Physics 570

Spinors Analysis as a Useful Tool in Studies of Spacetime Manifolds

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Prologue

There is a very important correspondence between 4-vectors and complex-valued 2-vectors, usually referred to as spinors. The principal reason for this is the homomorphism between the groups $SO(3,1)$ and $SL(2,\mathcal{C})$, which act on Minkowski 4-vectors and spinors, respectively. The purpose of this summary is to keep careful track of both the geometry and the formalism that underlies this correspondence. It begins with a relatively low-level introduction to why spinors should be expected to be relevant to questions in 4-dimensional geometry, with signature $\{+1,+1,+1,-1\}$, and then begins over again with a more serious discussion of the 4 different, related spinor spaces that one would like to attach to each point of an arbitrary manifold of this type. It pursues at considerable depth the questions surrounding the relationship of Lorentz transformations to (matrix) transformations on the spinor spaces, and on their tensor products. It then defines the prolongation of a covariant derivative to these spaces, and uses that to understand how symmetric (tensor) products of spinors can display the important properties of connections and curvatures in a much simpler way than can be performed with tensors themselves.

0. Initial Relation to Spacetime

For this pre-introductory, or motivational section, we do not distinguish—at least not very well or accurately—between coordinates on the underlying manifold and the components of vectors within the overlying vector spaces. This "sloppiness" will be corrected, at least somewhat, in the further sections. To simplify things at this moment, we will "confuse" the components of a 4-vector and the 4 (Minkowski) coordinates of the underlying manifold, making this more straight in the next section. We may think of this vector as having components which are presented as the elements of a real, 4×1 (column) matrix, or, alternatively, via the 4 elements of a real 1×4 (row) matrix, all with respect to some choice of basis in the vector space. Yet a different alternative is to use the 4 real degrees of freedom embodied in the elements of a complex-valued, 2×2, Hermitian matrix. For instance, taking our vector as simply $\widetilde{x}$, with components which are the coordinates, then the following is a presentation as such a matrix:

$$(x)^\mu \equiv x^\mu = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \quad \Longleftarrow \quad \widetilde{x} \Longrightarrow \quad \begin{pmatrix} z-t & x-iy \\ x+iy & -z-t \end{pmatrix} \equiv \mathbf{X} \, . \tag{0.1}$$

As this is quite a new way to be shown a presentation of these 4 degrees of freedom, we anticipate that this method will open up new ways of looking at them. We begin a search for such new ways by first looking at the (squared) "length," of our 4-vector. As it is given in Cartesian/Minkowski coordinates, the spacetime interval associated with it is just $\widetilde{x}^2 \equiv x^2 + y^2 + z^2 - t^2$, which turns out to be the same as the negative of the determinant of the matrix presentation:

$$x^T H x = x^\mu \, \eta_{\mu\nu} \, x^\nu \equiv \boldsymbol{\eta}(\widetilde{x}, \widetilde{x}) \equiv \widetilde{x} \cdot \widetilde{x} = -\det \mathbf{X} \, . \tag{0.2}$$

A second thing we notice is the appearance of the form of the standard 3 Pauli σ-matrices. By appending to them the 2×2 identity matrix, I_2, we can create a set of 4 (generalized) Pauli matrices, appropriate for 4-dimensional spacetime:

$$\begin{gathered} \sigma^\mu \equiv \begin{pmatrix} \vec{\sigma} \\ I_2 \end{pmatrix} \equiv \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ I_2 \end{pmatrix} , \quad \mathbf{X} \equiv \widetilde{x} \cdot \widetilde{\boldsymbol{\sigma}} = x^\mu \sigma_\mu = \begin{pmatrix} z-t & x-iy \\ x+iy & -z-t \end{pmatrix} , \\ \sigma_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma_y \equiv \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix} , \quad \sigma_z \equiv \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix} , \quad \mathbf{I}_2 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} . \end{gathered} \tag{0.3}$$

We can also use this approach to create a matrix form of the 4 basis 1-forms, say, for spacetime, or the basis tangent vectors. Using the current approach, with Cartesian coordinates, we may simply differentiate the earlier equation for $\mathbf{X}$ to create a basis for 1-forms, and use partial derivatives with respect to those variables to create the reciprocal basis:

$$\begin{aligned}
\underset{\approx}{\boldsymbol{\omega}} \equiv d\mathbf{X} = \sigma_\mu \, dx^\mu &= \begin{pmatrix} dz - dt & dx - i\,dy \\ dx + i\,dy & -dz - dt \end{pmatrix} \,, \\
\Longrightarrow \quad \eta_{\mu\nu}\, \underset{\approx}{\boldsymbol{\omega}}^\mu \otimes \underset{\approx}{\boldsymbol{\omega}}^\nu = \boldsymbol{\eta} &= \; -\det(\underset{\approx}{\boldsymbol{\omega}}) \,, \\
\widetilde{\mathbf{E}} \equiv \sigma^\mu \, \partial_{x^\mu} \equiv \sigma^\mu \, \partial_\mu &= \begin{pmatrix} \partial_z + \partial_t & \partial_x - i\,\partial_y \\ \partial_x + i\,\partial_y & -\partial_z + \partial_t \end{pmatrix} \,,
\end{aligned} \tag{0.5}$$

where, even though we are in fact considering only the coordinate basis sets for each of these, I use the symbols $\underset{\approx}{\boldsymbol{\omega}}$ and $\widetilde{\mathbf{E}}$ simply to remind us of what sort of objects are being described. I also note that the product used in this determinant is the symmetric part of the tensor product of two 1-forms, appropriate to create the metric tensor which is a symmetric, type [0,2] tensor.

We also need a reverse mechanism: Given an arbitrary Hermitian, 2×2 matrix, how do we find the associated (real) 4-vector? Although in principle one may clearly do it "by hand," 'tis better to have an algorithmic approach, which we create by using the properties of the traces of the Pauli matrices:

$$\tfrac{1}{2}\,\mathrm{tr}\;\sigma^\mu = \delta^\mu_4 \,, \; \tfrac{1}{2}\,\mathrm{tr}\,(\sigma_i \sigma^j) = \delta^j_i \quad \Longrightarrow \quad x_4 = -x^4 = \tfrac{1}{2}\,\mathrm{tr}\;\mathbf{X} \,, \quad \vec{x} = \tfrac{1}{2}\,\mathrm{tr}\,(\vec{\boldsymbol{\sigma}}\mathbf{X}) \,. \tag{0.6}$$

One of our most important interests in our study of tensorial objects over spacetime is their behavior under Lorentz transformations. Therefore, we want to determine how the action of a Lorentz transformation acting on $\widetilde{x}$ influences the form of its 2×2 matrix presentation, $\mathbf{X}$. One might easily suppose that the action on a matrix would be by some matrix transformation equation, i.e., if $\widetilde{x}' = L\widetilde{x}$ we might expect that there exists some 2×2 matrix, A, dependent on the choice of L, such that $\mathbf{X'} = A\mathbf{X}A^\dagger$, where we have inserted the matrix $A^\dagger$ on the side opposite to A since this is necessary to preserve the Hermitian character of $\mathbf{X}$. Moreover, since the (4-dimensional) interval of $\widetilde{x}$ is proportional to the determinant of $\mathbf{X}$, and Lorentz transformations preserve that interval, we must preserve the determinant of $\mathbf{X}$ under this transformation, i.e., we must insist that $\det(AA^\dagger) = +1$. In principle this allows the determinant of A to be some phase factor of modulus unity; however, as that extra phase factor would not appear in the form of the transformed matrix, it is not very useful, and we throw it away and summarize by saying

$$\widetilde{x}\,' = L\,\widetilde{x} \quad \Longrightarrow \quad \mathbf{X}' = A\mathbf{X}A^\dagger \ , \quad \text{and} \quad \det A = +1 \ . \tag{0.7}$$

Notice that an arbitrary 2×2, complex matrix has 8 (real) degrees of freedom, and the condition that its determinant should equal $+1$ is 2 real constraints on these quantities, leaving us with 6 degrees of freedom, as is desirable for the 6 degrees of freedom of the Lorentz group. [These groups of matrices have standard names. The group of matrices of the form A, above, is referred to by the name $\boldsymbol{SL}(2,\mathbb{C})$, which stands for the set of all 2×2 complex, invertible matrices with determinant equal to $+1$—this requirement on the determinant being the meaning of the $\boldsymbol{S}$ in front of the rest of the name, while the $\boldsymbol{L}$ simply reminds us that we are interested in "linear transformations," i.e. matrices on a vector space. On the other hand, in this notation the (usual) 3-dimensional, proper rotation group is referred to by the name $\boldsymbol{SO}(3,\mathbb{R})$, while the Lorentz group is often called $\boldsymbol{SO}(3,1)$, since there are 3 spacelike directions and 1 timelike direction. It is also quite common **not to write** $\mathbb{R}$ in the rotation group, that option being the default. This is even more true in the case of the Lorentz group, because the distinction between spacelike and timelike vectors, i.e., the reason why we refer to a signature with 3 plus signs and 1 minus sign, as indicated by the argument $(3, 1)$, is one that is lost when you allow general complex-valued transformations.]

Now, we may identify those 6 degrees of freedom much more precisely by proceeding, as usual, to look at the properties of the Lie algebra of generators of those elements of the group that are not too far from the identity transformation. Writing out the matrix $A = e^Q$ in terms of its generators, and remembering the requirement that $+1 = \det A = e^{\mathrm{tr}\,Q}$, we see that the only requirement on Q is that it should be traceless, i.e., have trace equal to zero. The standard Pauli matrices form a basis for all 2×2 traceless matrices; i.e., we may write any traceless, 2×2 matrix, such as Q, in the form $\vec{\alpha} \cdot \vec{\sigma}$, where $\vec{\alpha}$ is a 3-dimensional vector, with complex components. In order to calculate this exponential we need to know how to multiply the Pauli matrices:

the product of two arbitrary Pauli matrices may be written in the form

$$\begin{aligned} (\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma}) &= (\vec{a}\cdot\vec{b})\mathbf{I}_2 + i(\vec{a}\times\vec{b})\cdot\vec{\sigma} \\ \Longrightarrow \quad (\vec{\alpha}\cdot\vec{\sigma})^2 = \alpha^2\,\mathbf{I}_2 \quad &\text{and} \quad \vec{\sigma}\,(\vec{b}\cdot\vec{\sigma}) = \vec{b}\,\mathbf{I}_2 + i\,\vec{b}\times\vec{\sigma}\ . \end{aligned} \tag{0.8}$$

Therefore, after only a little algebra, we can convert the infinite sum in the exponential into simply 2 terms, with scalar coefficients, so that we may see that a transformation matrix not too far from the identity would look like

$$A = e^{Q} = e^{\vec{\alpha}\cdot\vec{\sigma}} = (\cosh\alpha)\,\mathbf{I}_2 + (\sinh\alpha)\,\hat{\alpha}\cdot\vec{\sigma}\ , \tag{0.9}$$

and the actual transformed Hermitian matrix would have the form

$$\mathbf{X}' = |\cosh\alpha|^2\,\mathbf{X} + \sinh\alpha\,(\hat{\alpha}\cdot\vec{\sigma})\mathbf{X} + (\sinh\alpha)^*\,\mathbf{X}(\hat{\alpha}^*\cdot\vec{\sigma}) + |\sinh\alpha|^2(\hat{\alpha}\cdot\vec{\sigma})\mathbf{X}(\hat{\alpha}^*\cdot\vec{\sigma})\ . \tag{0.10}$$

We may now take the trace of both sides of the expression, keeping only terms to lowest order in $|\alpha|$, which will give us, using Eq. (0.6):

$$t' = -\tfrac{1}{2}\,\mathrm{tr}\ \mathbf{X}' = t - (\vec{\alpha}+\vec{\alpha}^*)\cdot\vec{x} + O^2(|\alpha|)\ . \tag{0.11}$$

Knowing that a Lorentz boost would mix t and $\hat{v}\cdot\vec{x}$ so that $t' = t - \vec{v}\cdot\vec{x}$, to lowest order in v, while a rotation would leave t invariant, we may see that twice the real part of $\vec{\alpha}$ should correspond to $\vec{v}$, while the imaginary part must correspond to rotations since it would leave t invariant. On the other hand if we now first multiply Eq. (0.10) by $\vec{\sigma}$ and then take half the trace, and keep terms only to lowest order in $|\alpha|$, we will acquire

$$\begin{aligned} \vec{x}\,' = \tfrac{1}{2}\,\mathrm{tr}\,(\vec{\sigma}\,\mathbf{X}') &= \vec{x} + \tfrac{1}{2}\,\mathrm{tr}\,\{\vec{\sigma}\,(\vec{\alpha}\cdot\vec{\sigma})\mathbf{X}\} + \tfrac{1}{2}\,\mathrm{tr}\,\{\vec{\sigma}\,\mathbf{X}(\vec{\alpha}^*\cdot\vec{\sigma})\} + O^2(|\alpha|) \\ &= \vec{x} + \tfrac{1}{2}\,\mathrm{tr}\,\{\vec{\sigma}\,(\vec{\alpha}\cdot\vec{\sigma})\mathbf{X}\} + \tfrac{1}{2}\,\mathrm{tr}\,\{\mathbf{X}(\vec{\alpha}^*\cdot\vec{\sigma})\,\vec{\sigma}\} + O^2(|\alpha|) \\ &= \vec{x} + \tfrac{1}{2}\,\mathrm{tr}\,\{\vec{\alpha}\,\mathbf{X} + i\,(\vec{\alpha}\times\vec{\sigma})\mathbf{X} + \vec{\alpha}^*\,\mathbf{X} - i\,\mathbf{X}(\vec{\alpha}^*\times\vec{\sigma})\} + O^2(|\alpha|) \\ &= \vec{x} - (\vec{\alpha}+\vec{\alpha}^*)t + i(\vec{\alpha}-\vec{\alpha}^*)\times\vec{x} + O^2(|\alpha|)\ . \end{aligned} \tag{0.12}$$

