Notes on the Geometry of Spacetime,

and associated Vector, Tensor and matrix Notation and Conventions

by Daniel Finley, Fall, 2003

I. The Spacetime of Special Relativity

1. **Spacetime** is a 4-dimensional *manifold*, with points referred to as "events." We often label these points with a quadruplet of coordinates, for example (x, y, z, t) or (r, θ, φ, t), or $(x, y, u \equiv z + t, v \equiv z - t)$. Different *allowed observers* will ascribe these coordinates in different ways. The allowed observers in the spacetime of special relativity are often referred to as *inertial observers.* On the other hand, from the point of view of general relativity, any physical observer is allowed. (More will be said later about a manifold and such labelling of points on it by the use of coordinates.) Given two such points, i.e., two events, we indicate the differences of their coordinates by Δx, Δy, etc.; in the limit when these two points approach one another, we may treat this difference as infinitesimal, and denote it by dx, dy, etc.

   a. Spacetime also is provided with a notion of "distance," or "length," between pairs of events, often referred to as *the interval.* It is usual to denote this quantity by Δs^2, when the two events are well-separated; in the infinitesimal case, we will refer to it as ds^2. In either case, this is done even though it is not in general the square of anything; there are two distinct cases, and then it will be the square of something, which we will identify. (We will also associate with it a second-rank tensor, $\boldsymbol{\eta}$, as noted below.)

   The importance of the interval is that it is measured by all *inertial observers* to have the same value.

   Using Minkowski coordinates—also referred to as (4-dimensional) Cartesian coordinates, $\{x, y, z, t\} \equiv \{x^\mu \mid \mu = 1, 2, 3, 4\} \equiv \{x^\mu\}_1^4$, we may write

$$\Delta s^2 = \left\{ \begin{array}{c} (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - (\Delta t)^2 \equiv \eta_{\mu\nu}(\Delta x^\mu)(\Delta x^\nu) \ , \\ \text{or} \\ (\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2 - (\Delta t')^2 \equiv \eta_{\mu\nu}(\Delta x'^\mu)(\Delta x'^\nu) \ , \end{array} \right\} \mu, \nu = 1, 2, 3, 4 \ . \tag{1.1}$$

One may think about the interval as a quadratic sum of squares (of differences) of coordinates. The function, with components $\eta_{\mu\nu}$, that generates that quadratic sum is referred to as the metric for spacetime, and plays the role of a scalar product. From the form shown above one can "see" several of Finley's conventions:

i.) He uses "geometrized units," where the speed of light, c, has its value set equal to $+1$, which makes the SI units of meters and seconds interconvertible, and spatial and temporal coordinates have the same "dimensions," either meters or seconds. This approach emphasizes the underlying geometry, and the validity of the meaning of c for **all** types of phenomena. In MKS units its value is $c = 2.99792458 \times 10^8$ m/sec. It is perhaps also worth noting that we will use various other geometrized units, that come from setting Newton's gravitational constant equal to 1. [Recall that in MKS units, we have the value $G = 6.6726 \times 10^{-11}$ m^3/(kg-sec^2).]

Some useful conversion factors which result from this are, for example

(i). 1 solar mass = 1.47664 kilometer = 1.989 $\times 10^{33}$ g = 4.9255 microseconds;

(ii). the mass of a proton, $m_p = 0.93826$ GeV $= 6.764 \times 10^{-57}$ km $= 1.0888 \times 10^{13}$ Kelvins;

(iii). the charge on a proton, $e = 1.381 \times 10^{-39}$ km .

ii.) He uses the usual ("right") sign convention for the metric, where a $+1$ in the metric corresponds to a spatial direction, and -1 to a temporal direction. This leads to a so-called *"signature"* of the metric as $+2$, which is simply the sum of the diagonal elements, when it is diagonal.

iii.) He labels the coordinates so that indices $\{1, 2, 3\}$ are spatial, and time is labelled 4, so that it comes last in the sequence of coordinates. It is to be noted that Carroll, and also Hartle, label the temporal coordinate as 0, so that the temporal portion comes first in the sequence of coordinates. It is conceivable that Finley should change, for the purposes of the class, but it's not clear if he will.

iv.) Finley also uses the **Einstein summation convention**, which says that

any single term that contains the same index symbol twice, once as a subscript and once as a superscript is presumed to also contain (an unwritten) sum sign that indicates that a sum is to performed over all allowed values of that index.

This means that a correctly written mathematical product of symbols should **not** contain the same index occurring **three times**. If one does absolutely need the same index three or more times, as, for instance, in an eigenvalue equation with explicitly-presented matrix indices, then after the first two occurrences the others are indicated with the capital-letter version of the lower-case one that indicates the summation.

v.) Finley uses Greek, lower-case letters to take values from 1 to 4.

In addition, although we cannot see it in these equations, Finley uses Roman, lower-case letters to take values only from 1 to 3, representing the spatial portions of some otherwise 4-dimensional object. He also sometimes uses Roman, lower-case letters to simply indicate indices that take values from 1 to some yet-unspecified integer value, m, **and** he uses Roman, upper-case letters to indicate indices on 2x2 matrices [or the associated 2-dimensional vectors], which then run from 1 to 2.

b. The set of inequivalent (allowed) inertial observers can be labelled by the set of all possible Poincaré transformations from (the basis of) the "standard" observer into that of some other observer. Poincaré transformations include **all rotations, Lorentz boosts, 4-dimensional translations, and any of their products**; the set of all of these is 10-dimensional.

There will be some notes on the Lorentz (and Poincaré) groups, with more details.

c. Two distinct events may always be connected by a straight line; if they are

i.) **spacelike separated**, the interval along that straight line is positive, and its square root is called the proper length, $\Delta\ell$, between those two events. Its square **minimizes** the (square of the) length along arbitrary curves between the two events; or if they are ii.) **timelike separated**, the interval along that straight line is negative, and the square root of its negative is called the proper time, $\Delta\tau$, between those two events. Its square **maximizes** the (negative of the squared) length along arbitrary curves between the two points; or if they are

iii.) **null separated**, they both lie on the trajectory of some light ray.

The fact that the spacetime admits the interval, which allows the above statements, allows each observer to make a division of all displacement vectors, relative to her or his origin, into the (past) and (future) lightcones, the (past) and (future) timelike parts, and the spacelike part; this division is (of course) independent of which observer makes the measurements.

It should, however, be noted that in general relativity, this separation may well **not** be possible on a global scale, but only locally in some region "near" the current location of the observer.

2. **Worldlines** and related quantities:

a. The trajectory of any possible observer is the set of all events at which that observer is present. Such trajectories are called *worldlines.* We may easily think of the set of these events as a *path, or curve,* on the spacetime, which can be well described by the use of some single parameter that varies continuously and ever-increasing along the worldline. The "wristwatch-time" of the observer is of course a very reasonable choice for such a parameter. We think of this curve as a mapping of some range of real numbers, i.e., from some subset of the set of all real numbers, $\mathbb{R}$, into the spacetime. It is then straightforward to think of the curve as having, at each point, a tangent vector that indicates its direction at that point. Since any observer must always travel slower than does light, two nearby points on the worldline will always be timelike separated; therefore, we always suppose that we have chosen a proper scaling and a choice of origin for the observers "wristwatch-time" so that we may identify it with the locally-measured proper time, τ, at each event through which he lives.

b. Using the proper time as the parameter along the worldline, the tangent vector to the curve is well-defined, and will be referred to as the 4-velocity, since it is obviously a 4-dimensional vector. That tangent vector should have as components the rate of change, with respect to the proper time, τ, of the coordinates of the events, x^μ, along the worldline. For now we consider the 4-vector which has components, dx^μ (relative to some appropriate-chosen basis for the vector space in which the vector lives), and denote the vector itself by the symbol $d\widetilde{x}$. We may then take the ratio of this to the infinitesimal change of proper time in which it occurs; this is surely the desired tangent vector to the worldline, the 4-velocity, which we will call $\widetilde{u}$:

$$\widetilde{u} \equiv \frac{d\widetilde{x}}{d\tau} \ . \tag{1.2}$$

If we then rewrite Eqs. (1.1) in infinitesimal form and divide by the scalar $(d\tau)^2$, we acquire a statement that says that

$$\left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2 + \left(\frac{dz}{d\tau}\right)^2 - \left(\frac{dt}{d\tau}\right)^2 = -1 \ , \tag{1.2a}$$

where the last equality follows from the definition of $d\tau$. **However**, since we have used this relation to define a tangent vector, it seems reasonable to **extend** the definition of the tensor $\boldsymbol{\eta}$ to act as a "scalar product" for tangent vectors, where we mean by the symbols $\widetilde{u}^2 \equiv \widetilde{u} \cdot \widetilde{u}$ the sum of the squares of the spatial components minus the square of the temporal component, following the same "rule" we used earlier for the creation of the interval:

$$\widetilde{u}^2 \equiv (u^1)^2 + (u^2)^2 + (u^3)^2 - (u^4)^2 \equiv \boldsymbol{\eta}_{\mu\nu}\, u^\mu u^\nu \ . \tag{1.2b}$$

