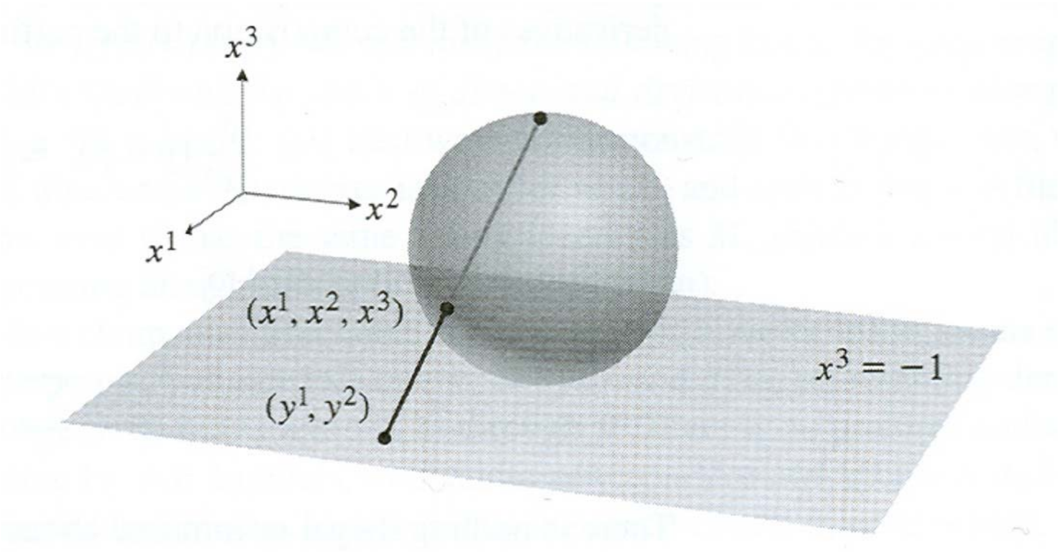
Stereographic Projection of the Sphere

I. Acceptable Charts for the Sphere

Stereographic Projection creates a map of the sphere onto a plane on which it sits. We place the sphere so that its South pole sits squarely on an infinite plane, which has $\{x, y\}$ coordinates with its (2-dimensional) origin at the point of contact. Then, we create a mapping between the sphere and the plane by drawing a straight line beginning at the North pole, passing through some point on the sphere, so that it ends on the plane. This constitutes a clearly-defined one-to-one mapping of all points on the sphere onto the plane, except, of course, for the North pole itself. See the figure below.

The figure is drawn in a 3-dimensional space. I will take that space to have coordinates (ξ, η, ξ), with their origin of coordinates at the center of the sphere; this is, perhaps unfortunately, different from the (x^1, x^2, x^3) coordinates the figure suggests. However, it will allow us to use considerably fewer superscripts and subscripts. Therefore from the point of view that sees the sphere as embedded within the 3-dimensional space, the arbitrary point on the sphere, through which our line passes, may be characterized as a point P, which has embedded



coordinates (ξ, η, ζ) subject to the constraint that they lie on a sphere of radius a, i.e., such that $\xi^2 + \eta^2 + \zeta^2 = a^2$.

We may also associate with this point the usual spherical coordinates,

$$\left.\begin{aligned} \theta = &\ \arccos(\zeta/a)\ , \\ \varphi = &\ \arctan(\eta/\xi)\ , \end{aligned}\right\} \quad \Longleftrightarrow \quad \left\{\begin{aligned} \xi &= a\sin\theta\,\cos\varphi\ , \\ \eta &= a\sin\theta\,\sin\varphi\ , \\ \zeta &= a\cos\theta\ . \end{aligned}\right.$$

It is clear from these definitions of θ and φ that they are not good coordinates at either of the north and south poles: (a) the value of φ is not defined when $\sin\theta = 0$, i.e., at the poles, and (b) the mapping is not invertible in a neighborhood of either pole.

The line then strikes the plane at $z = -a$ and some coordinates (x, y) on that plane. [In the figure, this pair of coordinates is referred to as (y^1, y^2), but, again, I feel these names will be more convenient.] The values of x and y at this point of striking will be this choice of (acceptable) coordinates for the point P on the sphere through which the line passed. (Notice that this is just an "unwrapping" and "stretching" of the sphere, minus the North pole, onto a plane. This mapping is then something like the usual Mercator projection. It is the one commonly used in complex variables, where it is useful to go in the other direction, i.e., to "roll up" the complex plane onto a sphere, so that the 'point at infinity' is more easily "seen.")

It is quite useful to create this mapping by drawing this straight line explicitly in the 3-dimensional space. We propose to create it as a parametrized curve, with parameter t, which varies from 0 to 1, as the line moves from the North pole to the (southern) plane. In that case the coordinate z along the line varies from $z = 1$, when $t = 0$, at the North pole, to $z = -1$, when $t = 1$, on that plane. As the parametrized equations for a straight line must be linear in the parameter, this immediately gives us the form for $z = z(t) = a(1 - 2t)$. Again, since the equations for $x(t)$ and $y(t)$ must also be linear, and since they should have value 0 at the North pole, where $t = 0$, then $x(t)$ and $y(t)$ must simply be proportional to t. However, the line which begins at the North pole, headed downward, will surely intersect the sphere again, at some arbitrary point on the sphere, which we have already labelled with symbols (ξ, η, ζ), subject to $\xi^2+\eta^2+\zeta^2=a^2$. Therefore, at that point, P, we have t_P given by $\zeta = a(1-2t_P)$, or $2t_P = 1-\zeta/a$. As the equations for $x(t)$ and $y(t)$ are simply proportional to t, and must have the values ξ and η at the point P, it follows that they are simply $x(t)=\xi(t/t_P)$ and $y(t)=\eta(t/t_P)$. Inserting the value of t_P just calculated, we have the desired parametric equations for this line in the 3-dimensional enveloping space:

$$\left.\begin{aligned} \frac{z}{a} &= 1-2t\ , \\ \frac{x}{a} &= \frac{2\xi/a}{1-\zeta/a}t = \frac{2\sin\theta\cos\varphi}{1-\cos\theta}t = 2t\cot(\theta/2)\,\cos\varphi\ , \\ \frac{y}{a} &= \frac{2\eta/a}{1-\zeta/a}t = \frac{2\sin\theta\sin\varphi}{1-\cos\theta}t = 2t\cot(\theta/2)\,\sin\varphi\ , \end{aligned}\right\} \qquad \begin{aligned} & t\in[0,1]\ , \\ \\ & t_P = \frac{1-\zeta}{2} = \sin^2(\theta/2)\le 1\ . \end{aligned}$$

Since the value of t at which the line hits the sphere is independent of φ, horizontal circles on the sphere, i.e., lines of constant θ, are mapped into circles on the plane; the sphere's equator is mapped onto the circle in the plane of radius $2a$. The South pole maps into the plane's origin, i.e., $(0,0)$. There is of course no point in the plane which receives the mapping of the North pole!

The desired coordinate chart for the sphere, i.e., a mapping of some appropriate open subset of that sphere onto the x,y-plane, i.e., onto $\mathbb{R}^2$, is effected by these formulae when $t=1$. For an arbitrary point on the sphere, P, we may think of it in terms of the usual spherical angles, θ and φ; however, as we have already seen they do not constitute a good, everywhere continuous and differentiable, mapping of the sphere. In principle we should now choose this mapping of those points, P, given by the above equations for x and y, evaluated at $t=1$. However, any constant, multiplicative scaling will do as well; because of the factors of 2 and a in those equations it is **much more convenient** to define our chart by dividing those quantities by $2a$, which is simply a (smooth) re-scaling of the lower plane which is obviously one-to-one. (Notice that if you don't like that particular 'scaling' approach, you can achieve the same effect by simply choosing our sphere to have radius $a=1/2$.) Therefore, we now define the (N-pole) chart via $x_1(P)=x/2a$ and $y_1(P)=y/2a$. We also note that these coordinates have a nice simple form when expressed in terms of a complex variable, $Z(P)\equiv x_1(P)+iy_1(P)=[x/a+iy/a]/2$:

for the point P on the sphere, identified in terms of ξ, η, ζ or θ, φ ,

$$x_1 = \frac{\xi/a}{1-\zeta/a} = \frac{\sin\theta}{1-\cos\theta}\cos\varphi\ , \qquad y_1 = \frac{\eta/a}{1-\zeta/a} = \frac{\sin\theta}{1-\cos\theta}\sin\varphi\ ,$$

$$Z_1 \equiv x_1 + iy_1 = \frac{\sin\theta}{1-\cos\theta}e^{i\varphi} = \cot(\theta/2)\,e^{i\varphi} \quad , \tag{1.1}$$

along with the inverse mapping,

$$\cot(\theta/2) = |Z_1| = \sqrt{{x_1}^2 + {y_1}^2} \quad , \quad \tan\varphi = \frac{y_1}{x_1} = \tan(phase(Z_1)) \quad ,$$

and we have used the subscripts 1 on the coordinates because these correspond to some choice of neighborhood which we will call U_1, which is any **open subset** of the sphere that excludes the North pole. The mapping is well-defined, invertible, continuous, etc., everywhere in U_1.

However, since it does not include the entire sphere, we **must** choose a second neighborhood U_2. Therefore, we do the same thing with another plane, which for simplicity we choose to be tangent to the North pole, and use lines of projection that begin at the South pole. Following through the same algebra as above, we find that these coordinates, $(x_2(P), y_2(P))$ are given by

$$Z_2 \equiv x_2 + iy_2 = \frac{\sin\theta}{1+\cos\theta}e^{i\varphi} = 2\tan(\theta/2)\,e^{i\varphi} \quad ,$$

along with the inverse mapping, (1.2)

$$\tan(\theta/2) = |Z_2| = \sqrt{{x_2}^2 + {y_2}^2} \quad , \quad \tan\varphi = \frac{y_2}{x_2} \quad .$$

In the overlap of regions U_1 and U_2, we would like to know how the two choices of coordinates relate to one another; the functions $x_1^i(x_2^a)$ and their inverse functions are referred to as the transition functions for this pair of charts. We insist that they must be differentiable, at least. By noting the following relationship,

$$x_1^2 + y_1^2 = \cot^2(\theta/2) = 1/\tan^2(\theta/2) = 1/(x_2^2 + y_2^2)\ ,$$

it is easy to show that the transition functions may be written in the form:

$$x_2 = \frac{x_1}{{x_1}^2 + {y_1}^2} \quad , \quad y_2 = \frac{y_1}{{x_1}^2 + {y_1}^2}\ , \qquad x_1 = \frac{x_2}{x_2^2 + y_2^2} \quad , \quad y_1 = \frac{y_2}{x_2^2 + y_2^2}\ , \tag{1.3a}$$

which are smooth everywhere in their region of definition, i.e., in the overlap of U_1 and U_2, which of course excludes both poles. Using the complex variable approach described above, the transition functions may be described much more elegantly simply as

$$Z_2 = \frac{1}{\overline{Z_1}} = \frac{Z_1}{|Z_1|^2} \quad , \tag{1.3b}$$

where the "overbar" denotes the complex conjugate.