Again we see that it is the real part of $\vec{\alpha}$ that generates mixing of $\vec{x}$ and t, i.e., that is an infinitesimal part of a boost, while, now, we see that the imaginary part generates rotations. More specifically, if we decompose $\vec{\alpha}$ into its real and imaginary parts we may identify things as follows: $\vec{\alpha} \equiv \vec{a} + i\vec{b} = \frac{1}{2}\{\lambda\hat{v} - i\theta\hat{e}\}$ so that the transformation is parameterized by

$$\frac{\lambda}{2}\hat{v}\cdot\vec{\sigma} - i\,\frac{\theta}{2}\hat{e}\cdot\vec{\sigma} \quad\longleftrightarrow\quad \lambda\hat{v}\cdot\vec{\mathfrak{K}} + \theta\hat{e}\cdot\vec{\mathfrak{J}}\,. \tag{0.13}$$

The next interesting thing to do is to consider what happens when we perform two such transformations in a row: let A_i correspond to L_i so that

$$\widetilde{x}'' = L_1\,\widetilde{x}' = L_1 L_2\,\widetilde{x} \quad\Longleftrightarrow\quad \mathbf{X}'' = A_1\mathbf{X}'A_1^\dagger = A_1(A_2\mathbf{X}A_2^\dagger)A_1^\dagger = (A_1A_2)\mathbf{X}(A_1A_2)^\dagger\,. \tag{0.14}$$

This calculation tells us that when A_1 corresponds to L_1 and A_2 corresponds to L_2, then we have also the relationship that A_1A_2 corresponds to L_1L_2. This is the requirement that is needed to satisfy the definition for a *representation* of a group of transformations, i.e., that the relationship in question should preserve the multiplication properties that are inherent in the group. Therefore we may conclude that the set of all the 2×2 matrices with determinant of $+1$, i.e., the set $\{A = e^{\vec{\alpha}\cdot\vec{\sigma}}\}$, constitutes a representation of the (proper, orthochronous) Lorentz group of transformations, which we could make more definite by using a symbol D for this mapping; i.e.,

$$\begin{aligned} R(\theta;\hat{e}) &= e^{\theta\,\hat{e}\cdot\vec{\mathfrak{J}}} \quad\text{and}\quad D[R(\theta;\hat{e})] = e^{-i(\theta/2)\hat{e}\cdot\vec{\sigma}}\,,\\ B(\vec{v}) &= e^{\lambda\,\hat{v}\cdot\vec{\mathfrak{K}}} \quad\text{and}\quad D[B(\vec{v})] = e^{(\lambda/2)\hat{v}\cdot\vec{\sigma}}\,. \end{aligned} \tag{0.15}$$

This representation has some peculiarities that should immediately be pointed out. They arise for two, related reasons. The first property is that the action of the matrices $D[L]$ is quadratic so that the action of $+D[L]$ on $\mathbf{X}$ is exactly the same as the action of $-D[L]$; this means that the representation in question is **not faithful**, but, rather, two to one: for every Lorentz transformation, L, there are two independent 2×2 matrices, $\pm D[L]$, which correspond to it. We would of course prefer to "normalize" this question by choosing the 2×2 identity matrix, $\mathbf{I}_2$, to be the representation of the identity Lorentz transformation. Then, as θ and/or

λ vary, continuously, away from zero, we would continue to choose that positive sign, at least as long as possible. What we will see below is that this approach does lead to some trouble, so that in fact that hope cannot be realized; the two to one representation is inherent in the nature of the geometry, and cannot be normalized away.

Secondly, because of the "extra" factor of one half in the representation, we can immediately see that the representation of a rotation by angle 2π is not quite what we would have expected:

$$D[R(2\pi;\hat{e})] = e^{-i\pi\hat{e}\cdot\vec{\sigma}} = \cos\pi\,\mathbf{I}_2 - i\sin\pi\,\hat{e}\cdot\vec{\sigma} = -\mathbf{I}_2\ . \tag{0.16}$$

Therefore if we begin at the identity Lorentz transformation, and choose the 2×2 identity matrix as its representation, then the representation of a complete rotation "all the way around," i.e., by angle 2π, about any direction, is represented by the negative of the identity matrix. Of course, it is obviously true that if one rotated by angle 4π he would indeed come back to the identity matrix again. This is the way that this $2\to 1$ mapping works so as to include all the elements on both sides: there are actually twice as many elements in the group $\boldsymbol{SL}(2,\mathbb{C})$ as there are in the Lorentz group, $\boldsymbol{SO}(3,1)$! This of course does not cause a problem with the actual action, on spacetime viewed as components of 2×2, Hermitian matrices, as described above, because it is a quadratic action. Nonetheless, when we consider linear actions, on the underlying 2-dimensional, complex vectors on which these representation matrices could be allowed to act, they will have the property that a complete rotation, by angle 2π, of the physical system—if any—which they represent would nonetheless change the sign of the representing vector!

I. Fundamental Spinor Spaces

Since we now have an action on 2×2, Hermitian matrices, which is quadratic in the fundamental representation matrices, we wonder what physical meaning there might be, if any, to the underlying, 2-dimensional, complex vector space on which these 2×2 matrices "live," as **linear** operators. To answer this question, we begin by creating a basic, 2-dimensional, complex vector space. The vectors in this space, say ξ, are quantities on which the action of the matrices $D(L)$ would take them from being viewed in, say, a frame S' to how they would be viewed in the frame S, those two being related as usual by the Lorentz transformation L, with $\xi = D(L)\,\xi'$. These objects are called *spinors.* As mentioned above, any physical systems which would be described by these spinors would be such that a complete rotation of that system, all the way around, so that the actual physical system would surely return to its original state, would nonetheless cause the 2-dimensional, complex vector, i.e., the spinor, representing its properties to have its sign changed.

It is customary to describe a spinor in terms of its components with respect to some chosen basis, that basis often never being discussed further. Therefore, we refer to the space of all spinors by the symbol V^2 and we denote the components of an arbitrary element of this space by a symbol such as $\{\xi^A \mid A = 1, 2\}$. [Sometimes the basis is described by giving names to a particular pair of non-parallel spinors. Very common names are o and i. We will, however, only very seldom, if ever, need to actually use these objects.] As these components are complex-valued, we might suppose that the complex conjugate of a spinor in V^2 could also be an element of V^2. This would however be quite wrong. Let us for instance consider the notion that α and β are spinors, in V^2, and that c is some complex number, so that, certainly $\alpha + c\beta$ is also an element of V^2, it being a vector space over the complex numbers. However, then the complex conjugate of this spinor would be $\overline{\alpha} + \overline{c}\overline{\beta}$ instead of what one might have expected, namely $\overline{\alpha} + c\overline{\beta}$. This tells us that complex conjugation of the vector space is an *anti-isomorphism* of V^2. A second reason this doesn't work is that there are allowed transformations of V^2 that can map the components of a spinor into their complex conjugates; therefore, these would simply be two different presentations of the same spinor. In summary, then, we decide that there should be a second space, which we refer to as the "complex-conjugated space," $\overline{V}^2$, with elements described by $\{\psi^{\dot{A}} \mid A = 1, 2\}$, and that complex-conjugation is to be considered as an anti-isomorphism between these two different spaces. We will make a constraint on the choices of bases in these two spaces, so that corresponding spinors, in the two spaces, will have components which are in fact simply complex conjugates of one another. One make describe this, for instance, by the bald statement that we suppose that the basis in question is real, or we can insist that whenever we make a change of basis in V^2 we make a corresponding, anti-holomorphic change of basis in $\overline{V}^2$. Since we will almost never make such a change of basis, this is not a problem of any practical importance.

Therefore, for a given, particular spinor, say $\xi^A \in V^2$, we may describe its anti-isomorphic image as having components which are simply its complex conjugates: $\overline{\xi^A} = \xi^{\dot{A}} \in \overline{V}^2$, where the overline means complex conjugation as usual.

Since the spinor spaces are certainly examples of vector spaces, it is reasonable that there should be associated with (each of) them dual spaces, which we label V^2_* and $\overline{V}_*^2$. In order to preserve the Einstein summation convention, that causes implied sums when there is a repeated upper and lower index, we use the notation ξ_A and $\xi_{\dot{A}}$ to refer to elements of these spaces. Again considering (linear) transformations of our spinors, if we make a transformation in V^2, we would expect that the spinors in V_2 would transform by the inverse transformation, so that the scalar that the two form would remain invariant under this transformation:

$$\xi'^{\,R} = A^R{}_C\,\xi^C \quad \text{and} \quad \zeta'_{\,S} = (A^{-1})^B{}_S\,\zeta_B \iff \xi'^{\,R}\zeta'_{\,R} = \xi^C\zeta_C\ . \tag{1.1}$$

However, since in ordinary spacetime the existence of the metric tensor allows us to create a mapping that maps tangent vectors and 1-forms—dual to one another—into each other, we might wonder if there is not also some appropriate mapping in the spinor space that allows this. We cannot be certain that our spinor spaces allow mappings between themselves and their duals, just as we cannot be certain that there are such mappings between the tangent bundle and the cotangent bundle of an arbitrary manifold; on the other hand we do know, there, that this happens when there is a metric allowed on the these bundles. On the other hand, in the current case, of Hermitian matrices over the spinor spaces, it is the determinant that creates lengths. As well, in our spaces with just 2 dimensions it is the Levi-Civita symbol,

ϵ_{AB}, which creates determinants. As it does indeed have just 2 indices, in fact it could also be used as a mapping sending ξ^A into ξ_A.

We will now show that, so long as our transformation matrices are restricted to have determinant +1, then this thought pattern does work, and we may satisfy Eq. (1.1) at the same time as we use ϵ_{AB} to lower indices. It is therefore usual to suppose the existence in each of our 4 (2-dimensional) vector spaces, of a numerical, index quantity, a Levi-Civita symbol, each with exactly the same numerical values for their components, namely +1 when the components take the (ordered) values 1, 2, the value −1 for the values of the indices as 2, 1, and 0 otherwise, i.e., 0 along the diagonal:

$$\begin{aligned} &\epsilon_{AB} \equiv \epsilon_{\dot{A}\dot{B}} \Longrightarrow \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} \Longleftarrow \epsilon^{AB} \equiv \epsilon^{\dot{A}\dot{B}} , \\ &\Longrightarrow \quad \epsilon_{BC}\,\epsilon^{AC} = \delta^A_B \Longrightarrow \quad (\epsilon_{AB})^{-1} = -\epsilon^{AB} . \end{aligned} \tag{1.2}$$

We then use these matrices as a metric in that they map spinors to the corresponding dual spinor, and vice versa. Since we intend to consider only transformations with determinant equal to +1 in these spaces, those transformations always preserve these Levi-Civita symbols, so that they play a distinguished role that is very similar to that of the standard spacetime (Lorentz) metric. However, because this symbol is skew-symmetric, rather than symmetric, it is important to be very careful as to the order of the indices when using it to "raise" and "lower" indices; the convention with regard to order that I use is the following one:

$$\begin{aligned} &\xi_A \equiv \epsilon_{AB}\,\xi^B , \quad \xi^C \equiv \xi_D\,\epsilon^{DC} , \\ &\xi^A = \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix} \Longrightarrow \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \xi_A = \begin{pmatrix} \xi^2 \\ -\xi^1 \end{pmatrix} , \end{aligned} \tag{1.3}$$

along with the same rules for "dotted indices." [Some people use the opposite convention, following Penrose! Our convention comes originally from von Neumann, Debever, and Plebański.]

It is therefore true that the invariant sum of the components of a spinor and its dual, i.e., $F_A\,F^A$*, is always identically zero, while for two, non-parallel spinors we have the following (doubtless unexpected) phenomenon:*

$$F_A\,G^A = -F^A\,G_A . \tag{1.4}$$

It turns out that the constraint that the transformation have determinant +1 will guarantee that these two approaches give exactly the same answer, as we will now show, where we show Eq. (1.1) on the left side of the chain of equalities below and our definitions for raising and lowering indices, from Eqs. (1.?), on the right hand side:

$$(A^{-1})^{B}{}_{R}\,\xi_{B} = \xi'{}_{R} = \epsilon_{RS}\,\xi'{}^{S} = \epsilon_{RS}\,A^{S}{}_{C}\,\xi^{C} = \epsilon_{RS}\,A^{S}{}_{C}\,\epsilon^{BC}\,\xi_{B}\;. \tag{1.5}$$

However, we now consider the equation for determinant of A:

$$\begin{aligned}\epsilon_{RS}\,A^{R}{}_{E}\,A^{S}{}_{F} = \epsilon_{EF}(\det A) \quad &\Longrightarrow \quad \epsilon_{RS}\,A^{S}{}_{F} = (\det A)\epsilon_{BF}\,(A^{-1})^{B}{}_{R}\\ \Longrightarrow \quad \epsilon_{RS}\,A^{S}{}_{F}\,\epsilon^{BF} &= (\det A)\,(A^{-1})^{B}{}_{R}\;.\end{aligned} \tag{1.6}$$

From this relationship with determinants we see that the two desires in Eq. (1.5) are indeed in agreement **provided** that we restrict ourselves to considering transformations which have unit determinant; i.e., we should only consider transformations from $\boldsymbol{SL}(2,\mathbb{C})$. This of course also has the extra advantage—not altogether unrelated—that all these transformations preserve the form of the Levi-Civita symbol.