We will then use this idea to define squares—and, by extension, scalar products—of any tangent vector, with the use of the "metric tensor," $\boldsymbol{\eta}$, or, more generally, $\mathbf{g}$.

$$\widetilde{u}_1 \cdot \widetilde{u}_2 \equiv \boldsymbol{\eta}_{\mu\nu}\, u_1^\mu u_2^\nu \ . \tag{1.2c}$$

c. Dynamical physics then allows the introduction of some important mechanical quantities, which are related to this tangent vector:

i.) 4-momentum vector, $\widetilde{p} \equiv m\widetilde{u}$, which has components $\vec{p}$, the usual 3-vector momentum, and E, the total energy of the particle whose worldline we have been considering, and also

ii.) the (net) 4-force, $\widetilde{K} \equiv d\widetilde{p}/d\tau$, which includes appropriate generalizations of the usual 3-vector force and also the power, i.e., the time-rate of change of the energy.

The components of these quantities all transform in the same way as do coordinate differences when one changes basis from one (inertial) observer to another, which is the meaning of the statement that they are 4-vectors.

II. Symbols to describe various Vector and Tensor Spaces

1. We first note that Finley uses the (common) useful mathematical notations that $\mathbb{R}$ stands for the set of all real numbers, and then $\mathbb{R}^n$ is the set of all "n-tuples" of real numbers, while $\mathbb{C}$ stands for the set of all complex numbers. Then $\mathbb{Z}$ denotes the set of all integers, and also $\mathbb{Z}^+$ stands for the set of all non-negative integers.

We use the symbol $\mathcal{M}$ to denote a manifold. If the manifold is arbitrary, it may be, for instance, of dimension n; however, our spacetime is of course a manifold of dimension 4. [For more details on how a manifold is defined, see the notes on the geometrical views on manifolds, vectors, differential forms, and tensors.]

We are also interested in various functions, mapping manifolds into the real numbers,$\mathbb{R}$, and denote the set of all of them by the symbol $\mathfrak{F}$, or $\mathfrak{F}(\mathcal{M})$ if it is necessary to specify which manifold. We usually are only interested in functions which (at least almost everywhere) possess arbitrarily many continuous derivatives. We refer to such a mapping as being either "of class $C^{(\infty)}$, or simply as "smooth."

2. In principle there are two, physically-different kinds of vectors, each of which has their corresponding vector space. These are the space of **"tangent vectors,"** $\mathfrak{T}^1$ (or sometimes just $\mathfrak{T}$), and the space of **"differential forms,"** Λ^1, also called cotangent vectors. Tangent vectors are the usual sort of vectors that one regularly uses in, say, freshman physics; however, differential forms are really ways to label, and "add," etc. hypersurfaces (of dimension $n-1$ in an n-dimensional manifold, so that ours will usually be 3-surfaces). This is like the usual, freshman-physics notion of the "normal" to a surface, this time, of course, in a 3-dimensional space. As it turns out, one may also characterize differential forms as (continuous) linear maps of tangent vectors into scalars, e.g., $\mathbb{R}$ or $\mathbb{C}$. Common mathematical language is to say that differential forms are *dual* to tangent vectors; i.e., Λ^1 is the dual vector space to $\mathfrak{T}^1$.

Geometrically, one should think of tangent vectors as (locally) tangent to 1-dimensional curves on $\mathcal{M}$, while differential forms are (locally) tangent to hypersurfaces, i.e., $n-1$-dimensional surfaces. From that point of view, the "action" of a hypersurface on a tangent vector, i.e., what it does to map that vector into a scalar, is to determine how many times the one intersects the other, that being the resultant scalar.

a. Because the spaces we are considering usually have a scalar product, the distinction between these two kinds becomes blurred, since the scalar product has the same effect, i.e., it also maps vectors into numbers. Therefore, this is the same as saying that one may use the metric (tensor) to map tangent vectors into differential forms, or vice versa, since both produce the same effect. We will discuss this in some detail later. Geometrically this is the same as characterizing a hypersurface by the vector which is normal to it.

3. Proceeding onward to notation, we will generalize the more familiar use of "arrows" over symbols, long used to denote ordinary, 3-dimensional vectors, by using an "over-tilde" to indicate a tangent vector, and an "under-tilde" to indicate a differential form:

i.) $\widetilde{v}$ is a tangent vector at some point on the manifold, and

ii.) $\underset{\sim}{\omega}$ is a differential form at some point, while

iii.) $\vec{p}$ will continue to be used for the usual 3-dimensional vectors.

iv.) The action of a differential form on a tangent vector will then be written as $\underset{\sim}{\alpha}(\widetilde{v}) \in \mathbb{R}$.

4. In any vector space, vectors are often described by giving their (scalar) components with respect to some choice of basis. Our most common spaces of interest will be 4-dimensional; therefore, most examples will come from there.

a. Finley uses the standard symbols $\{\widetilde{\mathbf{e}}_1, \widetilde{\mathbf{e}}_2, \widetilde{\mathbf{e}}_3, \widetilde{\mathbf{e}}_4\} \in \mathfrak{T}^1$ to denote an arbitrary basis for a space of tangent vectors; more compactly, we often write $\{\widetilde{\mathbf{e}}_\mu\}_{\mu=1}^4$ or just $\{\widetilde{\mathbf{e}}_\mu\}_1^4$ to mean the same thing.

Given any vector, $\widetilde{v}$, there always exist **unique**, *scalar* quantities, v^μ, such that

$$\widetilde{x} = x^\mu \, \widetilde{\mathbf{e}}_\mu \ . \tag{2.1}$$

The index on the symbol x^μ is a **superscript**; this will always be true when the vector $\widetilde{x}$ is a "tangent vector." We also refer to these indices as *"contravariant."*

b. We also need a choice for basis vectors for differential forms. Habitually, we will use the symbols $\{\underset{\approx}{\boldsymbol{\omega}}^\alpha\}_1^4 \in \Lambda^1$ for this basis. We will usually choose these basis elements so that the two basis sets, for tangent vectors and for differential forms, are *reciprocal bases*, which means that

$$\underset{\approx}{\boldsymbol{\omega}}^\alpha(\widetilde{\mathbf{e}}_\beta) = \delta^\alpha_\beta \ , \qquad \text{—the Kronecker delta: } \delta^\alpha_\beta = \begin{cases} 1, & \alpha = \beta, \\ 0, & \text{otherwise.} \end{cases} \tag{2.2}$$

c. For an arbitrary $\underset{\sim}{\mu} \in \Lambda^1$, its components are then the set of scalars $\{\underset{\sim}{\mu}_\alpha\}_1^4$ such that

$$\underset{\sim}{\mu} = \mu_\alpha \, \underset{\approx}{\boldsymbol{\omega}}^\alpha \ , \ \text{ and } \ \underset{\sim}{\mu}(\widetilde{v}) = \mu_\alpha v^\beta \, \underset{\approx}{\boldsymbol{\omega}}^\alpha(\widetilde{\mathbf{e}}_\beta) = \mu_\alpha v^\alpha \ . \tag{2.3}$$

The indices on the components of a 1-form are always lower indices, i.e., subscripts. We also refer to such indices as *"covariant"*. We also have the following set of useful relations, rather analogous to the behavior of "dot products":

$$x^\alpha = \underset{\approx}{\boldsymbol{\omega}}^\alpha(\widetilde{x}) \quad , \qquad \sigma_\beta = \underset{\sim}{\sigma}(\widetilde{\mathbf{e}}_\beta) \quad . \tag{2.4}$$

5. At any given point, $p \in \mathcal{M}$, there are also other interesting vector spaces. Tensor spaces, in general, are linear, continuous maps of some number, s, of tangent vectors and some number, r, of 1-forms into $\mathbb{R}$. One may also say that they are contravariant of type r and covariant of type s, or simply that they are "of type $[r, s]$." Since a tensor of type $[r, s]$ is a member of the tensor product of r copies of $\mathfrak{T}^1$ and s copies of Λ^1, the natural choice of basis is

$$\{\widetilde{\mathbf{e}}_{\mu_1} \otimes \widetilde{\mathbf{e}}_{\mu_2} \otimes \ldots \otimes \widetilde{\mathbf{e}}_{\mu_r} \otimes \underset{\approx}{\boldsymbol{\omega}}^{\lambda_1} \otimes \underset{\approx}{\boldsymbol{\omega}}^{\lambda_2} \otimes \ldots \otimes \underset{\approx}{\boldsymbol{\omega}}^{\lambda_s} \mid \mu_1, \ldots, \mu_r, \lambda_1, \ldots \lambda_s = 1, \ \ldots \ , n\} \ . \tag{2.5}$$

