text{tr}} + \rho_\Lambda) - k/R^2, \end{aligned} \tag{7b}$$

where we have used the subscript *tr* to indicate all the true matter, as opposed to the ones coming from the cosmological constant.

We may also divide the equation for the Hubble constant by itself, to create an equation for a collection of dimensionless quantities all of which must add to the value +1:

$$1 = \frac{\rho_{\text{tr}}}{\rho_c} - \frac{k}{\dot{R}^2} - \frac{\Lambda}{H^2} \equiv \Omega_{\text{tr}} + \Omega_k + \Omega_\Lambda, \quad \rho_c \equiv H^2/(8\pi/3). \tag{7c}$$

Separately, we can apply the Bianchi identity to the problem. We know that it says that the divergence of each side is equal to zero. Straightforward calculation shows that the 4 components of $T^{\mu\nu}{}_{;\nu} = 0$ amount to the three reasonably trivial equations $\nabla P = 0$, which just say that the pressure does not change in space, and then the fourth one

$$3(\rho + P)\frac{\dot{R}}{R} + \dot{\rho} = 0 ,$$

or

(8)

$$\frac{d}{dt}(\rho R^3) + P \frac{d}{dt}R^3 = 0 \quad \Leftrightarrow \quad 0 = \frac{d}{dt}E + P \frac{d}{dt}V .$$

This equation may be re-written in the following useful form as well. Were we to know the equation of state for this matter, i.e., the relationship $P = P(\rho)$, then we could presumably solve this equation, remove P , say, from the equations above, and have a simpler problem to deal with. Therefore, it is customary to divide the total energy density, ρ into contributions from matter, ρ_m and from radiation, ρ_r . The reason for doing this is that **we assume** that we can treat the two separately, i.e., that they do not interact, at least not in the last more than 10 billion years, and, furthermore, that the equations of state allow us to say that $P_m = 0$ while $P_r = \rho_r/3$. This allows to make statements about the dependence on $R(t)$ of these terms:

$$\begin{aligned} \rho_m R^3 &= \text{a constant} , & \rho_r R^4 &= \text{a constant} \\ \Rightarrow \rho(t) &= \rho_m(t) + \rho_r(t) = \rho_{m0} \left(\frac{R_0}{R(t)}\right)^3 + \rho_{r0} \left(\frac{R_0}{R(t)}\right)^4 , & P(t) &= \frac{1}{3} \rho_{r0} \left(\frac{R_0}{R(t)}\right)^4 \\ & \left(\frac{\dot{R}}{R}\right)^2 + \frac{k}{R^2} - \frac{8\pi}{3} \left\{ \rho_{m0} \left(\frac{R_0}{R}\right)^3 + \rho_{r0} \left(\frac{R_0}{R}\right)^4 \right\} & &= -\frac{1}{3}\Lambda \end{aligned} \quad (9)$$

This shows us the dependence of H^2 on terms that depend on $1/R^n$, where n has the values 0, 2, 3, and 4. It is possible, for instance, to think of this as an equation of the standard form that one takes in ordinary classical mechanics, where the $H^2 = (\dot{R}/R)^2$ term is a kinetic energy, the others are potential energies and the cosmological constant acts like a (constant) energy term.

On the other hand, it is often better not to look at this equation in this form, but, rather to use the ‘‘arc time,’’ η , as a variable instead. From Eq. (1a) we have the relationship

$$\frac{d\eta}{dt} = \frac{1}{R} \quad \Rightarrow \quad dR/d\eta = R\dot{R} = R^2 \frac{\dot{R}}{R} , \quad (10a)$$

so that multiplication of the so-called Friedmann equation, Eq. (9) above, by a factor of R^4 allows it to be written in the following form, which no longer has difficulties at $R = 0$, and is therefore much more useful, for instance, for computerized calculations, $dt/d\eta = R$ being the other member of a pair of calculations to determine R and t as functions of η :

$$\left(\frac{dR}{d\eta}\right)^2 = +B^2 + AR - kR^2 - \frac{1}{3}\Lambda R^4, \quad A \equiv 2\frac{4\pi}{3}\rho_{m_0}R_0^3, \quad B^2 \equiv 2\frac{4\pi}{3}\rho_{r_0}R_0^4, \quad (10b)$$

where A and B^2 are constants, with the quantities with subscript 0 being evaluated at some particular time, presumably now.