We also note, of course, that all the same sorts of statements are true for the "dotted" spinors, using the "dotted" transformation equations:

$$\xi'{}^{\dot{R}} = A^{\dot{R}}{}_{\dot{C}}\,\xi^{\dot{C}}\;, \quad \xi'{}_{\dot{R}} = (A^{-1})^{\dot{C}}{}_{\dot{R}}\,\xi_{\dot{C}} \quad . \tag{1.7}$$

II. Spinors with multiple indices: Hermitian, 2-index spinors

Having these 4 different spinor spaces, at each point, we may create many sorts of tensor products, which would contain quantities with components such as F^{AB}, $G^{A}{}_{B}$, C_{ABCD}, $R_{AB\dot{C}\dot{D}}$, and, of course $X^{A\dot{B}}$, which is an example of the Hermitian tensors we have already considered. We will actually look at each one of these as the appropriate physical motivation occurs. However, we will first re-consider, with more care, the mapping between Hermitian, 2nd-rank spinors and 4-vectors, or 1-forms, in spacetime, from §0. (We will see that several of these other sorts of spinor products also can play important roles in studies of physical phenomena.)

An Hermitian tensor $X^{A\dot{B}}$ is of course an element of the tensor product space $V^2 \otimes \overline{V}^{\,2}$. However, it is not an arbitrary such element, but rather one for which the process of complex conjugation and transposition brings us back to where we began:

$$\overline{X^{A\dot{B}}} = X^{B\dot{A}} \ . \tag{2.1}$$

Beginning with only one individual spinor, say k^A, one may create an Hermitian spinor by using also the corresponding complex-conjugated, or "dotted," spinor, $k^{\dot{A}} \equiv \overline{k^A} \in \overline{V}^{\,2}$. We first show that their matrix which presents their tensor product is Hermitian, and then that this matrix has zero determinant, which of course means that the 4-vector which that Hermitian spinor presents has zero length, i.e., it is a null vector:

$$\begin{aligned} K^{A\dot{B}} \equiv k^A k^{\dot{B}} = k^A \overline{k^B} = \overline{\overline{k^{\dot{A}}} k^B} = \overline{k^B k^{\dot{A}}} = \overline{K^{B\dot{A}}} \ , \\ \Longrightarrow \quad \det K = \epsilon_{AC} K^{A\dot{1}} K^{C\dot{2}} \ = \epsilon_{AC} k^A k^{\dot{1}} k^C k^{\dot{2}} = 0 \ . \end{aligned} \tag{2.2}$$

To create a more general sort of Hermitian spinor from 1-index spinors, we will have to have two of them that are not parallel. Therefore, suppose that k^A and ℓ^B are two non-parallel spinors, where the requirement of not being parallel is the same as the mathematical constraint that $k^A \ell_A \neq 0$. Then, there are a couple of simple ways to create an Hermitian spinor from this pair. The first method begins with a non-Hermitian, 2nd-rank spinor, such as $k^A \ell^{\dot{B}}$, and takes its Hermitian part. As the spinor is complex, and therefore the associated 4-vector is also complex, we may imagine this process as being quite the same as taking either the real or the imaginary part of that 4-vector. This is completely analogous to beginning with an arbitrary complex number, z, and taking either the real part, $(z+\overline{z})/2$, or the imaginary part, $(z-\overline{z})/(2i)$. To describe this for such a matrix it is useful to have a simple scalar to play either the role of $+1$, for the real part, or $+i$, for the imaginary part; therefore, we define the quantity ε such that $\varepsilon^2 = \pm 1$, and then consider the following, which incorporates both the idea of the real part and that of the imaginary part:

$$\begin{aligned} X^{A\dot{B}} \equiv \varepsilon(k^A \ell^{\dot{B}} \pm \ell^A k^{\dot{B}}) &= \begin{cases} 2\varepsilon\, k^{(A} \ell^{\dot{B})} & \text{for the + sign,} \\ 2\varepsilon\, k^{[A} \ell^{\dot{B}]} & \text{for the − sign,} \end{cases} \\ \Longrightarrow \ -\det X = \ -\tfrac{1}{2}\epsilon_{AC}\epsilon_{\dot{B}\dot{D}} X^{A\dot{B}} X^{C\dot{D}} &= -\tfrac{1}{2}\epsilon_{AC}\epsilon_{\dot{B}\dot{D}}\, \varepsilon(k^A \ell^{\dot{B}} \pm \ell^A k^{\dot{B}}) \varepsilon(k^C \ell^{\dot{D}} \pm \ell^C k^{\dot{D}}) \\ = -\tfrac{1}{2}\varepsilon^2 \{\pm (k^A \ell_A)(\ell^{\dot{B}} k_{\dot{B}}) \pm (\ell^A k_A)(k^{\dot{B}} \ell_{\dot{B}})\} &= +|k^A \ell_A|^2 \ . \end{aligned} \tag{2.3a}$$

We see that the length of the 4-vector, $\widetilde{x}$, associated with the Hermitian spinor, $\mathbf{X}$, is always positive, no matter the value of ϵ and whether or not we use the symmetric part or the skew-symmetric part of the tensor product of the two spinors; i.e., 4-vectors created in this way from a pair of 2-spinors will always be spacelike. To find other sorts of 4-vectors, we must look at a different way to create such Hermitian spinors, from two non-parallel spinors. To do this we take linear combinations of the null vectors, directly:

$$
\begin{aligned}
&X^{A\dot{B}} \equiv k^A k^{\dot{B}} \pm \ell^A \ell^{\dot{B}} \ , \\
\Longrightarrow \ -\det X = \ &-\tfrac{1}{2}\epsilon_{AC}\epsilon_{\dot{B}\dot{D}} X^{A\dot{B}} X^{C\dot{D}} = -\tfrac{1}{2}\epsilon_{AC}\epsilon_{\dot{B}\dot{D}}(k^A k^{\dot{B}} \pm \ell^A \ell^{\dot{B}})(k^C k^{\dot{D}} \pm \ell^C \ell^{\dot{D}}) \\
&= \mp\tfrac{1}{2}\{|k^A \ell_A|^2 + |\ell^A k_A|^2\} = \mp|(k^A \ell_A|^2 \ ,
\end{aligned}
\tag{2.3b}
$$

so that this method allows us the option of creating either timelike or spacelike 4-vectors, or even null ones if the two spinors were parallel.

We may therefore begin with two arbitrary, non-parallel 2-spinors, such as k^A and ℓ^B, with $k^A \ell_A \neq 0$, and create a proper set of 4 basis vectors needed for tangent vectors, or 1-forms, in spacetime. Beginning from the simpler geometric quantities, i.e., spinors, is often very useful when considering eigenvector problems! Although they are in principle arbitrary, there is no obvious reason why we should not normalize them in some convenient way; therefore, we choose to normalize them so that $k_A \ell^A = +1$.

Before doing this, however, let us retreat slightly and re-consider Eqs. (0.5), which gave a quick glance at basis 1-forms and tangent vectors for simple, flat spaces. Instead, now let us suppose that we have given, as vector fields over some neighborhood of our manifold, an orthonormal basis for 1-forms, and also a null basis, where we quickly recall their properties:

$$
\{\underset{\sim}{\omega}^{\hat{\alpha}}\}_1^4, \text{ where } g^{\alpha\beta} \equiv g(\underset{\sim}{\omega}^{\hat{\alpha}}, \underset{\sim}{\omega}^{\hat{\beta}}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \equiv \eta^{\alpha\beta} \quad , \tag{2.4a}
$$

and the null one,

$$
\{\underset{\sim}{\theta}^{\alpha}\}_1^4, \text{ where } g^{\mu\nu} \equiv g(\underset{\sim}{\theta}^{\mu}, \underset{\sim}{\theta}^{\nu}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \equiv \nu^{\mu\nu} \quad . \tag{2.4b}
$$

(When either the $\underset{\sim}{\omega}^{\hat\alpha}$ or the $\underset{\sim}{\theta}^{\mu}$ may be used in a formulation, we will sometimes use the notation, $\underset{\sim}{\omega}^{\alpha}$, indicating that we may use either the orthonormal basis, the null basis, or, for that matter, any other basis of 1-forms one might wish.) It might also be useful to note the transformation between the two bases,

$$\underset{\sim}{\theta}^{\mu} = M^{\mu}{}_{\hat\alpha}\,\underset{\sim}{\omega}^{\hat\alpha}\,, \text{ where } M^{\mu}{}_{\hat\alpha} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i & 0 & 0 \\ 1 & -i & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \quad . \tag{2.5}$$

As an aside, we notice that $m \equiv \det(M) = +i$, which tells us that the 4-dimensional Levi-Civita symbol is different in the 2 bases. Using the standard definitions, indeed, we find that

$$\begin{aligned} \eta^{1234} &= +1 = -\eta_{1234}\,, \qquad \text{in the orthonormal basis,} \\ \eta^{1234} &= +i = \eta_{1234}\,, \qquad \text{in the null basis.} \end{aligned} \tag{2.6}$$

We may now re-constitute Eq. (0.5) in a mode appropriate to an arbitrary manifold, by presenting either of these sets of bases as the elements of a 2×2 matrix, where of course we have used the relationship between the two as given above:

$$\underset{\sim}{\boldsymbol{\omega}}^{A\dot B} \equiv \sigma_{\alpha}^{A\dot B}\,\underset{\sim}{\omega}^{\hat\alpha} \equiv \begin{pmatrix} \underset{\sim}{\omega}^{\hat 3} - \underset{\sim}{\omega}^{\hat 4} & \underset{\sim}{\omega}^{\hat 1} - i\underset{\sim}{\omega}^{\hat 2} \\ \underset{\sim}{\omega}^{\hat 1} + i\underset{\sim}{\omega}^{\hat 2} & -\underset{\sim}{\omega}^{\hat 3} - \underset{\sim}{\omega}^{\hat 4} \end{pmatrix} \equiv \sqrt{2}\begin{pmatrix} \underset{\sim}{\theta}^{4} & \underset{\sim}{\theta}^{2} \\ \underset{\sim}{\theta}^{1} & -\underset{\sim}{\theta}^{3} \end{pmatrix} \quad . \tag{2.7}$$

We will refer to the $\{\underset{\sim}{\omega}^{A\dot B}\,|\; {\scriptstyle A,\,\dot B} = 1,2\}$ as a spinorial basis, where the σ_α are the same matrices discussed in §0, and have the form that is usually called the "Pauli sigma matrices," i.e., $((\sigma^{\alpha\,A\,\dot B})) \equiv (\sigma_x, \sigma_y, \sigma_z, I_2)^T$. On the other hand, it is also very handy to be able to write these equations in reverse: Eqs. (2.7) gives us the spinorial basis forms in terms of the standard cotangent basis forms, via those Pauli matrices, which, in this role, are often referred to as "bridge spinors," which actually accomplish the mapping between $V^2 \otimes \overline{V}^{\,2}$ and $\mathcal{T}$. What we want now is to go the other way, again using the bridge spinors, but now in reverse. Since this involves beginning with a matrix, with 1-form elements, and ending up with just 1-forms, we will have to take *matrix traces* of some sort. The algebra is performed by a couple of important quadratic identities that basically talk about obtaining identities in one or the other of the "spaces" where they have their being, i.e., in the Hermitian spinor space or in the tangent bundle over spacetime:

$$\mathrm{tr}\,(\sigma^\alpha\,\sigma^T_\beta) \equiv \sigma^{\alpha A\dot{R}}\sigma_{\beta A\dot{R}} = -2\,\delta^\alpha_\beta\ , \qquad \widetilde{\sigma}^{A\dot{B}}\cdot\widetilde{\sigma}_{C\dot{D}} \equiv \sigma^{\alpha A\dot{B}}\sigma_{\alpha C\dot{D}} = -2\,\delta^A_C\delta^{\dot{B}}_{\dot{D}}\ , \tag{2.8}$$

where we are sometimes displaying the bridge spinors as just a matrix (with 4-vector indices), and sometimes displaying them as explicit 4-vectors (with matrix indices), and sometimes with all their indices displayed. In the first equality, we take the matrix product, and matrix trace, and acquire an identity operator in the 4-vector space; in the second equality, we take the 4-vector scalar product, and acquire an identity operator in the matrix space. Next, since these products involve bridge spinors which have their indices both up and down, we need to understand the form of the bridge spinor with lower indices:

$$\underset{\approx}{\boldsymbol{\omega}}_{A\dot{B}} = \epsilon_{AC}\,\epsilon_{\dot{B}\dot{D}}\,\underset{\approx}{\boldsymbol{\omega}}^{\alpha C\dot{D}} = \underset{\approx}{\omega}^{\hat{\alpha}}\,\sigma_{\alpha\,A\dot{B}} = \begin{pmatrix} -\underset{\approx}{\omega}^{\hat{3}} - \underset{\approx}{\omega}^{\hat{4}} & -\underset{\approx}{\omega}^{\hat{1}} - i\underset{\approx}{\omega}^{\hat{2}} \\ -\underset{\approx}{\omega}^{\hat{1}} + i\underset{\approx}{\omega}^{\hat{2}} & +\underset{\approx}{\omega}^{\hat{3}} - \underset{\approx}{\omega}^{\hat{4}} \end{pmatrix} = -\sqrt{2}\begin{pmatrix} \underset{\approx}{\theta}^3 & \underset{\approx}{\theta}^1 \\ \underset{\approx}{\theta}^2 & -\underset{\approx}{\theta}^4 \end{pmatrix}\ . \tag{2.9}$$