Some physically-interesting examples of tensors are given by

a. an ordinary tangent vector, which is of type [1,0],

b. a differential form, or 1-form, which is of type [0,1],

c. the metric tensor, of type [0,2], and symmetric

d. the electromagnetic field tensor, also of type [0,2] but skew-symmetric,

e. the stress-energy tensor, of type [1,1], and

f. the curvature of the manifold, of type [1,3].

g. Of special interest are the tensor spaces made up of combinations of skew-symmetric tensor products of a number, p, of 1-forms. These objects are often called *p-forms.* We use a special skew-symmetric version of the tensor product in these spaces, referred to as a *Grassmann product*, or, more commonly, just a *"wedge" product*, since it is denoted with the symbol $\wedge$ between the two objects for which this is the product. If we define the wedge product of two basis 1-forms as

$$\underset{\approx}{\omega}^\alpha \wedge \underset{\approx}{\omega}^\beta \equiv \underset{\approx}{\omega}^\alpha \otimes \underset{\approx}{\omega}^\beta - \underset{\approx}{\omega}^\beta \otimes \underset{\approx}{\omega}^\alpha \ , \tag{2.6}$$

then a basis for the vector space of 2-forms, Λ^2, is just

$$\{\underset{\approx}{\omega}^\alpha \wedge \underset{\approx}{\omega}^\beta \mid \alpha, \beta = 1, \ldots , n \ ; \ \alpha < \beta\} \ . \tag{2.7}$$

We then use the associativity of the tensor product to extend this definition of the wedge product to Λ^p, and obtain sets of basis vectors accordingly; for instance, for Λ^3 we have

$$\begin{aligned} \underset{\approx}{\omega}^\alpha \wedge \underset{\approx}{\omega}^\beta \wedge \underset{\approx}{\omega}^\gamma \ &\equiv \underset{\approx}{\omega}^\alpha \otimes \underset{\approx}{\omega}^\beta \otimes \underset{\approx}{\omega}^\gamma - \underset{\approx}{\omega}^\beta \otimes \underset{\approx}{\omega}^\alpha \otimes \underset{\approx}{\omega}^\gamma + \underset{\approx}{\omega}^\beta \otimes \underset{\approx}{\omega}^\gamma \otimes \underset{\approx}{\omega}^\alpha \\ &\quad - \underset{\approx}{\omega}^\alpha \otimes \underset{\approx}{\omega}^\gamma \otimes \underset{\approx}{\omega}^\beta + \underset{\approx}{\omega}^\gamma \otimes \underset{\approx}{\omega}^\alpha \otimes \underset{\approx}{\omega}^\beta - \underset{\approx}{\omega}^\gamma \otimes \underset{\approx}{\omega}^\beta \otimes \underset{\approx}{\omega}^\alpha \ , \\ &\left\{\underset{\approx}{\omega}^\alpha \wedge \underset{\approx}{\omega}^\beta \wedge \underset{\approx}{\omega}^\gamma \mid \alpha, \beta, \gamma = 1, \ldots , n \ ; \ \alpha < \beta < \gamma\right\} \ . \end{aligned} \tag{2.8}$$

The vector space of all p-forms is denoted by Λ^p, and has dimension $\binom{n}{p} = n(n-1)\ldots(n-p+1)/p!$. Therefore, over a manifold of dimension n, the variety of p-forms extends only from $p = 1$ to $p = n$, although often one also takes $\mathfrak{F}$, the space of all smooth functions, as though they were 0-forms, Λ^0. It will also allow us to introduce a very important mapping, called the *exterior derivative*, $d : \Lambda^p \rightarrow \Lambda^{p+1}$, which is a generalization of the ordinary 3-dimensional notions of "gradient," "curl," and "divergence."

For **p-forms** of rank higher than 1, we will also use an "under-tilde" so that such a symbol does not automatically tell us "the value of p." It is very unlikely that this will cause much confusion, however, since we will only truly discuss relatively few distinct p-forms; each one of interest will generally have its own particular symbol, never used for anything else. More discussion is given elsewhere.

h. Area and volume forms, and the totally anti-symmetric tensor density, related to Levi-Civita's symbol, $\epsilon^{\mu\nu\lambda\eta}$. (Hodge) duality is a very useful map from $\Lambda^p \rightarrow \Lambda^{n-p}$, which is created by the Levi-Civita symbol and the metric. [More discussion about the Levi-Civita symbol will be given later.]

For higher-rank tensors that are not p-forms, we will put their special symbol in boldface letters when not explicitly indicating their indices; an example will be the metric tensor, of type [0,2], denoted $\boldsymbol{\eta}$.

III. Matrix presentations for the components of geometrical objects

1. Matrices are not, *a priori* geometrical objects but, rather, arrays of scalar quantities along with (standard) rules concerning their display, and their manipulation to create new matrices. One must therefore have (rather arbitrary) conventions/rules that relate the matrix arrays of scalar quantities with the arrays of scalar quantities that form the set of components of some geometrical object.

2. Having agreed on a specific choice of basis, it is convenient, and very conventional, to display the set of components of a vector by means of a matrix with only one column, usually referred to as a column-vector. However, since we have two sorts of vectors, we generalize this convention—not done by all authors—so that we represent our geometrical vectors so that

a. contravariant components are represented via **column-vectors**, i.e., **matrices with only one column**, and

b. covariant components are represented via row-vectors,

where **row-vectors** are actually **matrices with only one row**.

An additional problem, however, is that different choices of basis would generate different sets of scalar quantities for the very same vector; therefore $\widetilde{x}$ is not **equal** to the set of scalars, x^μ, so that, instead of writing an equality, I use the symbol $\Longrightarrow$, which is read as "is represented by." When this symbol is used, it should remind us that we must know which particular choice of basis has been made, and what the choice of ordering is, before we may understand whatever comes next. Finally, then, examples might be

$$\begin{aligned} \widetilde{V} \quad &\Longrightarrow \quad V^a \quad \Longrightarrow \quad \begin{pmatrix} V^1 \\ V^2 \\ \vdots \end{pmatrix} , \\ \underset{\sim}{\Upsilon} \quad &\Longrightarrow \quad \Upsilon_b \quad \Longrightarrow \quad (\,\Upsilon_1 \quad \Upsilon_2 \quad \ldots\,) \ . \end{aligned} \tag{3.1}$$

3. We also need conventions to describe the generic elements of a matrix. My standard convention for matrices is that the entries within matrices are labelled by their **row** and their **column**, with the row index coming first. We use this convention independent of whether the indices for the components are upper or lower, i.e., whether they are being used to form a presentation of tensorial objects that transform contravariantly or covariantly. Therefore, for example, we could easily have (different) matrices with elements denoted in the following ways: $F^a{}_b\,,\ G_{ab}\,,\ H_a{}^b\,,\ J^{ab}$.

In each case the row index is the one that comes first.

4. We also use matrices to display the components of other geometrical quantities, especially those sorts of tensors that relate two vectors at once—called second-rank tensors. Here we give appropriate **conventions** for two, very common examples, the metric tensor and the electromagnetic field tensor.

a. The metric tensor is a bilinear mapping that takes two tangent vectors and gives a scalar—their "scalar product." This makes it an element of $\Lambda^1 \otimes \Lambda^1$, a basis for which is the set of all $\underset{\sim}{\omega}^\alpha \otimes \underset{\sim}{\omega}^\beta$. Therefore we can define the components in either of two equivalent ways:

$$\boldsymbol{\eta} = \eta_{\alpha\beta}\, \underset{\sim}{\omega}^\alpha \otimes \underset{\sim}{\omega}^\beta \quad , \qquad \text{or} \quad \eta_{\alpha\beta} = \boldsymbol{\eta}\,(\widetilde{\mathbf{e}}_\alpha, \widetilde{\mathbf{e}}_\beta) \quad , \tag{3.2}$$

and then we can use a matrix representation to display these components:

$$\boldsymbol{\eta} \Longrightarrow \quad \eta_{\mu\nu} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} , \quad \mu,\nu = 1,2,3,4 \quad , \tag{3.3}$$

Notice that $\boldsymbol{\eta}$ is that particular metric tensor that is commonly used in special relativity, for an "orthonormal" metric. In a more general context, I would use the symbol $\mathbf{g} = g_{\alpha\beta}\, \underset{\sim}{\boldsymbol{\omega}}^\alpha \otimes \underset{\sim}{\boldsymbol{\omega}}^\beta$ to refer to the metric tensor, that determines the interval.