In the case that the cosmological constant, Λ , is non-zero, this equation involves elliptic functions and/or numerical calculations. On the other hand, for the case with $\Lambda = 0$, then it may be integrated fairly easily, giving the following results:

$$\begin{aligned} k = +1 &\Rightarrow \begin{cases} R = A(1 - \cos \eta) + B \sin \eta, \\ t = A(\eta - \sin \eta) + B(1 - \cos \eta), \\ \implies R(\eta = \pi) = 2A, \text{ max. } R \text{ if } A \gg B. \end{cases} \\ k = -1 &\Rightarrow \begin{cases} R = A(\cosh \eta - 1) + B \sinh \eta, \\ t = A(\sinh \eta - \eta) + B(\cosh \eta - 1), \\ \implies R(t) \approx t \text{ for very long times.} \end{cases} \\ k = 0 &\Rightarrow \begin{cases} R = B\eta + \frac{1}{2}A\eta^2, \\ t = \frac{1}{2}B\eta^2 + \frac{1}{6}A\eta^3. \end{cases} \end{aligned} \quad (11)$$

In all 3 cases we have the approximate behavior at very early times, so long as $B \neq 0$:

$$R(t) \approx \sqrt{2Bt} + O(t). \quad (12a)$$

If, for some reason, we would have $B = 0$, then there are other possibilities:

$$\begin{aligned} B = 0 = \Lambda, \quad A \neq 0 &\implies R \propto t^{2/3} + O(t), \\ k = B = 0 = A, \quad \Lambda \neq 0 &\implies R = R_0 e^{\pm\sqrt{-\Lambda/3}t}. \end{aligned} \quad (12b)$$

When Λ is not zero, Maple can nicely handle computerized integrations of the pair of equations. Just below the listing for this handout, on the main class webpage, there are references to two Maple files, presented as html-documents, that perform two such calculations. One of them is for $k = +1$ with constants chosen so that the universe collapses back to its initial singularity again; the other is for $k = -1$ and constants chosen so that the universe expands forever.

Comments on the Geodesic Equations

The equations for a *timelike* geodesic, with tangent vector $\tilde{u} = u^{\hat{\mu}} \tilde{e}_{\hat{\mu}}$ are the following:

$$\begin{aligned}
(u^{\hat{r}})' - \frac{\sqrt{1-kr^2}}{rR} [(u^{\hat{\theta}})^2 + (u^{\hat{\phi}})^2] + \frac{\dot{R}}{R} u^{\hat{r}} u^{\hat{t}} &= 0, \Rightarrow \{R(t)u^{\hat{r}}\}^2 + \left(\frac{L}{r}\right)^2 = Q^2, \text{ a constant,} \\
(u^{\hat{\theta}})' + \frac{\sqrt{1-kr^2}}{rR} u^{\hat{r}} u^{\hat{\theta}} - \frac{\cot\theta}{rR} (u^{\hat{\phi}})^2 + \frac{\dot{R}}{R} u^{\hat{\theta}} u^{\hat{t}} &= 0, \\
&\Rightarrow \{R(t)r u^{\hat{\theta}}\}^2 + \left(\frac{C}{\sin\theta}\right)^2 = L^2, \text{ a constant,} \\
(u^{\hat{\phi}})' + \frac{\sqrt{1-kr^2}}{rR} u^{\hat{r}} u^{\hat{\phi}} + \frac{\cot\theta}{rR} u^{\hat{\theta}} u^{\hat{\phi}} + \frac{\dot{R}}{R} u^{\hat{\phi}} u^{\hat{t}} &= 0, \Rightarrow \{R(t)r \sin\theta u^{\hat{\phi}}\} = C, \text{ a constant,} \\
(u^{\hat{t}})' + \frac{\dot{R}}{R} [(u^{\hat{r}})^2 + (u^{\hat{\theta}})^2 + (u^{\hat{\phi}})^2] &= 0, \Rightarrow (u^{\hat{t}})^2 = \left(\frac{dt}{d\tau}\right)^2 = 1 + \frac{Q^2}{R^2(t)},
\end{aligned} \tag{13}$$

where we recall that

$$\begin{aligned}
\tilde{u} &\implies \left(\frac{R}{\sqrt{1-kr^2}} r', rR\theta', rR\sin\theta\varphi', t' \right), \\
-1 &= (\tilde{u})^2 = (u^{\hat{r}})^2 + (u^{\hat{\theta}})^2 + (u^{\hat{\phi}})^2 - (u^{\hat{t}})^2.
\end{aligned} \tag{14}$$