One can now sort through the algebra fairly easily to obtain

$$\underset{\approx}{\omega}^{\hat{\alpha}} = -\tfrac{1}{2}\,\sigma^{\alpha A\dot{B}}\,\underset{\approx}{\boldsymbol{\omega}}_{A\dot{B}} = -\tfrac{1}{2}\ \mathrm{tr}\,\left(\sigma^\alpha\underset{\approx}{\omega}^T\right)\ . \tag{2.10}$$

We may consider an arbitrary 1-form, $\underset{\sim}{Y} = Y_{\hat{\mu}}\,\underset{\approx}{\omega}^{\hat{\mu}} \in \Lambda^1$ and its associated tangent vector, $\widetilde{Y} = Y^{\hat{\nu}}\,\widetilde{e}_{\hat{\nu}} \in \mathcal{T}$, with their components described via an orthonormal basis in each place, reciprocal to one another, and therefore related via the usual metric mapping: $Y_{\hat{\mu}} = g_{\hat{\mu}\hat{\nu}}\,Y^{\hat{\nu}} = \eta_{\hat{\mu}\hat{\nu}}\,Y^{\hat{\nu}}$, and relate these components to their components in the vector space of Hermitian spinors:

$$\begin{aligned} \eta^{\hat{\mu}\hat{\nu}}Y_{\hat{\nu}} = Y^{\hat{\mu}} &= \underset{\approx}{\boldsymbol{\omega}}^{\mu}(\widetilde{Y}) = -\tfrac{1}{2}\sigma^{\mu A\dot{B}}\,\underset{\approx}{\boldsymbol{\omega}}_{A\dot{B}}(\widetilde{Y}) = -\tfrac{1}{2}\mathbf{Y}_{A\dot{B}}\,\sigma^{A\dot{B}}_{\mu}\ , \\ \epsilon^{AC}\epsilon^{\dot{B}\dot{D}}\,Y_{C\dot{D}} = Y^{A\dot{B}} &= \underset{\approx}{\boldsymbol{\omega}}^{A\dot{B}}(\widetilde{Y}) = \sigma^{A\dot{B}}_{\mu}\,\underset{\approx}{\boldsymbol{\omega}}^{\hat{\mu}}(\widetilde{Y}) = \sigma^{A\dot{B}}_{\mu}\,Y^{\hat{\mu}}\ . \end{aligned} \tag{2.11}$$

Another minor sort of remark, or memory of formulae, may also be reasonably placed here. Since we have been concentrating almost completely on 1-forms, rather than tangent vectors, there is nonetheless some reason to recall what would be the similar structures for tangent vectors. We suppose, in particular, that $\{\widetilde{e}_{\hat{\alpha}}\}^4_1$ is the orthonormal basis of tangent vectors reciprocal to our orthonormal basis of 1-forms $\{\underset{\sim}{\omega}^{\hat\beta}\}_1^4$; as well take $\{\widetilde{f}_\mu\}_1^4$ as the basis reciprocal to our null basis forms $\{\underset{\sim}{\theta}^\nu\}_1^4$. Then we may use the standard mapping between 1-forms and tangent vectors implemented by the metric to construct spinorial quantities such as $\widetilde{\mathbf{e}}^{A\dot B}$, the 2×2 matrices with elements that are (linear combinations of the) basis vectors $\{\widetilde{e}_\mu\}_1^4$:

$$\widetilde{\mathbf{e}}^{A\dot B} \equiv \left\{\left(\underset{\sim}{\boldsymbol{\omega}}^{A\dot B}\right)_{\hat\mu}\eta^{\hat\nu\hat\mu}\right\}\widetilde{e}_{\hat\nu} = \eta^{\hat\nu\hat\mu}\,\sigma_\mu^{A\dot B}\widetilde{e}_{\hat\nu} = \sigma^{\nu\,A\dot B}\widetilde{e}_{\hat\nu}\ ,$$
$$\widetilde{\mathbf{e}}^{A\dot B} = \begin{pmatrix} \widetilde{e}_{\hat 3}+\widetilde{e}_{\hat 4} & \widetilde{e}_{\hat 1}-i\widetilde{e}_{\hat 2} \\ \widetilde{e}_{\hat 1}+i\widetilde{e}_{\hat 2} & -\widetilde{e}_{\hat 3}+\widetilde{e}_{\hat 4}\end{pmatrix} = \sqrt{2}\begin{pmatrix}\widetilde{f}_{\hat 3} & \widetilde{f}_{\hat 1} \\ \widetilde{f}_{\hat 2} & -\widetilde{f}_{\hat 4}\end{pmatrix}\ ,\quad \widetilde{\mathbf{e}}_{A\dot B} = -\sqrt{2}\begin{pmatrix}\widetilde{f}_4 & \widetilde{f}_2 \\ \widetilde{f}_1 & -\widetilde{f}_3\end{pmatrix}\ . \tag{2.12}$$

We may then also generalize the statements about the determinant and its relation to "lengths" of 4-vectors, by re-writing those equations in the form where we use a determinant of 1-forms to define the desired 2nd rank tensor, **g**:

$$\mathbf{g} = \eta_{\alpha\beta}\,\underset{\sim}{\omega}^{\hat\alpha}\otimes\underset{\sim}{\omega}^{\hat\beta} = \nu_{\mu\nu}\,\underset{\sim}{\theta}^\mu\otimes\underset{\sim}{\theta}^\nu = -\epsilon_{\dot C\dot D}\,\underset{\sim}{\omega}^{1\dot C}\otimes\underset{\sim}{\omega}^{2\dot D} = -\hat\epsilon_{AB}\epsilon_{\dot C\dot D}\,\underset{\sim}{\omega}^{A\dot C}\otimes\underset{\sim}{\omega}^{B\dot D} = -\det(\underset{\sim}{\omega}^{A\dot B}) \tag{2.13}$$

and, again, the "product" in the determinant is taken as the symmetric part of the tensor product.

We can also use our knowledge of how to create spacetime vectors from spinors to create a null basis of 1-forms specific to a single, given **pair** of spinors, that might, for instance, be a pair of "eigenspinors" for some interesting spinorial object:

$$\begin{aligned}\underset{\sim}{\theta}^3 \equiv \underset{\sim}{k} \equiv -\tfrac{1}{\sqrt{2}}k_A k_{\dot B}\,\underset{\sim}{\boldsymbol{\omega}}^{A\dot B}\ ,\ \underset{\sim}{\theta}^4 \equiv \underset{\sim}{\ell} \equiv \tfrac{1}{\sqrt{2}}\ell_A\ell_{\dot B}\,\underset{\sim}{\boldsymbol{\omega}}^{A\dot B}\ ,\\ \underset{\sim}{\theta}^1 \equiv \underset{\sim}{m} \equiv \tfrac{1}{\sqrt{2}}k_A\ell_{\dot B}\,\underset{\sim}{\boldsymbol{\omega}}^{A\dot B}\ ,\ \underset{\sim}{\theta}^2 \equiv \overline{\underset{\sim}{m}} \equiv \tfrac{1}{\sqrt{2}}\ell_A k_{\dot B}\,\underset{\sim}{\boldsymbol{\omega}}^{A\dot B}\ ,\end{aligned} \tag{2.14}$$

where the minus sign in the definition of $\underset{\sim}{k}$ makes later signs much more convenient. As is usual in our null tetrads, two of the null vectors are complex rather than real; therefore the associated matrix, $k^A\ell^{\dot B}$ is not Hermitian.

It should be noted, as usual you should think by now, that there are various naming conventions here: some other people use $\underset{\sim}{\ell}$, $\underset{\sim}{n}$ with $\underset{\sim}{m}$ as the complex one, while yet some others follow the original names created by Sachs, which are $\underset{\sim}{k}$, $\underset{\sim}{m}$, with $\underset{\sim}{t}$ as the complex one. The identifications I have given, between $\underset{\sim}{\theta}^\alpha$ and the 1-forms made from k_A and ℓ_B are also customary; however, if you take the identifications we have made, between the bridge spinors and the Pauli matrices, then these identifications require some specific identifications for this pair of non-parallel spinors as well. Those identifications are that

$$k_A \Longrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \quad \ell_A \Longrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} . \tag{2.15}$$

On the other hand, we may define a complete set of basis 1-forms for the spacetime from any (useful and interesting) pair of non-parallel spinors, from, say V^2, as is done above. We may even, then, define that null basis set as above; however, it will then either cause a change in the presentation of the bridge spinors, or the invocation of a Lorentz transformation between the new null basis and the older one.

Having said all that, let us try to yet look at the question of what happened to the "phase" of the spinor when we mapped it into a null vector. Clearly k_A and $e^{i\alpha}k_A$ map into the same 1-form (or corresponding null tangent vector), so that the information about the phase of k_A is lost when this mapping is made. However, as it turns out it is actually "hidden" in the orientation of the 2-frame in the transverse plane. To see how this comes about, let us now suppose that we have made a complete choice of spinor basis and corresponding null basis, of 1-forms, in spacetime. Then let us make a transformation where we modify the phase of k_A by an extra phase factor; i.e., we consider the change whereby $k_A \;\to\; e^{+i\alpha}k_A$. Since ℓ_A must stay properly normalized, this induces a change of phase of it, in the opposite direction: $\ell_A \;\to\; e^{-i\alpha}\ell_A$. These changes of course change neither of the real 1-forms $\underset{\sim}{k}$ nor $\underset{\sim}{\ell}$. On the other hand, the transverse plane is spanned by the complex, null 1-forms $\underset{\sim}{m}$ and its conjugate $\overline{\underset{\sim}{m}}$, and under this change they are transformed as follows: $\underset{\sim}{m} \;\to\; e^{+2i\alpha}\,\underset{\sim}{m}$ and $\overline{\underset{\sim}{m}} \;\to\; e^{-2i\alpha}\,\overline{\underset{\sim}{m}}$. This accomplishes a rotation, by angle 2α of the frame in that plane. It is therefore common to say that the phase of the basis spinor is disclosed, or made manifest, in the behavior of a *"flag"* in the spacetime. To describe this geometrically, we take as *"the flagpole,"* the null vector

$\widetilde{k} \equiv \left(\nu^{\alpha\beta} k_\beta\right) \widetilde{f}_\alpha$, where the associated 1-form was defined by $\underset{\sim}{k} \equiv k_\beta \underset{\sim}{\theta}^\beta$. We then take the real part of the null vector associated with $\underset{\sim}{m}$, and think of it as a single vector perpendicular to the entire 2-plane spanned by $\widetilde{k}$ and $\widetilde{\ell}$. The 2-plane spanned by this vector and the original null vector $\widetilde{k}$ is called a "flag," and as the underlying spinor changes its phase, we can see that the flag rotates about its flagpole.