b. As already suggested, the existence on our manifold of a metric "blurs" the (calculational) distinction between 1-forms and tangent vectors; more precisely, it allows a mapping between them: since $\boldsymbol{\eta} : \mathcal{T}^1 \otimes \mathcal{T}^1 \to \mathbb{R}$, the result of giving it one vector but not the other would be some tensor which, when given the second vector, would give you a scalar; however, that is exactly what a 1-form is, so that we see that there is a different behavior for the metric, namely $\boldsymbol{\eta} : \mathcal{T}^1 \to \Lambda^1$. We use the notation $\boldsymbol{\eta}(\widetilde{v}, \cdot)$ to indicate this mapping, where the centered dot is a "placeholder" waiting the arrival of a second tangent vector so that the metric may compute the scalar product of the two; therefore, $\boldsymbol{\eta}(\widetilde{v}, \cdot)$ is actually a 1-form, able and willing to give a scalar for any vector given it, and in a continuous, linear fashion. We may "explain" all this in index notation as follows:

$$\begin{gathered} \forall\, \widetilde{v} = v^\alpha\, \widetilde{\mathbf{e}}_\alpha \in \mathcal{T}^1 , \quad \exists!\, \underset{\sim}{v} = v_\beta\, \underset{\sim}{\boldsymbol{\omega}}^\beta \in \Lambda^1 \quad \text{where} \quad v_\beta \equiv \eta_{\beta\alpha}\, v^\alpha , \\ \text{so that } \underset{\sim}{v}(\widetilde{w}) = v_\beta\, w^\beta = \eta_{\beta\alpha}\, v^\alpha w^\beta = \eta_{\alpha\beta}\, v^\alpha w^\beta = \boldsymbol{\eta}(\widetilde{v}, \widetilde{w}) \;\equiv\; \widetilde{v} \cdot \widetilde{w} . \end{gathered} \tag{3.4}$$

This process is usually just referred to as "lowering an index," from contravariant to covariant. As well, at the end of the equation one sees that it is also common practice to simply use a "centered *dot*" to indicate the scalar product.

c. Clearly the process should have a reverse as well, since we may "undo" the process of lowering the index by using the inverse matrix for the matrix presenting $\boldsymbol{\eta}$, which we can refer to by the symbol $\boldsymbol{\eta}^{-1}$, although we will usually use exactly the same symbol for the inverse, but with the indices raised:

$$\boldsymbol{\eta}^{-1} \Longrightarrow \quad \eta^{\mu\nu} \quad \text{where} \quad \eta^{\mu\nu}\eta_{\nu\lambda} \equiv \delta^\mu_\lambda \Longleftarrow \quad \mathbf{I}_4 . \tag{3.5}$$

Having this inverse, we immediately have two different functions that it can perform for us:

i.) The first is simply that it maps 1-forms to tangent vectors:

$$\begin{gathered}\boldsymbol{\eta}^{-1}:\Lambda^1\to\boldsymbol{\mathfrak{T}}^1\ \text{, where }\ \forall\,\underset{\sim}{\alpha}=\alpha_\mu\,\underset{\sim}{\boldsymbol{\omega}}^\mu\in\Lambda^1\ ,\quad \exists!\,\widetilde{\alpha}=\alpha^\mu\,\widetilde{\mathbf{e}}_\mu\in\boldsymbol{\mathfrak{T}}^1\\ \text{with }\alpha^\mu\equiv\eta^{\mu\nu}\alpha_\nu\text{ and }\forall\underset{\sim}{\beta}\in\Lambda^1\ ,\quad \underset{\sim}{\beta}(\widetilde{\alpha})=\beta_\mu\,\alpha^\mu=\beta_\mu\eta^{\mu\nu}\alpha_\nu\ .\end{gathered}\tag{3.6}$$

This process, of course, is referred to as "raising an index," from covariant to contravariant.

ii.) The second function is already put into evidence in the last phrases of the equation above, namely we may use $\boldsymbol{\eta}^{-1}$ as a scalar product in the vector space of differential forms, Λ^1:

$$\boldsymbol{\eta}^{-1}:\Lambda^1\times\Lambda^1\to\mathbb{R}\ ,\quad\text{via}\quad \boldsymbol{\eta}^{-1}(\underset{\sim}{\alpha},\underset{\sim}{\beta})\equiv\eta^{\mu\nu}\,\alpha_\mu\beta_\nu\ \equiv\ \underset{\sim}{\alpha}\cdot\underset{\sim}{\beta}\ \in\mathbb{R}\ .\tag{3.7}$$

Do note that **all of the above concerning raising and lowering indices** applies equally well when using some (vector) basis where the metric has quite different form than does the simple, Minkowski presentation of $\eta_{\mu\nu}$. Under those circumstances, we might well refer to the metric as $\mathbf{g}$, although, in special relativity, this would still be a matrix with the same signature and significance relative to the interval.

d. The electromagnetic tensor is a skew-symmetric, second-rank tensor, and therefore is actually a 2-form, i.e., an element of Λ^2. Therefore we use the (skew-symmetric) wedge products that provide a basis for Λ^2:

$$\underset{\sim}{F}\equiv\frac{1}{2}F_{\alpha\beta}\,\underset{\sim}{\boldsymbol{\omega}}^\alpha\wedge\underset{\sim}{\boldsymbol{\omega}}^\beta\quad ,\tag{3.8}$$

where the $\frac{1}{2}$ is a convention making it consistent with a different possible definition, namely $\underset{\sim}{F}=F_{\alpha\beta}\underset{\sim}{\boldsymbol{\omega}}^\alpha\otimes\underset{\sim}{\boldsymbol{\omega}}^\beta$, and reminding us that the components are skew-symmetric, i.e., $F_{\alpha\beta}=-F_{\beta\alpha}$.

In 4 dimensions, any skew-symmetric, second-rank tensor is specified by exactly 6 degrees of freedom; therefore, it is customary to label those 6 degrees of freedom via a pair of

3-dimensional vectors. In the case of the electromagnetic tensor, we refer to those two 3-vectors as "the electric field," $\vec{E}$, and "the magnetic field," $\vec{B}$, justifying the following representations, with respect to the bases vectors noted in Eq. (3.8) above:

$$\underset{\sim}{F} \Longrightarrow F_{\mu\nu} = \begin{pmatrix} 0 & B^z & -B^y & E_x \\ -B^z & 0 & B^x & E_y \\ B^y & -B^x & 0 & E_z \\ -E_x & -E_y & -E_z & 0 \end{pmatrix} , \quad \mu,\nu = 1,2,3,4 \quad , \quad \text{(Finley's order)} \tag{3.9}$$

Just as an additional help, I note that when one has these matrices the choice for the ordering of the indices creates a considerable difference in the appearance of the displayed matrix. Therefore, the electromagnetic tensor, expressed as a matrix (as above), but with the indices ordered according to Carroll's order, i.e., $(0,1,2,3)$, gives the following appearance:

$$F_{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B^z & -B^y \\ E_y & -B^z & 0 & B^x \\ E_z & B^y & -B^x & 0 \end{pmatrix} , \quad \alpha,\beta = 0,1,2,3 \quad , \quad \text{(Carroll's order)} \tag{3.9$'$}$$

Following our discussion of raising and lowering indices, we may now consider various different "index locations" for the components of the electromagnetic tensor, all presented, as usual, in Finley's index ordering:

$$\begin{aligned} F^\mu{}_\nu \equiv \eta^{\mu\lambda}\, F_{\lambda\nu} \Longrightarrow & \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & B^z & -B^y & E_x \\ -B^z & 0 & B^x & E_y \\ B^y & -B^x & 0 & E_z \\ -E_x & -E_y & -E_z & 0 \end{pmatrix} \\ & = \begin{pmatrix} 0 & B^z & -B^y & E_x \\ -B^z & 0 & B^x & E_y \\ B^y & -B^x & 0 & E_z \\ E_x & E_y & E_z & 0 \end{pmatrix} , \end{aligned} \tag{3.10}$$

$$\begin{aligned} F_\mu{}^\nu \equiv F_{\mu\lambda}\, \eta^{\lambda\nu} \Longrightarrow & \begin{pmatrix} 0 & B^z & -B^y & E_x \\ -B^z & 0 & B^x & E_y \\ B^y & -B^x & 0 & E_z \\ -E_x & -E_y & -E_z & 0 \end{pmatrix} \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \\ & = \begin{pmatrix} 0 & B^z & -B^y & -E_x \\ -B^z & 0 & B^x & -E_y \\ B^y & -B^x & 0 & -E_z \\ -E_x & -E_y & -E_z & 0 \end{pmatrix} . \end{aligned} \tag{3.11}$$

IV. Display of Matrix Arithmetic without explicit indices

1. Basic Notions:

We sometimes want to conserve ink, and simplify the life of the reader, by not explicitly writing out the indices on symbols representing matrices. In fact, this justifies the fact that there are very standard conventions concerning matrix multiplication. To proceed, we first give notation to relate the statement that some matrix is named A, and the statement that the elements of that matrix are labelled by the symbols $A^\alpha{}_\mu$ through the following convention:

$$A = ((A^\alpha{}_\nu)) \qquad \text{means } A \text{ is the matrix with components } A^\alpha{}_\nu. \tag{4.1}$$

Note that the *transpose* of a matrix requires switching its rows and columns; therefore, we could have the following examples, where we use an upper ^{T} to denote the transpose of a given matrix:

$$A = ((A^\alpha{}_\nu)) \quad \Longleftrightarrow \quad A^T = ((A_\nu{}^\alpha)) \quad \text{and} \quad B = ((B_{ab})) \quad \Longleftrightarrow \quad B^T = ((B_{ba}))\ . \tag{4.2}$$

We may then state succinctly the rules concerning matrix multiplication, of two matrices, A and B, say:

$$C \equiv AB\ , \quad \Longleftrightarrow \quad C^a{}_b = A^a{}_e B^e{}_b\ , \tag{4.3}$$

where the symbol $\Longleftrightarrow$ is to be read as meaning "if and only if."

It is perhaps also worth noting that a matrix with a contravariant row index and a covariant column index—the most usual form we see—is one that is presenting an operator that maps tangent vectors into tangent vectors; i.e., we have the following relationship between a linear operator and its matrix presentation:

$$A : \mathfrak{T}^1 \to \mathfrak{T}^1 \quad \Longleftrightarrow A = ((A^\mu{}_\nu)) \ \text{ or, equivalently } \ A \Longrightarrow \quad A^\mu{}_\nu\ . \tag{4.4}$$

On the other hand an operator, B, that, for instance, is presented by a matrix with two covariant indices maps tangent vectors into 1-forms:

$$B : \mathfrak{T}^1 \to \Lambda^1 \quad \Longleftrightarrow B \Longrightarrow \quad B_{\mu\nu}\ . \tag{4.5}$$

A slightly different example begins with the tensor which has as components the (usual) Kronecker delta symbol, $\delta^a{}_b$, which has either the value 1 or 0, depending on whether a equals b, or not; it should surely be conceived of as denoting the components of the identity matrix, I. Therefore we can have the parallel statements, as above, one using the conventions of matrix multiplication, and the second using the rules for indices:

$$\begin{aligned} AB = I = BA \qquad &\Longleftrightarrow \qquad B = A^{-1} \;, \\ A^\beta{}_\nu \, B^\nu{}_\alpha = \delta^\beta{}_\alpha \;, \qquad &B^\mu{}_\alpha \, A^\alpha{}_\nu = \delta^\mu{}_\nu \;. \end{aligned} \tag{4.6}$$

Yet more structure enters when we work out products of three matrices:

$$\begin{aligned} W_{\mu\nu} = A^\alpha{}_\mu \, A^\beta{}_\nu \, M_{\alpha\beta} \quad &\Longleftrightarrow \quad W = A^T \, M \, A \;, \\ J^\mu{}_\nu = (A^{-1})^\mu{}_\alpha \, A^\beta{}_\nu \, F^\alpha{}_\beta \quad &\Longleftrightarrow \quad J = A^{-1} \, F \, A \quad . \end{aligned} \tag{4.7}$$

In the first line of Eqs. (4.7) the transpose of the matrix A is necessary because of the ordering of the indices. Matrix multiplication always sums the column indices of the matrix on the left with the row indices of the matrix on the right. Of the two summations indicated on that line, via the Einstein summation convention, the second sum has the correct ordering for matrix multiplication, and so is indicated simply by the matrix product GA. However, the first sum has the opposite ordering, necessitating the use of the transpose on the matrix A, so that the matrix multiplication rules represent correctly the desired summation, namely $A^T \, G$.

2. Examples generated by **change of basis vectors**:

It is probably important to preface these remarks that the entire "scheme" of how geometric objects transform when their components are considered with respect to different choices of a basis set is the place where more classical (à la 1930's to, perhaps, 1960's) treatments, and definitions, of tensor analysis begin. Therefore, this material is really more important than its location here might immediately suggest!

We begin by considering two distinct choices for basis elements of the cotangent space (of differential forms), namely $\{\underset{\sim}{\boldsymbol{\omega}}^\alpha\}$ and $\{\underset{\sim}{\boldsymbol{\sigma}}^\mu\}$. Since either set forms a basis, we may immediately write down each member of the one choice of basis in terms of the members of the other; i.e., there exists uniquely a (square array) of scalar quantities $A^\alpha{}_\mu$ and, alternatively, the quantities $B^\mu{}_\alpha$ such that

$$\underset{\sim}{\boldsymbol{\omega}}^\beta = A^\beta{}_\nu \, \underset{\sim}{\boldsymbol{\sigma}}^\nu \,, \qquad \underset{\sim}{\boldsymbol{\sigma}}^\mu = B^\mu{}_\alpha \, \underset{\sim}{\boldsymbol{\omega}}^\alpha \quad . \tag{4.8}$$

Using the elements of these arrays, we may phrase the fact that if one goes "backward," she should surely arrive at the place she started; i.e., we must have the relationships

$$A^\beta{}_\nu \, B^\nu{}_\alpha = \delta^\beta{}_\alpha \,, \qquad B^\mu{}_\alpha \, A^\alpha{}_\nu = \delta^\mu{}_\nu \,. \tag{4.9}$$

Continuing, we can easily see that it would be possible, and probably advisable, to consider these square arrays as square matrices, where we give the (square) array involving the quantities $A^\alpha{}_\mu$ the name A, and do the similar thing of giving the name B to the square array with element $B^\nu{}_\alpha$.

It is also a very useful notational tool to now also create new sorts of matrices, which contain elements which are 1-forms, via the following presentations, we may rewrite Eqs. (4.8) and (4.9) in pure matrix form, where Ω and $\mathit{\Sigma}$ are column vectors with the respective 1-form basis elements as their elements:

$$\begin{gathered} ((\underset{\sim}{\boldsymbol{\omega}}^\mu)) = \begin{pmatrix} \underset{\sim}{\boldsymbol{\omega}}^1 \\ \underset{\sim}{\boldsymbol{\omega}}^2 \\ \underset{\sim}{\boldsymbol{\omega}}^3 \\ \underset{\sim}{\boldsymbol{\omega}}^4 \end{pmatrix} \Longleftarrow \quad \Omega \,, \quad ((\underset{\sim}{\boldsymbol{\sigma}}^\nu)) = \begin{pmatrix} \underset{\sim}{\boldsymbol{\sigma}}^1 \\ \underset{\sim}{\boldsymbol{\sigma}}^2 \\ \underset{\sim}{\boldsymbol{\sigma}}^3 \\ \underset{\sim}{\boldsymbol{\sigma}}^4 \end{pmatrix} \Longleftarrow \quad \mathit{\Sigma} \,, \\ \Longrightarrow \quad \Omega = A\,\mathit{\Sigma} \,, \qquad \mathit{\Sigma} = B\,\Omega \,, \qquad AB = I = BA \,, \text{ or } \; B = A^{-1} \,. \end{gathered} \tag{4.10}$$

Knowing that the geometrical objects themselves, i.e., the local tangent planes to physical hypersurfaces (3-surfaces in our case), are independent of any choice of basis, we can induce transformations of various other associated objects. We begin with the components of an arbitrary 1-form $\underset{\sim}{\tau}$:

$$\begin{gathered} \underset{\sim}{\tau} = \tau_\alpha \, \underset{\sim}{\boldsymbol{\omega}}^\alpha \qquad \text{or} \quad \underset{\sim}{\tau} \Longrightarrow \quad T \equiv ((\tau_\alpha)) \text{ , relative to the basis } \{\underset{\sim}{\boldsymbol{\omega}}^\alpha\} \\ \underset{\sim}{\tau} = \tau'_\mu \, \underset{\sim}{\boldsymbol{\sigma}}^\mu \qquad \text{or} \quad \underset{\sim}{\tau} \Longrightarrow \quad T' \equiv ((\tau'_\mu)) \text{ , relative to the basis } \{\underset{\sim}{\boldsymbol{\sigma}}^\mu\} \\ \underset{\sim}{\boldsymbol{\sigma}}^\mu = (A^{-1})^\mu{}_\alpha \, \underset{\sim}{\boldsymbol{\omega}}^\alpha \qquad \Longrightarrow \qquad \tau'_\mu = A^\alpha{}_\mu \, \tau_\alpha \;\; \text{or} \qquad T' = TA \,, \end{gathered} \tag{4.11}$$

where we have agreed that the matrix presentation of the components of a 1-form, such as $\underset{\sim}{\tau}$, being covariant, would be taken as a row-vector, which insures that the order indicated above for the matrix multiplication is indeed the correct order.