III. Spinors with multiple indices: symmetric spinors, and 2-forms

Having described in reasonable detail the relation between spinors and 1-forms, let us now proceed onward to the rest of the Grassmann algebra of p-forms over the manifold. We next look at 2-forms, and begin by defining the collections of 2×2 matrices with entries which are basis sets for 2-forms, created as wedge products of the basis set for 1-forms we have already been using. We will see that such 2-forms are represented by symmetric, 2nd-rank spinors. We begin by considering wedge products of our basic matrix-valued 1-forms, which gives us 2×2 matrices, with entries which are individual 2-forms:

$$\begin{aligned} \underset{\sim}{S}^{AB} &\equiv \hat{\underset{\sim}{\boldsymbol{\omega}}}^{A\dot{C}} \wedge \underset{\sim}{\boldsymbol{\omega}}^{B}{}_{\dot{C}} = \tfrac{1}{2}\, \sigma_\alpha^{A\dot{C}}\, \sigma_\beta^{B\dot{D}} \epsilon_{\dot{C}\dot{D}}\, \underset{\sim}{\omega}^\alpha \wedge \underset{\sim}{\omega}^\beta \equiv \hat{S}^{AB}{}_{\alpha\beta}\, \underset{\sim}{\omega}^\alpha \wedge \underset{\sim}{\omega}^\beta \in \Lambda^2 \quad , \\ \underset{\sim}{S}^{\dot{A}\dot{B}} &\equiv \hat{\underset{\sim}{\boldsymbol{\omega}}}^{C\dot{A}} \wedge \underset{\sim}{\boldsymbol{\omega}}_{C}{}^{\dot{B}} = \tfrac{1}{2}\, \sigma_\alpha^{C\dot{A}} \sigma_\beta^{D\dot{B}} \epsilon_{CD}\, \underset{\sim}{\omega}^\alpha \wedge \underset{\sim}{\omega}^\beta \equiv \hat{S}^{\dot{A}\dot{B}}{}_{\alpha\beta}\, \underset{\sim}{\omega}^\alpha \wedge \underset{\sim}{\omega}^\beta \in \Lambda^2 \quad . \end{aligned} \tag{3.1}$$

We write them out more explicitly, but use the null basis 1-forms for this presentation, since the matrices appear much simpler in that form:

$$\begin{aligned} \underset{\sim}{S}^{AB} &= \begin{pmatrix} 2\underset{\sim}{\theta}^4 \wedge \underset{\sim}{\theta}^2 & \underset{\sim}{\theta}^1 \wedge \underset{\sim}{\theta}^2 + \underset{\sim}{\theta}^3 \wedge \underset{\sim}{\theta}^4 \\ \underset{\sim}{\theta}^1 \wedge \underset{\sim}{\theta}^2 + \underset{\sim}{\theta}^3 \wedge \underset{\sim}{\theta}^4 & 2\underset{\sim}{\theta}^3 \wedge \underset{\sim}{\theta}^1 \end{pmatrix} \quad , \\ \underset{\sim}{S}^{\dot{A}\dot{B}} &= \begin{pmatrix} 2\underset{\sim}{\theta}^4 \wedge \underset{\sim}{\theta}^1 & -\underset{\sim}{\theta}^1 \wedge \underset{\sim}{\theta}^2 + \underset{\sim}{\theta}^3 \wedge \underset{\sim}{\theta}^4 \\ -\underset{\sim}{\theta}^1 \wedge \underset{\sim}{\theta}^2 + \underset{\sim}{\theta}^3 \wedge \underset{\sim}{\theta}^4 & 2\underset{\sim}{\theta}^3 \wedge \underset{\sim}{\theta}^2 \end{pmatrix} = \overline{\underset{\sim}{S}^{AB}} \quad , \end{aligned} \tag{3.2}$$

where the statement concerning complex conjugation requires us to recall that $\underset{\sim}{\theta}^3$ and $\underset{\sim}{\theta}^4$ are real, while $\underset{\sim}{\theta}^2$ is the complex conjugate of $\underset{\sim}{\theta}^1$. One immediately sees that the matrices, $\underset{\sim}{S}^{AB}$ and $\underset{\sim}{S}^{\dot{A}\dot{B}}$ are symmetric as matrices, i.e., symmetric in their spinor indices. Therefore, each of them has exactly **three** linearly-independent 2-forms between them, and there is no overlap between the two sets, so that between the two of them they contain the entirety of a basis for

2-forms over M^4, i.e., for Λ^2, analogous to the way that the $\utilde{\theta}^\alpha$ form a basis for Λ^1. There is in fact an important extra feature to this splitting of the 6 basis 2-forms into two groups of 3. Working through the definition of the Hodge dual for 2-forms, in this null basis, one can in fact show that

$$\begin{aligned} &\text{(a)} \qquad *\,\utilde{S}^{AB} = \utilde{S}^{AB} \quad , \text{ so that those 3 are self-dual,} \\ &\text{(b)} \qquad *\,\utilde{S}^{\dot{A}\dot{B}} = -\utilde{S}^{\dot{A}\dot{B}} \quad , \text{ so that those 3 are anti-self-dual.} \end{aligned} \tag{3.3}$$

Having now a good understanding of these basis 2-forms, we may now generalize the quadratic identities for Pauli matrices, Eqs. (2.8), that we used earlier.

$$\begin{aligned} &\sigma^{\alpha A\dot{C}}\sigma^{\beta B}{}_{\dot{C}} = -g^{\alpha\beta}\epsilon^{AB} + S^{\alpha\beta AB} , \\ &\Longrightarrow \quad \begin{cases} \sigma^{\alpha[A\dot{C}}\sigma^{\beta B]}{}_{\dot{C}} = -g^{\alpha\beta}\epsilon^{AB} = \sigma^{(\alpha A\dot{C}}\sigma^{\beta)B}{}_{\dot{C}} , \\ \sigma^{\alpha(A\dot{C}}\sigma^{\beta B)}{}_{\dot{C}} = S^{\alpha\beta AB} = \sigma^{[\alpha A\dot{C}}\sigma^{\beta]B}{}_{\dot{C}} . \end{cases} \end{aligned} \tag{3.4}$$

This also gives us sufficient data to work out various general relations between the wedge products of these matrices of 2-forms themselves:

$$\begin{aligned} &\utilde{S}^{AB} \wedge \utilde{S}_{CD} = 2\delta^A_{(C}\delta^B_{D)}\utilde{V} = -\utilde{S}^{\dot{A}\dot{B}} \wedge \utilde{S}_{\dot{C}\dot{D}} \quad , \\ &S_{\alpha\beta}{}^{AB}S^{\alpha\beta}{}_{CD} = 4\delta^A_{(C}\delta^B_{D)} \\ &S^{\alpha\beta}{}_{AB}S_{\gamma\delta}{}^{AB} = 2\delta^\alpha_{[\gamma}\delta^\beta_{\delta]} + 2\eta^{\alpha\beta\sigma\tau}g_{\gamma\sigma}g_{\delta\tau} \quad , \\ &S_{\alpha\beta}{}^{\dot{A}\dot{B}}S^{\alpha\beta}{}_{\dot{C}\dot{D}} = 4\delta^{\dot{A}}_{(\dot{C}}\delta^{\dot{B}}_{\dot{D})} \\ &S^{\alpha\beta}{}_{\dot{A}\dot{B}}S_{\gamma\delta}{}^{\dot{A}\dot{B}} = 2\delta^\alpha_{[\gamma}\delta^\beta_{\delta]} + 2\eta^{\alpha\beta\sigma\tau}g_{\gamma\sigma}g_{\delta\tau} \quad , \\ &S_{\alpha\beta}{}^{AB}S^{\alpha\beta}{}_{\dot{C}\dot{D}} = 0 \quad , \end{aligned} \tag{3.5}$$

where $\utilde{V}$ is the volume form for our spacetime:

$$\utilde{V} \equiv \utilde{\omega}^1 \wedge \utilde{\omega}^2 \wedge \utilde{\omega}^3 \wedge \utilde{\omega}^4 \text{ is the “volume form” for the space.} \tag{3.6}$$

As a supplement to the comments concerning duality for 2-forms, I append the statements for 3-forms and 4-forms:

$$*\utilde{\omega}^{A\dot{B}} = \tfrac{1}{3}\utilde{\omega}^{A\dot{C}} \wedge \utilde{S}_{\dot{C}}{}^{\dot{B}} = \tfrac{1}{3}\utilde{S}^{AC} \wedge \utilde{\omega}_C{}^{\dot{B}} , \qquad \text{and } *\utilde{V} = +1 \quad . \tag{3.7}$$

Since the two (matrix) sets of 2-forms contain all 6 of the basis 2-forms, we are not surprised when told that the "converse" of these statements is most simply written as the following equality:

$$\underset{\sim}{\omega}^{A\dot{B}} \wedge \underset{\sim}{\omega}^{C\dot{D}} = \epsilon^{\dot{B}\dot{D}} \underset{\sim}{S}^{AC} + \epsilon^{AC} \underset{\sim}{S}^{\dot{B}\dot{D}} \quad . \tag{3.8}$$

which allows us to split an arbitrary 2-form into its two parts, i.e., its self-dual part and its anti-self-dual part:

$$\underset{\sim}{F} = \hat{F}_{\alpha\beta}\, \underset{\sim}{\omega}^{\alpha} \wedge \underset{\sim}{\omega}^{\beta} = F_{AB}\, \underset{\sim}{S}^{AB} + F_{\dot{A}\dot{B}}\, \underset{\sim}{S}^{\dot{A}\dot{B}} \quad , \tag{3.9}$$

where

$$F_{AB} = \tfrac{1}{8} S^{\alpha\beta}{}_{AB}\, F_{\alpha\beta} = \overline{F_{\dot{A}\dot{B}}} \quad , \tag{3.10}$$

so that

$$\begin{aligned} &F_{AB}\, \underset{\sim}{S}^{AB} \quad \text{is a self-dual 2-form,} \\ &F_{\dot{A}\dot{B}}\, \underset{\sim}{S}^{\dot{A}\dot{B}} \quad \text{is an anti-self-dual 2-form.} \end{aligned} \tag{3.11}$$

This gives us the promised "explanation" of the physical meaning of symmetric, 2nd-rank spinors, namely that those in $V^2 \otimes V^2$ are the image of the (3 independent degrees of freedom of the) self-dual parts of 2-forms over spacetime, while those in $\overline{V^2} \otimes \overline{V^2}$, i.e., those with 2 dotted indices, are the image of the anti-self-dual parts.

A choice for an explicit labelling for the 3 degrees of freedom of any such symmetric, 2nd-rank spinor, say F_{AB}, is as follows:

$$F_{AB} = \frac{1}{4} \begin{pmatrix} C^{+} & -C^{0} \\ -C^{0} & -C^{-} \end{pmatrix} , \qquad C^{\pm} \equiv C^{x} \pm iC^{y} \ , \quad C^{0} \equiv C^{z} \ . \tag{3.12}$$

Since these are only 2×2 matrices, one may always decompose a symmetric, 2nd-rank spinor, such as F_{AB} in terms of two 1-spinors, k_A and ℓ_B, and a scalar λ, which amounts to a magnitude:

$$F_{AB} = 2\lambda\, k_{(A}\ell_{B)}, \text{ along with } \begin{cases} k^{A}\ell_{A} = 1\,, & \text{so that } \det(F_{AB}) \neq 0\,, \\ k^{A}\ell_{A} = 0\,, & \text{so that } \det(F_{AB}) = 0\,. \end{cases} \tag{3.13}$$

These quantities may be thought of as "eigenspinors," with eigenvalues $\pm\lambda$, since one has the following obvious consequences:

$$F_{AB}\, k^B = \lambda\, k_A \quad , \qquad F_{AB}\, \ell^B = -\lambda\, \ell_A \quad . \tag{3.14}$$

Having such a decomposition of our self-dual 2-form, for example, we may use these two eigenspinors to create a specially-oriented tetrad back in spacetime, following the model given in Eqs. (3.2), which puts the original 2-form in its optimally-simple form.

IV. Indicial Approach to the Spinor Transformation Equations

Having now more details about the place(s) where the various combinations of spinors exist, we may now return to our original understandings of the matrices forming the spinor transformations and put more matrix indices into those equations. We began with Eqs. (0.7), relating the transformations of 1-forms and of Hermitian, 2×2 matrices over spacetime. We now propose to rewrite those equations from an "earlier" point of view, with the fundamental proposition that the *unimodular* matrices A, used in §0, are the same as the basic transformation matrices for our fundamental spinor space, V^2, as described in Eq. (1.3): Therefore, our Hermitian matrices would transform as follows, where, for example, we pick a (null) 1-form, $\mathbf{Y}$, created from an arbitrary $k^A \in V^2$:

$$\begin{aligned} Y'^{\,R\dot{S}} = k'^{\,R} k'^{\,\dot{S}} = A^R{}_B\, \overline{A^S{}_E}\, k^B k^{\dot{E}} = A^R{}_B\, A^{\dot{S}}{}_{\dot{E}}\, Y^{B\dot{E}} \\ \text{or, in matrix form} \quad Y' = A\, Y\, \overline{A^T} = A\, Y\, A^\dagger \; . \end{aligned} \tag{4.1}$$

At this point a minor aside is of some value, relative to the equations describing the Lie algebra for these matrices, A, involving representations in terms of the Pauli sigma matrices, as noted at Eqs. (0.9) and (0.13), and the equations describing the use of the Pauli sigma matrices to form a basis set for 1-forms, already at Eqs. (0.3). In principle these two sets of objects are quite different:

a. the basis 1-forms are objects from the product spinor space, $V^2 \otimes \overline{V}^2$, i.e., a spinorial tensor of the form $\sigma^{A\dot{B}}$. We used the "bridge spinors" to create this mapping. As they are a standard set of 2×2 matrices, it was most convenient to describe them by saying that they had the shape of the (standard) Pauli sigma matrices, at least in that particular frame, while

b. the identification of the generators of the Lie algebra, $\theta\hat{e}\cdot\boldsymbol{\mathfrak{J}} + \lambda\hat{v}\cdot\boldsymbol{\mathfrak{K}}$, must clearly have their presentation in terms of objects that live in the product spinor space, $V^2 \otimes V^2_*$, i.e., a spinorial tensor of the form $\sigma^A{}_B$, so that the use of the actual matrices $\widetilde{\alpha}\cdot\widetilde{\sigma}$ to describe them is again simply a reasonable way to present the desired form of the matrices in terms of a standard, well-defined set of matrices.