We may then consider the reciprocal bases for tangent vectors, relative to the two bases $\{\underset{\sim}{\boldsymbol{\omega}}^\alpha\}$ and $\{\underset{\sim}{\boldsymbol{\sigma}}^\mu\}$, respectively, as follows:

$$\underset{\sim}{\boldsymbol{\omega}}^\alpha(\widetilde{\mathbf{e}}_\beta) = \delta^\alpha{}_\beta \qquad \text{and} \quad \underset{\sim}{\boldsymbol{\sigma}}^\mu(\widetilde{\mathbf{f}}_\nu) = \delta^\mu{}_\nu \; , \tag{4.12}$$

$$\underset{\sim}{\boldsymbol{\sigma}}^\mu = (A^{-1})^\mu{}_\alpha \, \underset{\sim}{\boldsymbol{\omega}}^\alpha \qquad \Longrightarrow \qquad \widetilde{\mathbf{f}}_\mu = A^\alpha{}_\mu \, \widetilde{\mathbf{e}}_\alpha \; , \text{ or } \; F = E\,A \; ,$$

where we have named the row vectors (with vector entries) called $F = ((\widetilde{\mathbf{f}}_\mu))$ and $E = ((\widetilde{\mathbf{e}}_\mu))$. This gives us the capability to look at the components of an arbitrary tangent vector, $\widetilde{v}$:

$$\widetilde{v} = v^\alpha \, \widetilde{\mathbf{e}}_\alpha \qquad \text{or} \quad \widetilde{v} \longrightarrow \quad V \equiv ((v^\alpha)) \; , \text{ relative to the basis } \{\widetilde{\mathbf{e}}_\alpha\}$$

$$\widetilde{v} = v'^\mu \, \widetilde{\mathbf{f}}_\mu \qquad \text{or} \quad \widetilde{v} \longrightarrow \quad V' \equiv ((v'^\mu)) \; , \text{ relative to the basis } \{\widetilde{\mathbf{f}}_\mu\} \tag{4.13}$$

$$\underset{\sim}{\boldsymbol{\sigma}}^\mu = (A^{-1})^\mu{}_\alpha \, \underset{\sim}{\boldsymbol{\omega}}^\alpha \qquad \Longrightarrow \qquad v'^\mu = (A^{-1})^\mu{}_\alpha \, v^\alpha \;\; \text{or} \qquad V' = A^{-1}\,V \; ,$$

where we have agreed that the matrix representation of the components of a tangent vector, such as $\widetilde{v}$, would be taken as a column-vector, so that the order indicated above for the matrix multiplication is again the correct order.

Notice that the components of a 1-form, conventionally taken as lower indices, are such that they transform according to the same matrix, $A = B^{-1}$, as do the tangent vector basis elements, $\{\widetilde{\mathbf{e}}_\alpha\}$; it is for this reason that they are referred to as "covariant," in the sense that they transform in the same manner. On the other hand, we see that the components of a tangent vector transform in the inverse manner to the basis elements for tangent vectors, for which reason they are referred to as "contravariant."

Following the reasoning above, one may now generalize and discover the transformation laws of more complicated objects, i.e., higher-rank tensors. This time, again, I give two examples. Firstly, we consider the metric tensor, $\mathbf{g} \in \Lambda^1 \otimes \Lambda^1$:

$$\mathbf{g} = g_{\alpha\beta} \, \underset{\sim}{\boldsymbol{\omega}}^\alpha \otimes \underset{\sim}{\boldsymbol{\omega}}^\beta \qquad \text{or} \quad \mathbf{g} \longrightarrow \quad G \equiv ((g_{\alpha\beta})) \; , \text{ relative to the basis } \{\underset{\sim}{\boldsymbol{\omega}}^\alpha\}$$

$$\mathbf{g} = g'_{\mu\nu} \, \underset{\sim}{\boldsymbol{\sigma}}^\mu \otimes \underset{\sim}{\boldsymbol{\sigma}}^\nu \qquad \text{or} \quad \mathbf{g} \longrightarrow \quad G' \equiv ((g'_{\mu\nu})) \; , \text{ relative to the basis } \{\underset{\sim}{\boldsymbol{\sigma}}^\mu\} \tag{4.14}$$

$$\underset{\sim}{\boldsymbol{\sigma}}^\mu = (A^{-1})^\mu{}_\alpha \, \underset{\sim}{\boldsymbol{\omega}}^\alpha \qquad \Longrightarrow \qquad g'_{\mu\nu} = A^\alpha{}_\mu \, A^\beta{}_\nu \, g_{\alpha\beta} \;\; \text{or} \qquad G' = A^T\,G\,A \; ,$$

The transpose of the matrix in the last line is necessary because of the ordering of the indices.

As a second (important) example, consider the type [1,1] tensor created in Eqs. (3.8-9), but in the [1,1] format described in Eq. (3.10). As given there it has components $F^\alpha{}_\beta$, which gives us

$$F'^\mu{}_\nu = (A^{-1})^\mu{}_\alpha \, A^\beta{}_\nu \, F^\alpha{}_\beta \qquad \text{or} \quad F' = A^{-1}\, F\, A \quad . \tag{4.15}$$

We are not surprised that this sort of transformation looks "familiar." It is a similarity transformation, as one might have encountered in classical or quantum mechanics. Therefore we consider for a moment the following.

3. The study of matrix transformations has a long history.

Transformations of the type appropriate for F, as given in Eqs. (4.15), are called *similarity transformations*, $F' = A^{-1}\, F\, A$; they preserve all the eigenvalues of the matrix, and, therefore, also its determinant. For *normal matrices*, this sort of transformation can be used to bring the matrix into diagonal form. (Normal matrices are defined as those which commute with their transpose; both symmetric and skew-symmetric matrices are examples, although there are many more less common ones.)

On the other hand, transformations of the type appropriate for the metric matrix G, as given in Eqs. (4.14) above, are called *congruency transformations*: $G' = A^T\, G\, A$. They do not preserve the determinant; instead they preserve the *signature of the matrix*, which is a particular set of +1's, -1's, and 0's for that matrix. Sylvester's theorem says every symmetric matrix has a congruency transformation which will not only diagonalize it, but in fact bring it to have only +1, -1, or 0 in all the places on the diagonal. This set, independent of order, is called the signature. (Sometimes, only the sum of these quantities is referred to, also, as the signature; this is completely understandable only when the dimension of the manifold is a priori known.)

Therefore, in a spacetime, the equivalence principle of Einstein assures us that there is always a change of basis for $\mathcal{T}$, effected by

an invertible matrix, M, such that the components of the metric can be converted from $g_{\alpha\beta}$ to just those of an arbitrary inertial frame of special relativity, $\eta_{\mu\nu}$:

$$G = M^T \, H \, M \,, \quad \textbf{or} \qquad g_{\alpha\beta} = M^\mu{}_\alpha \, \eta_{\mu\nu} \, M^\nu{}_\beta \quad , \tag{4.16}$$

where the symbol H is a capital Greek η.

This fact assures us that we may always, in some neighborhood of a given point in spacetime, find a special basis set for which the components are what they would have been were the spacetime flat and special relativity valid. It is only to be noted that this basis set may not be defined everywhere when the spacetime is not flat.

5. Comments on Determinants of Matrices.

In addition to rules for matrix multiplication, matrices also come equipped with a definition of various scalars created from them. The only important ones we are likely to use are *the trace*, tr A, and *the determinant*, det A.

a. The definition of trace must be given so that it will be invariant under tensor transformation rules, i.e., it should be a scalar; therefore, we can write

$$\text{tr } A \equiv A^\alpha{}_\alpha \quad = g^{\alpha\beta} A_{\beta\alpha} \quad , \tag{4.17}$$

where $g^{\alpha\beta}$ are the components of the inverse metric tensor, if the manifold admits one to exist there.

If there is no metric tensor, or if it has no inverse, then the trace will not be an invariant quantity. In addition we should explicitly note that $\sum_{\alpha=1}^4 A_{\alpha\alpha}$ is **not** the trace, since it would have a different value in every distinct coordinate basis.

b. The definition of the determinant of an $n \times n$ matrix is independent of the properties of the underlying manifold. It involves taking products of n elements, one from each row and from each column, in certain orders, with certain signs.

The Levi-Civita symbol, $\epsilon^{b_1 b_2 \ldots b_n} \equiv \epsilon_{b_1 b_2 \ldots b_n}$, has n indices, is skew-symmetric under the interchange of any two of those indices, and is such that $\epsilon^{1234} = +1$. **It is not a tensor quantity!**

It has been defined precisely so that it creates determinants via summations, as expressed by the **Fundamental Theorem of Determinants** as written below. In fact this theorem is simply a mathematical re-phrasing of the language one learned long ago about how to take determinants, involving "signed minors," etc.:

$$\epsilon^{b_1 b_2 \ldots b_n} A^{a_1}{}_{b_1} A^{a_2}{}_{b_2} \ldots A^{a_n}{}_{b_n} = \epsilon^{a_1 a_2 \ldots a_n} \det(A) \,. \tag{4.18}$$

The following rather complicated (numerical) relations involving the values of a product of Levi-Civita symbols, are worth writing down, since they are occasionally of use, especially in those cases where one or more pair of the indices are being summed. Their proof is very straightforward but quite lengthy, and will be omitted here:

$$\epsilon^{a_1 \ldots a_n} \epsilon_{b_1 \ldots b_n} = \begin{vmatrix} \delta^{a_1}_{b_1} & \ldots & \delta^{a_1}_{b_n} \\ \vdots & \ddots & \vdots \\ \delta^{a_n}_{b_1} & \ldots & \delta^{a_n}_{b_n} \end{vmatrix} \equiv \delta^{a_1 \ldots a_n}_{b_1 \ldots b_n} \quad , \tag{4.19}$$

where the vertical bars imply the determinant of a matrix. For instance, in the simple case of 2 dimensions, the above equation simply says

$$\begin{aligned} \epsilon^{a_1 a_2} \epsilon_{b_1 b_2} &= \begin{vmatrix} \delta^{a_1}_{b_1} & \delta^{a_1}_{b_2} \\ \delta^{a_2}_{b_1} & \delta^{a_2}_{b_2} \end{vmatrix} = \delta^{a_1}_{b_1} \delta^{a_2}_{b_2} - \delta^{a_1}_{b_2} \delta^{a_2}_{b_1} \equiv \delta^{a_1 a_2}_{b_1 b_2} \\ \Longrightarrow \quad \epsilon^{a_1 a_2} \epsilon_{a_1 b_2} &= \delta^{a_1}_{a_1} \delta^{a_2}_{b_2} - \delta^{a_1}_{b_2} \delta^{a_2}_{a_1} = 2\, \delta^{a_2}_{b_2} - \delta^{a_2}_{b_2} = \delta^{a_2}_{b_2} \,. \end{aligned}$$

As well, we will regularly consider some $q \times q$ submatrix of the general matrix given above, and the $q \times q$ determinant made with Kronecker delta entries, as above. We will denote such a determinant with the *generalized Kronecker delta symbol*, that has q upper and q lower indices. Using that notation, we may then write out an expression, easily provable from the general one above, for partial sums of products of Levi Civita symbols. Writing $q+q' \equiv n$, it follows that

$$\epsilon^{a_1\dots a_q c_1\dots c_{q'}}\,\epsilon_{b_1\dots b_q c_1\dots c_{q'}} \quad = \delta^{a_1\dots a_q c_1\dots c_{q'}}_{b_1\dots b_q c_1\dots c_{q'}} \equiv \quad (q')!\,\delta^{a_1\dots a_q}_{b_1\dots b_q} \quad . \tag{4.20}$$

In 4 dimensions we have the following explicit values:

$$\begin{gathered} \epsilon^{1234} = +1 = \epsilon^{2143} = \epsilon^{4321} = \epsilon^{1423} = \dots \\ \epsilon^{2134} = -1 = \epsilon^{1243} = \epsilon^{4312} = \epsilon^{4123} = \dots \\ \epsilon^{\alpha\beta\gamma\delta} = 0 \text{ whenever 2 indices are equal.} \end{gathered} \tag{4.21}$$

One should notice that $\epsilon^{\alpha\beta\gamma\delta}$ is **not a tensor**, but we will discuss shortly how to create a tensor from it.

V. The volume 4-form and its relationship to Hodge duality.

1. Over 4-dimensional manifolds, there are 5 distinct spaces of p-forms, Λ^p:

i. Λ^0 is just the space of continuous (C^∞) functions, also denoted by $\mathcal{F}$. We say that it has dimension 1, since no true "directions" are involved.

ii. Λ^1 is the space of 1-forms, already considered; it has as many dimensions as the manifold, so for 4-dimensional spacetime, it has dimension 4.

iii. Λ^2 is the space of 2-forms, i.e., skew-symmetric tensors, or linear combinations of wedge products of 1-forms; therefore in general it has dimension $\hat{n}(n-1)$, which becomes 6 for 4-dimensional spacetime. A basis can be created by taking all wedge products of the basis set for 1-forms: $\{\underset{\sim}{\boldsymbol{\omega}}^\alpha \wedge \underset{\sim}{\boldsymbol{\omega}}^\beta \mid \alpha,\beta = 1,\dots 4; \alpha<\beta\}$.

iv. Λ^3 is the space of 3-forms, i.e., linear combinations of wedge products of 1-forms, three at a time; in general it has dimension $\binom{n}{3} = \frac{1}{6}\,n(n-1)(n-2)$, which becomes 4 for 4-dimensional spacetime.

v. Λ^4 is the space of 4-forms; in general it has dimension $\binom{n}{4}$. For 4-dimensional spacetime, this is a 1-dimensional space; i.e., every 4-form is proportional to every other; we refer to some particular choice of basis for this space as *the volume form.* (In a general n-dimensional space, the volume form is always an n-form.)

vi. Over n-dimensional spacetime, it is impossible to have more than n things skew all at once; therefore, the volume form is always the last in the sequence of basis sets for p-forms. So, in 4 dimensions, there is no Λ^p for $p \geq 5$.

vii. The union of all n of the non-zero vector spaces Λ^p is sometimes referred to as the entire Grassmann algebra of p-forms over a manifold, and is denoted simply by Λ.

2. Working in the usual (local) Minkowski coordinates, where it is reasonable to choose $\{dx, dy, dz, dt\}$ as a basis for 1-forms, we choose the particular 4-form

$$\underset{\sim}{V} \equiv dx \wedge dy \wedge dz \wedge dt \ , \qquad \text{the (standard) volume form} \tag{5.1}$$

as our choice of a volume form.

More generally, if $\{\underset{\approx}{\boldsymbol{\omega}}^\alpha\}_1^4$ is an arbitrary basis for 1-forms, we may define **the very important tensor quantity** $\eta^{\alpha\beta\gamma\delta}$, which gives the "components" of the volume form relative to an arbitrary choice of basis:

$$\underset{\approx}{\boldsymbol{\omega}}^\alpha \wedge \underset{\approx}{\boldsymbol{\omega}}^\beta \wedge \underset{\approx}{\boldsymbol{\omega}}^\gamma \wedge \underset{\approx}{\boldsymbol{\omega}}^\delta \equiv \eta^{\alpha\beta\gamma\delta}\, \underset{\sim}{V} \quad , \tag{5.2}$$

$$\underset{\sim}{V} = \frac{1}{4!}\eta_{\alpha\beta\gamma\delta}\, \underset{\approx}{\boldsymbol{\omega}}^\alpha \wedge \underset{\approx}{\boldsymbol{\omega}}^\beta \wedge \underset{\approx}{\boldsymbol{\omega}}^\gamma \wedge \underset{\approx}{\boldsymbol{\omega}}^\delta \equiv \frac{1}{4!}\{g_{\alpha\rho}g_{\beta\sigma}g_{\gamma\tau}g_{\delta\varphi}\eta^{\rho\sigma\tau\varphi}\}\, \underset{\approx}{\boldsymbol{\omega}}^\alpha \wedge \underset{\approx}{\boldsymbol{\omega}}^\beta \wedge \underset{\approx}{\boldsymbol{\omega}}^\gamma \wedge \underset{\approx}{\boldsymbol{\omega}}^\delta \ . \tag{5.3}$$

This tensor is completely skew-symmetric, i.e., it changes sign when any two indices are interchanged, and so must be proportional to the *Levi-Civita symbol*, $\epsilon^{\alpha\beta\gamma\delta}$, used for determinants. One verifies that the following defines tensors of type [4,0] and [0,4], respectively, related as usual by raising/lowering of indices via the metric tensor, where we again must recall that the symbol H is a capital Greek η, and therefore stands for the basic matrix that represents the metric when it is diagonal and has only $+1$'s and -1's along that diagonal:

$$\begin{gathered} \eta^{\alpha\beta\gamma\delta} = \frac{1}{m}\,\epsilon^{\alpha\beta\gamma\delta} \quad , \qquad \eta_{\alpha\beta\gamma\delta} = (-1)^s\, m\, \epsilon_{\alpha\beta\gamma\delta} \ , \\ \text{where} \qquad m \equiv \det(M) \ , \qquad G = M^T\, H\, M \ , \\ \text{and} \quad g \equiv \det(G) = m^2 \, \det(H) = (-1)^s\, m^2 \ , \end{gathered} \tag{5.4}$$

and one chooses $s = 0$ or 1, as the number of timelike directions.