This means that there are no particular spinor indices associated to the actual Pauli matrices; rather, we must know the physical intent/content of the objects being considered in order to know how they should transform. Because of this, we can quickly note that although both the elements of $\underset{\approx}{\boldsymbol{\omega}}^{A\dot{B}}$ and of $\left(\theta\hat{e}\cdot\boldsymbol{\mathfrak{J}} + \lambda\hat{v}\cdot\boldsymbol{\mathfrak{K}}\right)^A{}_B$ are given in terms of the Pauli sigma matrices in a particular choice of basis, if we were to transform both of these quantities to some different frame they would transform differently so that they would no longer appear to use the *same* set of matrices for their presentations.

Returning, now, to Eqs. (0.15), which give the explicit formulation of the representations, within $\boldsymbol{SL}(2,\mathbb{C})$, of rotations and boosts, the description above suggests that if the set of all unimodular matrices, A, constitutes a set of transformation equations for V^2, while the set of all of the complex conjugates of these matrices constitutes a set of transformation equations for the complex-conjugate spinor space, $\overline{V}^2$, these being in fact two different representations of the Lorentz group. Choosing a convention and agreeing that the representation usually labelled as $D(0,\frac{1}{2})$ is the one given in Eqs. (0.13) and (0.15), the alternative one is the complex conjugate representation, $D(\frac{1}{2},0)$:

$$
\begin{aligned}
D(0,\tfrac{1}{2}) &\Longrightarrow \begin{cases} \vec{\boldsymbol{\mathfrak{J}}} \Longrightarrow -\frac{i}{2}\vec{\sigma} \\ \vec{\boldsymbol{\mathfrak{K}}} \Longrightarrow +\frac{1}{2}\vec{\sigma} \end{cases} \Longrightarrow \begin{cases} D^A{}_B[R(\theta;\hat{e})] = (e^{-i(\theta/2)\,\hat{e}\cdot\vec{\sigma}})^A{}_B \\ D^A{}_B[B(\lambda;\hat{v})] = (e^{+(\lambda/2)\hat{v}\cdot\vec{\sigma}})^A{}_B \end{cases}, \\
D(\tfrac{1}{2},0) &\Longrightarrow \begin{cases} \vec{\boldsymbol{\mathfrak{J}}} \Longrightarrow -\frac{i}{2}\vec{\sigma} \\ \vec{\boldsymbol{\mathfrak{K}}} \Longrightarrow -\frac{1}{2}\vec{\sigma} \end{cases} \Longrightarrow \begin{cases} D^{\dot{A}}{}_{\dot{B}}[R(\theta;\hat{e})] = (e^{-i(\theta/2)\,\hat{e}\cdot\vec{\sigma}})^{\dot{A}}{}_{\dot{B}} \\ D^A{}_B[B(\lambda;\hat{v})] = (e^{-(\lambda/2)\hat{v}\cdot\vec{\sigma}})^{\dot{A}}{}_{\dot{B}} \end{cases},
\end{aligned}
\tag{4.2}
$$

where it is true that if one needed to do so, for clarity of notation, one could have written $(D^{(0,1/2)})^{A}{}_{B}$ and $(D^{(1/2,0)})^{\dot{A}}{}_{\dot{B}}$, respectively. On the other hand, if you do believe that one obtains the matrices for the representation $D(1/2,0)$ from those for $D(0,1/2)$ by complex conjugation, as is indeed stated above, you might wonder. This wonderment occurs because σ_y is imaginary while the other two are real, so that complex conjugation actually does strange things to some quantity such as $\hat{v}\cdot\vec{\sigma}$; however, this strangeness may be cleaned up by an appropriate change of basis. More precisely, notice that a change of basis that amounts to a rotation about the $\hat{y}$-axis by angle $\theta=\pi$ would change the sign of $\hat{x}$ and $\hat{z}$ but leave invariant the sign of $\hat{y}$; coupled with a complex-conjugation this would change the sign of the entire vector, $\vec{\sigma}$, making it appear much less strange. Using the generic expansion given in Eq. (0.9) and the formulation for $D(0,1/2)$ given in Eqs. (0.15), we see that such a rotation would be given simply by the matrix $i\sigma_y$, with its inverse $-i\sigma_y=(+i\sigma_y)^{-1}$; therefore, we may quickly write that if we first complex conjugate the matrix elements, and then perform this transformation we would have the following effects on the generators, and on the matrix elements:

$$(i\sigma_y)\,\overline{\begin{pmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\end{pmatrix}}\,(-i\sigma_y)=-\begin{pmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\end{pmatrix}\,,\quad\Longrightarrow\quad\begin{cases}(i\sigma_y)\,(\overline{e^{-i(\theta/2)\,\hat{e}\cdot\vec{\sigma}}})(-i\sigma_y)=e^{-i(\theta/2)\,\hat{e}\cdot\vec{\sigma}}\\[1ex] (i\sigma_y)\,(\overline{e^{+(\lambda/2)\,\hat{v}\cdot\vec{\sigma}}})(-i\sigma_y)=e^{-(\lambda/2)\,\hat{v}\cdot\vec{\sigma}}\end{cases}\tag{4.3}$$

We may now completely finish up our description of the representations above in more detail: We want to find the expicit relations between the matrix $\mathbf{A}\in SL(2,\mathcal{C})$ and the matrix $L\in SO(3,1)$. We first set down the defining equations for an arbitrary tangent vector, $\widetilde{Y}$, the associated 1-form, $\underset{\sim}{Y}$, and for their Hermitian matrix presentations with either upper or lower indices, which are just completions of the details already mentioned in Eqs. (4.1):

$$\begin{gathered}Y'^{\,\mu}=L^{\mu}{}_{\alpha}\,Y^{\alpha}\,,\quad Y'{}_{\nu}=(L^{-1})^{\alpha}{}_{\nu}\,Y_{\alpha}=L_{\nu}{}^{\alpha}\,Y_{\alpha}\,,\qquad g_{\alpha\beta}=L^{\mu}{}_{\alpha}\,L^{\nu}{}_{\beta}\,g_{\mu\nu}\,,\\ Y'^{\,R\dot{S}}=A^{R}{}_{C}\,A^{\dot{S}}{}_{\dot{E}}\,Y^{C\dot{E}}\,,\quad Y'{}_{R\dot{S}}=(A^{-1})^{C}{}_{R}\,(A^{-1})^{\dot{E}}{}_{\dot{S}}\,Y_{C\dot{E}}=A_{R}{}^{C}\,A_{\dot{S}}{}^{\dot{E}}\,Y_{C\dot{E}}\,,\\ \eta_{\mu\nu}\,L^{\mu}{}_{\alpha}\,L^{\nu}{}_{\beta}=\eta_{\alpha\beta}\quad\Longrightarrow\quad L_{\kappa}{}^{\alpha}=(L^{-1})^{\alpha}{}_{\kappa}\ \text{ or }\ (HLH^{-1})^{T}=L^{-1}\,,\\ +1=\det A\quad\Longrightarrow\epsilon_{AB}\,A^{A}{}_{M}\,A^{B}{}_{N}=\epsilon_{MN}\Longrightarrow-A_{B}{}^{M}=+(A^{-1})^{M}{}_{B}\ \text{ or }\ (\epsilon A\epsilon^{T})^{T}=A^{-1}\,.\end{gathered}\tag{4.4}$$

These relations allow us to "factor out" the components of the 1-form itself, from which we may infer the equality of the action of the Lorentz transformation on the spacetime indices of the sigma matrices and the action of the spinorial transformation matrices on the matrix indices of those same sigma matrices:

$$L^\mu{}_\alpha\, \sigma_\mu^{R\dot{S}} = A^R{}_C\, A^{\dot{S}}{}_{\dot{E}}\, \sigma_\alpha^{C\dot{E}} \quad \text{or} \quad L^\mu{}_\alpha\, \sigma_\mu = A\, \sigma_\alpha\, \overline{A}^T = A\, \sigma_\alpha\, A^\dagger \quad , \tag{4.5}$$

Using now the equation for the trace of the product of two sigma matrices with different spacetime indices, from the first line of Eqs. (3.5), we may resolve this equality for the matrix elements of the Lorentz transformation desired:

$$L^\mu{}_\beta = -\hat{A}^R{}_C\, \sigma_\beta^{C\dot{E}}\, A^{\dot{S}}{}_{\dot{E}}\, \sigma^\mu_{R\dot{S}}\ , \text{ or suppressing spinor indices} \quad L^\mu{}_\beta = -\tfrac{1}{2}\mathrm{tr}(A\sigma_\beta A^\dagger \sigma^{\mu T}) \quad . \tag{4.6}$$

We had two earlier discussions using two basis spinors, k^A and ℓ^B. We could use them to create a null basis in spacetime, as discussed following Eqs. (2.13), and, also, we could use them to determine a complete set of eigenspinors for an arbitrary symmetric, second-rank spinor, as in the discussion near Eqs. (3.9). We may now use that same thought to create a somewhat different parametrization for the Lorentz transformations. In both those cases there is an ambiguity that arises in their choice: Having chosen some such pair of spinors, one still has the freedom to change to a new pair, k'_A and/or ℓ'_B, related to the original ones by any one, *or more*, of the following transformations:

$$A: \quad \begin{cases} A_\sigma : \begin{cases} k_A{}' = e^\sigma k_A \\ l_A{}' = e^{-\sigma} l_A{}' \end{cases} & , \\ A_\rho : \begin{cases} k_A{}' = k_A \\ l_A{}' = l_A - \rho k_A \end{cases} & , \\ A_\eta : \begin{cases} k_A{}' = k_A - \eta l_A \\ l_A{}' = l_A \end{cases} & . \end{cases} \tag{4.7}$$

Treating each of these 3 parameters, σ, ρ, and η as a complex number, this gives us a parametrization of a 6-real-parameter group, identifying the ambiguity with which we could have chosen this pair of spinors so as to constitute a spinorial basis, and, thereby, a null basis for spacetime. Therefore they constitute 6 degrees of freedom for the choice of such a basis. This freedom in choice of a basis then should correspond to the 6 degrees of freedom for Lorentz transformations, since they also preserve the character of a basis set.

Each of these transformations has the property that it preserves some 4-vector of zero length, while the standard "rotations" and "boosts" each preserve two vectors, one of which is spacelike and one timelike—for rotations—while for boosts both of them are spacelike. If we choose the unprimed versions of these two spinors as a basis for spinorial matrices, then we may write down the 2×2 matrices associated with these 3 sorts of Lorentz transformations as follows:

$$A: \quad \begin{cases} A_\sigma = \begin{pmatrix} e^\sigma & 0 \\ 0 & e^{-\sigma} \end{pmatrix} \quad , & \sigma\text{-transformations, for } \sigma \in \mathcal{C} \\ A_\rho = \begin{pmatrix} 1 & -\rho \\ 0 & 1 \end{pmatrix} \quad , & \rho\text{-transformations, for } \rho \in \mathcal{C} \\ A_\eta = \begin{pmatrix} 1 & 0 \\ -\eta & 1 \end{pmatrix} \quad , & \eta\text{-transformations, for } \eta \in \mathcal{C} \end{cases} \tag{4.8}$$

Of course we can use, for instance, Eqs. (4.6) to move these transformations into spacetime. Perhaps the simplest way to do this is to write them in a matrix format that makes changes on the null basis sets for 1-forms, $\underset{\sim}{\theta}^\alpha$:

$$\underset{\sim}{\theta}'^\alpha \equiv L^\alpha{}_\nu \, \underset{\sim}{\theta}^\nu \; : \qquad \begin{cases} L_\sigma = \begin{pmatrix} e^{2i\,\mathrm{Im}(\sigma)} & 0 & 0 & 0 \\ 0 & e^{-2i\,\mathrm{Im}(\sigma)} & 0 & 0 \\ 0 & 0 & e^{2\,\mathrm{Re}(\sigma)} & 0 \\ 0 & 0 & 0 & e^{-2\,\mathrm{Re}(\sigma)} \end{pmatrix} \quad , \\ L_\rho = \begin{pmatrix} 1 & 0 & -\overline{\rho} & 0 \\ 0 & 1 & -\rho & 0 \\ 0 & 0 & 1 & 0 \\ \rho & \overline{\rho} & -\rho\overline{\rho} & 1 \end{pmatrix} \quad , \\ L_\eta = \begin{pmatrix} 1 & 0 & 0 & -\eta \\ 0 & 1 & 0 & -\overline{\eta} \\ \overline{\eta} & \eta & 1 & -\eta\overline{\eta} \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad . \end{cases} \tag{4.9}$$

One sees that the σ-transformations amount to re-scalings of the (affine) parameters along the null directions, while the other transformations are of the form as to preserve one of the real null-directions while "rotating" all the other directions "around" that one.

As these have been constructed from our simple, spinor presentations of $\boldsymbol{SL}(2,\mathbb{C})$, so that the (4-dimensional) basis vectors are all null vectors, I also present here their form if we choose the more-standard orthonormal, Cartesian-type basis vectors, $\underset{\sim}{\omega}^{\hat{\alpha}}$, related to the null ones via a matrix M given in Eq. (2.5), where we denote the actual matrices, relative to this orthonormal basis, by the symbol LL instead of just L:

$$
\underset{\sim}{\omega}'^{\hat{\alpha}} \equiv LL^{\hat{\alpha}}{}_{\hat{\nu}}\, \underset{\sim}{\omega}^{\hat{\nu}} \ : \quad
\begin{cases}
LL_\sigma = \begin{pmatrix} \cos(2r) & -\sin(2r) & 0 & 0 \\ \sin(2r) & \cos(2r) & 0 & 0 \\ 0 & 0 & \cosh(2s) & \sinh(2s) \\ 0 & 0 & \sinh(2s) & \cosh(2s) \end{pmatrix} , \\[2ex]
LL_\rho = \begin{pmatrix} 1 & 0 & -p & -p \\ 0 & 1 & +q & +q \\ +p & -q & 1-\frac{1}{2}|\rho|^2/2 & -\frac{1}{2}|\rho|^2 \\ -p & +q & \frac{1}{2}|\rho|^2 & 1+\frac{1}{2}|\rho|^2 \end{pmatrix} , \\[2ex]
LL_\eta = \begin{pmatrix} 1 & 0 & -n & +n \\ 0 & 1 & -m & +m \\ +n & +m & 1-\frac{1}{2}|\eta|^2 & \frac{1}{2}|\eta|^2 \\ +n & +m & -\frac{1}{2}|\eta|^2 & 1+\frac{1}{2}|\eta|^2 \end{pmatrix} ,
\end{cases}
\qquad (4.10)
$$

$$
\sigma \equiv s + ir\,, \quad \rho \equiv p + iq\,, \quad \eta \equiv n + im\ .
$$

Another interesting comment concerns the generators of these transformations, which we present reverting back to the null basis set: Hopefully it is straightforward to see that if we write any one of these transformations as the exponential of its logarithm, i.e., we write $L_i \equiv e^{Q_i}$, then we have

$$
\begin{aligned}
Q_\sigma &= \begin{pmatrix} \sigma-\overline{\sigma} & 0 & 0 & 0 \\ 0 & -\sigma+\overline{\sigma} & 0 & 0 \\ 0 & 0 & \sigma+\overline{\sigma} & 0 \\ 0 & 0 & 0 & -\sigma-\overline{\sigma} \end{pmatrix} , \\
Q_\rho &= \begin{pmatrix} 0 & 0 & -\overline{\rho} & 0 \\ 0 & 0 & -\rho & 0 \\ 0 & 0 & 0 & 0 \\ \rho & \overline{\rho} & 0 & 0 \end{pmatrix} , \\
Q_\eta &= \begin{pmatrix} 0 & 0 & 0 & -\eta \\ 0 & 0 & 0 & -\overline{\eta} \\ \overline{\eta} & \eta & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} .
\end{aligned}
\qquad (4.11)
$$

The re-scaling transformation has a generator which is diagonal, in this null basis, while the other two are "parabolic." This means that the third power of the generator vanishes; therefore, the exponential series terminates after the second-order term.

V. Higher-order Representations of $\boldsymbol{SL}(2,\mathbb{C})$

A different way to create the original, 4×4 matrix, transformations on spacetime from the 2×2 spinorial transformations is to start looking at higher-order spinor transformations. We have already described carefully the two basic representations, $D(0, 1/2)$—corresponding to transformations on V^2—and $D(1/2, 0)$—corresponding to transformations on $\overline{V}^2$. We may therefore begin to take various products of them. The representations $D(0, n/2)$, for any integer n, correspond to transformations acting on spinorial tensors with n indices, completely symmetric under interchange of any of them, such as, say, $F_{A_1 A_2 \ldots A_n}$. These spinors must be symmetric because any pair of skew-symmetric indices in spinor space can only really take on one value, either $1, 2$, or $2, 1$, and, therefore, must be proportional to the Levi-Civita symbol there. We can demonstrate this easily, using, first, a second-rank spinorial quantity:

$$Z_{AB} = \tfrac{1}{2}\left(Z_{AB} + Z_{BA}\right) + \tfrac{1}{2}\left(Z_{AB} - Z_{BA}\right) = \tfrac{1}{2}\left(Z_{AB} + Z_{BA}\right) + \tfrac{1}{2}\epsilon_{AB}\,\epsilon^{CD}\,Z_{CD}\,. \tag{5.1}$$

This tells us that that portion of a tensor that has a pair of indices for which the tensor is skew symmetric must be proportional to just the Levi-Civita symbol, with the proportionality actually being the trace, $\epsilon^{AB}\,Z_{AB}$. As well, it is clear that, in a 2-dimensional space, a non-zero tensor cannot be skew-symmetric on an entire triplet of indices; therefore, we can always reduce an arbitrary n-th rank spinorial tensor to one which is totally symmetric in all n indices, plus some others which are products of some number of Levi-Civita symbols multiplied by tensors of lower order, i.e., of order $n-2$, $n-4$, and so forth.

An entirely other way to approach this question comes from thinking about the representation theory for our transformation group. Since all the matrices have determinant one, the Levi-Civita symbol is actually an invariant tensor, and therefore may be said to correspond to a 1-dimensional representation, $D(0,0)$, where every such transformation in the original group, $\boldsymbol{SL}(2,\mathbb{C})$ is represented by just the scalar $+1$. It is also obvious, I suggest, that the summed out trace, $Z^{AB}\,\epsilon_{AB}$, is also an invariant, and therefore corresponds to transformation by the representation $D(0,0)$. However, we now consider the product of $D(0,1/2) \otimes D(0,1/2)$, which must equal the direct sum, $D(0,1) \oplus D(0,0)$. In geometrical language this statement about representations says that an arbitrary tensor with two indices of the same kind, i.e., an element of the direct product space, $V^2 \otimes V^2$, should decompose into a subspace with dimension 3, corresponding to the representation $D(0,1)$, and a subspace with dimension 1, corresponding to the representation $D(0,0)$. The subspace of dimension 3 is of course the subspace symmetric on the pair of indices, as described in Eq. (4.1) above, and the subspace of dimension 1 is the skew-symmetric part, made up of the invariant Levi-Civita tensor multiplied by some scalar quantity, the "trace" of that second-rank tensor. Likewise consideration of the decomposition of the product $D(0,1) \otimes D(0,1/2) = D(0,3/2) \oplus D(0,1/2)$ should start us with $F_{AB}W_C$, where F_{AB} is symmetric, and end us with a 3-index, symmetric tensor and a Levi-Civita symbol picking up two indices and an additional one left over, for the $D(0,1/2)$.

Let us now go on to more complicated forms. We of course begin with the representation $D(1/2,1/2) = D(0,1/2) \otimes D(1/2,0)$, appropriate for the matrix presentation of real 4-vectors (or 4-dimensional 1-forms). To see the details of the representation in more detail, we should try to create the 4-dimensional generators, beginning of course from the presentations given earlier for the two 2-dimensional representations. To do this, we will have to "back up" slightly, and consider what happens to the generators when considering a product representation. We understand that the matrices representing the group elements of a product representation correspond to direct products of the individual representations of that group element. However, one must be somewhat more careful with respect to the generators. The following is the right approach for the generators, for any Lie group, G, with Lie algebra, $\mathcal{G}$. Let us suppose that we have been an element $g \in G$ such that there exists $Q \in \mathcal{G}$ such that $G = e^Q$. Further let us consider the situation when we have two distinct representations of G, which we label by $D_1(g)$ and $D_2(g)$. Therefore, we also can find, and label, the representations of the generator Q, simply by $Q^{(1)}$ and $Q^{(2)}$, so that

$$D_1(g) = e^{Q^{(1)}} \ , \quad D_2(g) = e^{Q^{(2)}} \ . \tag{5.2}$$

Then we find that the generators of the product representation are related to those of the individual representations in the following way:

$$\{D_1 \otimes D_2\}(g) \equiv [D_1(g)] \otimes [D_2(g)] \equiv e^{Q^{(1\oplus 2)}} \ , \quad Q^{(1\oplus 2)} \equiv Q^{(1)} \otimes I_2 + I_1 \otimes Q^{(2)} \ , \tag{5.3}$$

where I_i indicates the matrix which is the identity matrix in representation D_i. To verify this we consider the following, setting, first $T \equiv Q^{(1)}$:

$$\begin{aligned} e^{T\otimes I_2} &= I_{1\otimes 2} + T \otimes I_2 + \frac{1}{2!}(T \otimes I_2)(T \otimes I_2) + \frac{1}{3!}(T \otimes I_2)(T \otimes I_2)(T \otimes I_2) + \ \ldots \\ &= \left\{ I_1 + T + \frac{1}{2!}T^2 + \frac{1}{3!}T^3 + \ \ldots \ \right\} \otimes I_2 = D_1(g) \otimes I_2 \ . \end{aligned} \tag{5.4a}$$

We have an entirely analogous construction for the exponential of $I_1 \otimes Q^{(2)}$, so that we may finally write

$$\begin{aligned} e^{Q^{(1)}\otimes I_2 + I_1 \otimes Q^{(2)}} &= e^{Q^{(1)}\otimes I_2}\, e^{I_1\otimes Q^{(2)}} = \{D_1(g) \otimes I_2\}\, \{I_1 \otimes D_2(g)\} \\ &= D_1(g) \otimes D_2(g) \equiv \{D_1 \otimes D_2\}(g) \ , \end{aligned} \tag{5.4b}$$

where we have used the "obvious" fact that $Q^{(1)} \otimes I_2$ commutes with $I_1 \otimes Q^{(2)}$.

With that notion in hand, we may immediately use Eq. (4.2) to write down the generators for the representation $D(1/2, 1/2)$:

$$\vec{\mathfrak{J}} \Longrightarrow \ -\tfrac{i}{2}\{\vec{\sigma} \otimes I_2 + I_2 \otimes \vec{\sigma}\} \ , \qquad \vec{\mathfrak{K}} \Longrightarrow \ \tfrac{1}{2}\{\vec{\sigma} \otimes I_2 - I_2 \otimes \vec{\sigma}\}$$

$$\begin{aligned} &\mathfrak{J}^x \Longrightarrow -\frac{i}{2}\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} , \ \mathfrak{J}^y \Longrightarrow +\frac{1}{2}\begin{pmatrix} 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix} , \ \mathfrak{J}^z \Longrightarrow i\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \\ &\mathfrak{K}^x \Longrightarrow \frac{1}{2}\begin{pmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{pmatrix} , \ \mathfrak{K}^y \Longrightarrow +\frac{i}{2}\begin{pmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{pmatrix} , \ \mathfrak{K}^z \Longrightarrow \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \end{aligned} \tag{5.5}$$

You should now immediately point out that we already know what the generators $\vec{\mathfrak{J}}$ and $\vec{\mathfrak{K}}$ look like in 4 dimensions, as they act on tangent vectors in spacetime, say, and the ones above are not those at all. This is certainly true; however, as usual, the problem here is that these are given with respect to a different set of basis vectors. With some algebra it is not too hard to show the following:

$$
W\,\vec{\mathcal{M}}^{(1/2,1/2)}\,W^{-1} = \vec{\mathcal{M}}^{(3+1)}\ ,
$$
$$
W = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & -1 \\ i & 0 & 0 & i \\ 0 & -1 & -1 & 0 \\ 0 & +1 & -1 & 0 \end{pmatrix}\ , \tag{5.6}
$$

where $\mathcal{M}$ is meant to indicate either $\mathcal{J}$ or $\mathcal{K}$, while the representation we just created is labelled by the superscript $(1/2,1/2)$ and the more customary representation is labelled by $(3+1)$, i.e., that one is our original $\{x,y,z,t\}$ representation for these generators. This tells us, as expected, that the standard matrices for $D(1/2,1/2)$ are simply given with respect to a different basis, built on $(x\pm iy)/\sqrt{2}$ and $(\pm z-t)/\sqrt{2}$, a choice of new basis which is not too surprising.

VI. Differential Relationships

Denoting by ∇ the covariant derivative operator for an arbitrary direction, then $\nabla\,\underset{\sim}{\omega}^{A\dot{B}}$ is the tensor product of two 1-forms, which then must contain the same information as does $\nabla\,\underset{\sim}{\omega}^{\mu}$; recall the connection coefficients are defined so that

$$
\begin{aligned}
\nabla_\alpha \underset{\sim}{\omega}^{\beta} &= -\Gamma^{\beta}{}_{\gamma\alpha}\underset{\sim}{\omega}^{\gamma}\ ,\\
\nabla_\alpha\,\underset{\sim}{\omega}^{A\dot{B}} &= \hat{\Gamma}^{\beta}_{\gamma\alpha}\,\sigma_\beta{}^{A\dot{B}}\,\sigma^{\gamma}{}_{C\dot{D}}\,\underset{\sim}{\omega}^{C\dot{D}} = \hat{\Gamma}_{\beta\gamma\alpha}\,\sigma^{[\beta\,A\dot{B}}\sigma^{\gamma]\,C\dot{D}}\,\underset{\sim}{\omega}_{C\dot{D}}\\
&= \tfrac{1}{4}\Gamma_{\beta\gamma\alpha}\left(\underset{\sim}{S}^{\beta\gamma\,AC}\epsilon^{\dot{B}\dot{D}} + \underset{\sim}{S}^{\beta\gamma\,\dot{B}\dot{D}}\epsilon^{AC}\right)\underset{\sim}{\omega}_{C\dot{D}}\\
&\equiv \Gamma^{AC}{}_{\alpha}\underset{\sim}{\omega}_{C}{}^{\dot{B}} + \Gamma^{\dot{B}\dot{D}}{}_{\alpha}\,\underset{\sim}{\omega}^{A}{}_{\dot{D}}\quad ,
\end{aligned} \tag{6.1}
$$

where we have defined the components of the 2 spinor-valued connections by

$$
\begin{aligned}
\underset{\sim}{\Gamma}^{AC} &\equiv \Gamma^{AC}{}_{\alpha}\,\underset{\sim}{\omega}^{\alpha} \equiv -\tfrac{1}{4}\Gamma_{\beta\gamma\alpha}\,S^{\beta\gamma\,AC}\underset{\sim}{\omega}^{\alpha}\quad ,\\
\underset{\sim}{\Gamma}^{\dot{B}\dot{D}} &\equiv \Gamma^{\dot{B}\dot{D}}{}_{\alpha}\underset{\sim}{\omega}^{\alpha} \equiv -\tfrac{1}{4}\Gamma_{\beta\gamma\alpha}S^{\beta\gamma\,\dot{B}\dot{D}}\underset{\sim}{\omega}^{\alpha}\quad .
\end{aligned} \tag{6.2}
$$

Making explicit the compatibility of the connection with the spinor/vector transformation mapping, i.e., $d\sigma^{\alpha\,A\dot{B}} = 0$, we may now re-write the first structure equations, *and* the corresponding equations for self-dual 2-forms:

$$
d\underset{\sim}{\omega}^{A\dot{B}} + \underset{\sim}{\omega}^{A\dot{D}}\wedge\underset{\sim}{\Gamma}^{\dot{B}}{}_{\dot{D}} + \underset{\sim}{\omega}^{C\dot{B}}\wedge\underset{\sim}{\Gamma}^{A}{}_{C} = 0\quad , \tag{6.3}
$$

$$
\begin{aligned}
d\underset{\sim}{S}^{AB} + \underset{\sim}{S}^{AC} \wedge \underset{\sim}{\Gamma}^{B}{}_{C} + \underset{\sim}{S}^{BC} \wedge \underset{\sim}{\Gamma}^{A}{}_{C} &= d\underset{\sim}{S}^{AB} + 2\underset{\sim}{S}^{C(A} \wedge \underset{\sim}{\Gamma}^{B)}{}_{C} = 0 \quad , \\
d\underset{\sim}{S}^{\dot{A}\dot{B}} + \underset{\sim}{S}^{\dot{C}\dot{A}} \wedge \underset{\sim}{\Gamma}^{\dot{B}}{}_{\dot{C}} + \underset{\sim}{S}^{\dot{C}\dot{B}} \wedge \underset{\sim}{\Gamma}^{\dot{A}}{}_{\dot{C}} &= d\underset{\sim}{S}^{\dot{A}\dot{B}} + 2\underset{\sim}{S}^{(\dot{A}\dot{C}} \wedge \underset{\sim}{\Gamma}^{\dot{B})}{}_{\dot{C}} = 0 \quad ,
\end{aligned}
\tag{6.4}
$$

where the 1-forms $\underset{\sim}{\Gamma}^{AB}$ and $\underset{\sim}{\Gamma}^{\dot{A}\dot{B}}$ are **the** connection 1-forms for the "un-dotted" and "dotted" indices, respectively. From the point of view of the spaces of self-dual and anti-self-dual 2-forms over M^4, this means that we have a connection over them, individually and separately!

Therefore, thinking back to the original spinor spaces, this means that we can extend (or prolong) the connection to those spaces by the following:

$$
\begin{aligned}
\nabla k^{A} &\equiv dk^{A} + k^{B}\underset{\sim}{\Gamma}^{A}{}_{B} \quad , \\
\nabla l^{\dot{A}} &\equiv dl^{\dot{A}} + l^{\dot{C}}\underset{\sim}{\Gamma}^{\dot{A}}{}_{\dot{C}} \quad .
\end{aligned}
\tag{6.5}
$$

This allows us to also prolong the original definition of the exterior derivative to a much more "sophisticated" version, usually called "Cartan's exterior derivative operator," $D : \Lambda^p \to \Lambda^{p+1}$, which *also* "notices" whether and what kind of indices an object may have and inserts an appropriate connection 1-form:

$$
\begin{aligned}
D\,\underset{\sim}{\omega}^{\alpha} &\equiv d\underset{\sim}{\omega}^{\alpha} + \underset{\sim}{\Gamma}^{\alpha}{}_{\beta} \wedge \underset{\sim}{\omega}^{\beta} \quad , \\
D\,\underset{\sim}{\Omega}^{\alpha}{}_{\beta} &\equiv d\underset{\sim}{\Omega}^{\alpha}{}_{\beta} + \underset{\sim}{\Gamma}^{\alpha}{}_{\mu} \wedge \underset{\sim}{\Omega}^{\mu}{}_{\beta} - \underset{\sim}{\Gamma}^{\nu}{}_{\beta} \wedge \underset{\sim}{\Omega}^{\alpha}{}_{\nu} \quad , \\
D\,\underset{\sim}{\omega}^{A\dot{B}} &\equiv d\underset{\sim}{\omega}^{A\dot{B}} + \underset{\sim}{\Gamma}^{A}{}_{C} \wedge \underset{\sim}{\omega}^{C\dot{B}} + \underset{\sim}{\Gamma}^{\dot{B}}{}_{\dot{D}} \wedge \underset{\sim}{\omega}^{A\dot{D}} \quad , \\
D\,\underset{\sim}{S}^{AC} &\equiv d\underset{\sim}{S}^{AC} + \underset{\sim}{\Gamma}^{A}{}_{B} \wedge \underset{\sim}{S}^{BC} + \underset{\sim}{\Gamma}^{C}{}_{B} \wedge \underset{\sim}{S}^{AB} \quad .
\end{aligned}
\tag{6.6}
$$

Šince the bundle of self-dual 2-forms now has "its own" connection, $\underset{\sim}{\Gamma}^{AB}$, we may define its curvature 2-form, $\underset{\sim}{\Omega}^{AB}$, and likewise for the anti-self-dual 2-forms:

$$
\begin{aligned}
\underset{\sim}{\Omega}^{A}{}_{B} &\equiv d\underset{\sim}{\Gamma}^{A}{}_{B} + \underset{\sim}{\Gamma}^{A}{}_{C} \wedge \underset{\sim}{\Gamma}^{C}{}_{B} \quad , \\
\underset{\sim}{\Omega}^{\dot{A}}{}_{\dot{B}} &\equiv d\underset{\sim}{\Gamma}^{\dot{A}}{}_{\dot{B}} + \underset{\sim}{\Gamma}^{\dot{A}}{}_{\dot{C}} \wedge \underset{\sim}{\Gamma}^{\dot{C}}{}_{\dot{B}} \quad .
\end{aligned}
\tag{6.7}
$$

and the Bianchi identities take the very simple form

$$
D\,\underset{\sim}{\Omega}^{A}{}_{B} = 0 \qquad , \qquad D\,\underset{\sim}{\Omega}^{\dot{A}}{}_{\dot{B}} = 0 \quad . \tag{6.8}
$$

However, since the curvatures are 2-forms, we may of course write them in terms of their self-dual and anti-self-dual parts, totally analogous to the case for $\underset{\sim}{F}^{AB}$. We define the quantities R_{ABCD} and $R_{AB\dot{C}\dot{D}}$ by the equations

$$\begin{aligned} \underset{\sim}{\Omega}_{AB} &\equiv -\hat{R}_{ABCD}\,\underset{\sim}{S}^{CD} - \hat{R}_{AB\dot{C}\dot{D}}\,\underset{\sim}{S}^{\dot{C}\dot{D}} \quad , \\ \underset{\sim}{\Omega}_{\dot{A}\dot{B}} &\equiv -\hat{R}_{\dot{A}\dot{B}\dot{C}\dot{D}}\,\underset{\sim}{S}^{\dot{C}\dot{D}} - \hat{R}_{\dot{A}\dot{B}CD}\,\underset{\sim}{S}^{CD} \quad , \end{aligned} \tag{6.9}$$

where it is clear that the reality of M^4 generates the relations

$$\overline{R_{AB\dot{C}\dot{D}}} = R_{\dot{A}\dot{B}CD} \qquad , \qquad \overline{R_{ABCD}} = R_{\dot{A}\dot{B}\dot{C}\dot{D}} \quad . \tag{6.10}$$

Also, the inverse equations are

$$\begin{aligned} R_{ABCD} &\equiv \tfrac{1}{16} S^{\alpha\beta}{}_{AB} S^{\gamma\delta}{}_{CD} R_{\alpha\beta\gamma\delta} \quad , \\ R_{AB\dot{C}\dot{D}} &\equiv \tfrac{1}{16} S^{\alpha\beta}{}_{AB} S^{\gamma\delta}{}_{\dot{C}\dot{D}} R_{\alpha\beta\gamma\delta} \quad . \end{aligned} \tag{6.11}$$

We may re-write the commutation relations for two covariant derivatives in terms of these quantities:

$$D \wedge D\, k^A = k^B \underset{\sim}{\Omega}^A{}_B \quad , \quad D \wedge D\, l_B = -l_A \underset{\sim}{\Omega}^A{}_B \quad , \quad D \wedge D\, m^{\dot{A}} = m^{\dot{B}} \underset{\sim}{\Omega}^{\dot{A}}{}_{\dot{B}} \quad , \quad \text{etc.} \tag{6.12}$$

From the symmetry evidenced by Eqs. (6.11), it is straightforward to see that one has $R_{ABCD} = R_{CDAB}$, as well as that $R_{ABCD} = R_{BACD} = R_{ABDC} = R_{BADC}$. **However**, there is no apparent reason to think that R_{ABCD} and R_{ADCB} are equal. Instead, we calculate that

$$\epsilon^{BD}\epsilon^{AC} R_{ABCD} = \tfrac{1}{16} S^{\alpha\beta\, CD} S^{\gamma\delta}{}_{CD} R_{\alpha\beta\gamma\delta} = \cdots = \tfrac{1}{4}\mathcal{R} \text{ , the Ricci scalar} \tag{6.13}$$

so that we may define the completely symmetric quantity

$$C_{ABCD} \equiv R_{ABCD} + \tfrac{1}{12}\epsilon_{A(C}\epsilon_{D)B}\mathcal{R} \quad , \tag{6.14}$$

where C_{ABCD} is indeed symmetric under interchange of any pair of indices, and therefore has 5 independent components. It is the spinorial "image" of the conformal or Weyl tensor, in the sense that a decomposition of the Weyl tensor alone would have resulted in only these quantities:

$$C_{\alpha\beta\gamma\delta} = \tfrac{1}{4} S_{\alpha\beta}{}^{AB} S_{\gamma\delta}{}^{CD} C_{ABCD} + \tfrac{1}{4} S_{\alpha\beta}{}^{\dot{A}\dot{B}} S_{\gamma\delta}{}^{\dot{C}\dot{D}} C_{\dot{A}\dot{B}\dot{C}\dot{D}} \quad . \tag{6.15}$$

For reference, the relation between these two tensors back in M^4 is given by

$$R^{\alpha\beta}{}_{\gamma\delta} \equiv C^{\alpha\beta}{}_{\gamma\delta} + \hat{\delta}^{\alpha\beta\epsilon}_{\gamma\delta\zeta} (\mathcal{R}^{\zeta}{}_{\epsilon} - \tfrac{1}{4} \delta^{\zeta}_{\epsilon} \mathcal{R}) - \tfrac{1}{12} \mathcal{R} \delta^{\alpha\beta}_{\gamma\delta} \quad . \tag{6.16}$$

Likewise, the quantities, $R_{AB\dot{C}\dot{D}} = \overline{R_{\dot{A}\dot{B}CD}}$ correspond to a 3×3, Hermitian matrix that, therefore, contains some 9 real-valued, linearly-independent quantities, which may be best understood by beginning with Eq. (3.11) to conclude that

$$R_{AB\dot{C}\dot{D}} = \tfrac{1}{16} S^{\alpha\beta}{}_{AB} S^{\gamma\delta}{}_{\dot{C}\dot{D}} R_{\alpha\beta\gamma\delta} = \cdots = \tfrac{1}{4} \sigma^{\alpha}{}_{A\dot{C}} \sigma^{\beta}{}_{B\dot{D}} \left(\mathcal{R}_{\alpha\beta} - \tfrac{1}{4} g_{\alpha\beta} \mathcal{R} \right) \quad . \tag{6.17}$$

It is also valuable to perform all the necessary arithmetic to write out explicitly the spinorial component version of the Bianchi identities, given as 2-form equations in Eqs. (3.8). The results are the following:

$$\begin{aligned} &3 \nabla^{F}{}_{\dot{D}} C_{FABC} = \nabla_{A}{}^{\dot{A}} R_{BC\dot{B}\dot{A}} + \nabla_{B}{}^{\dot{B}} R_{CA\dot{D}\dot{B}} + \nabla_{C}{}^{\dot{C}} R_{AB\dot{D}\dot{C}} \quad , \\ &8 \nabla^{C\dot{D}} R_{AC\dot{B}\dot{D}} + \nabla_{A\dot{B}} \mathcal{R} = 0 \quad . \end{aligned} \tag{6.18}$$