The matrix M is of course the congruency transformation that Sylvester's theorem asserts exists, that puts the metric into its normal form. Of course the values of $\eta^{\alpha\beta\gamma\delta}$ depend on the basis chosen; however, let us consider quickly the problem for an orthonormal tetrad, or triad, in (4-dimensional) spacetime or 3-dimensional space where the matrix M, above, is just the identity matrix, so that $m = 1$:

a. when the metric components are just $\eta_{\mu\nu}$ as they would be with Minkowski coordinates $\{x, y, z, t\}$, we must choose $s = 1$, which then implies that $\eta^{1234} = +1 = -\eta_{1234}$;

b. or, when we are in ordinary, Cartesian coordinates, $\{x, y, z\}$, in 3-dimensional space, we choose $s = 0$, and we simply have $\eta^{123} = +1 = \eta_{123}$.

3. the (Hodge) dual, $\quad * : \Lambda^p \longrightarrow \Lambda^{n-p}$

Let $\underset{\sim}{\alpha}$ be an arbitrary p-form; then we denote the (Hodge) dual by ${}^*\underset{\sim}{\alpha}$, an $(n-p)$-form. They are related as follows, where we, habitually, use $p' \equiv n - p$ as a useful symbol:

$$\begin{aligned}
\underset{\sim}{\alpha} &= \frac{1}{p!}\,\alpha_{b_1\ldots b_p}\,\underset{\sim}{\boldsymbol{\omega}}^{b_1} \wedge \ldots \wedge \underset{\sim}{\boldsymbol{\omega}}^{b_p} \quad , \\
{}^*\underset{\sim}{\alpha} &\equiv \frac{i^{pp'+s}}{p!(p')!}\alpha^{b_1\ldots b_p}\,\eta_{b_1\ldots b_p c_1\ldots c_{p'}}\,\underset{\sim}{\boldsymbol{\omega}}^{c_1} \wedge \ldots \wedge \underset{\sim}{\boldsymbol{\omega}}^{c_{p'}} \equiv \frac{1}{(p')!}\,({}^*\alpha)_{c_1\ldots c_{p'}}\,\underset{\sim}{\boldsymbol{\omega}}^{c_1} \wedge \ldots \wedge \underset{\sim}{\boldsymbol{\omega}}^{c_{p'}} \ , \qquad (5.5) \\
\text{or} \quad (*\underset{\sim}{\alpha})_{c_1\ldots c_{p'}} &= \frac{i^{pp'+s}}{p!}\alpha^{b_1\ldots b_p}\eta_{b_1\ldots b_p\,c_1\ldots c_{p'}} = \frac{i^{pp'+s}}{p!}\alpha_{a_1\ldots a_p}g^{a_1b_1}\ldots g^{a_pb_p}\,\eta_{b_1\ldots b_p\,c_1\ldots c_{p'}} \ .
\end{aligned}$$

The factors of $i \equiv \sqrt{-1}$ have been inserted in **just such a way that** the dual of the dual brings one back to where she started:

$$ {}^*\{{}^*\underset{\sim}{\alpha}\} = \underset{\sim}{\alpha} \quad . \qquad (5.6)$$

There are various conventions concerning the i's in the definition. My convention, using the factors of i, allows for eigen-2-forms of the * operator, since Eq. (5.6) obviously tells us that the eigenvalues of the duality operator, *, are just ± 1. Many authors omit this extra factor, which causes the eigenvalues to be $\pm i$, but which does not insert factors of i in the process of taking the dual of some tensor. As it turns out, later, there is considerable value in having such extra i's when one wants to look at tensors as complex objects.

Since the definition of (Hodge) duality appears quite complicated, it is worthwhile writing it down for **all** plausible exemplars that may occur, in our 4-dimensional spacetime. We do this for the standard Minkowski tetrad, $\{dx, dy, dz, dt\} = \{\underset{\sim}{\boldsymbol{\omega}}^\mu\}_1^4$, and the bases of each distinct space of p-forms, Λ^p:

$$
\begin{aligned}
&\Lambda^1 \leftrightarrow \Lambda^3 : \ * \begin{pmatrix} dx \\ dy \\ dz \\ dt \end{pmatrix} = - \begin{pmatrix} dy \wedge dz \wedge dt \\ dz \wedge dx \wedge dt \\ dx \wedge dy \wedge dt \\ dx \wedge dy \wedge dz \end{pmatrix} , \\
&\Lambda^0 \leftrightarrow \Lambda^4 : \ *1 = -i dx \wedge dy \wedge dz \wedge dt = -i\mathcal{V} , \\
&\Lambda^2 \leftrightarrow \Lambda^2 : \ * \begin{pmatrix} dx \wedge dy \\ dy \wedge dz \\ dz \wedge dx \end{pmatrix} \quad = \quad -i \begin{pmatrix} dz \wedge dt \\ dx \wedge dt \\ dy \wedge dt \end{pmatrix} , \quad .
\end{aligned}
\tag{5.7}
$$

As an example, consider the electromagnetic 2-form, defined in Eqs. (3.8), from which we have:

$$
\begin{aligned}
\underset{\sim}{F} \Longrightarrow \quad F_{\mu\nu} &= \begin{pmatrix} 0 & B^z & -B^y & E_x \\ -B^z & 0 & B^x & E_y \\ B^y & -B^x & 0 & E_z \\ -E_x & -E_y & -E_z & 0 \end{pmatrix} , \\
{}^*\underset{\sim}{F} \Longrightarrow \quad ({}^*F)_{\mu\nu} &= -\, i \begin{pmatrix} 0 & -E^z & E^y & B_x \\ E^z & 0 & -E^x & B_y \\ -E^y & E^x & 0 & B_z \\ -B_x & 0 - B_y & -B_z & 0 \end{pmatrix} .
\end{aligned}
\tag{5.8}
$$

Note that the map from $\underset{\sim}{F}$ to $i^*\underset{\sim}{F}$ is accomplished by sending $\vec{B} \to -\vec{E}$ and $\vec{E} \to +\vec{B}$; this is an approach originally discovered by Maxwell and Hertz. They saw this because of the intriguing properties of the *self-dual part* of this tensor, which can be completely characterized by a single, 3-dimensional but complex vector, $\vec{C} \equiv \vec{B} + i\vec{E}$:

$$
\underset{\sim}{F} + *\underset{\sim}{F} = \begin{pmatrix} 0 & C^z & -C^y & -iC_x \\ -C^z & 0 & C^x & -iC_y \\ C^y & -C^x & 0 & -iC_z \\ iC_x & iC_y & iC_z & 0 \end{pmatrix} . \tag{5.9}
$$

Notice that $*\left(\underset{\sim}{F} + {}^*\underset{\sim}{F}\right) = \underset{\sim}{F} + {}^*\underset{\sim}{F}$; i.e., this tensor is self-dual, corresponding to an eigenvalue of $+1$ under the duality operation.

As a final completion of the picture, we also give details for **3-dimensional, Euclidean space**, with Cartesian basis, $\{dx, dy, dz\} = \{\underset{\sim}{\omega}^a\}_1^3$:

$$\Lambda^1 \leftrightarrow \Lambda^2 : \; * \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = - \begin{pmatrix} dy \wedge dz \\ dz \wedge dx \\ dx \wedge dy \end{pmatrix} \quad , \qquad \Lambda^0 \leftrightarrow \Lambda^3 : \; *1 = dx \wedge dy \wedge dz \qquad . \tag{5.10}$$

As a useful example here, let us begin with that 3×3 submatrix of the electromagnetic 2-form that contains only spatial parts, and consider it as a 2-form in 3-dimensional space, with basis $\{dx, dy, dz\}$:

$$\begin{aligned} \begin{pmatrix} 0 & B^z & -B^y \\ -B^z & 0 & B^x \\ B^y & -B^x & 0 \end{pmatrix} \Longleftarrow \quad & \underset{\sim}{B} = B^z\, dx \wedge dy + B^y\, dz \wedge dx + B^x\, dy \wedge dz \\ \Longrightarrow \quad & -*\underset{\sim}{B} \; = B^z dz + B^y dy + B^x dx \; . \end{aligned} \tag{5.11}$$

This shows us that in order to properly move the usual, 3-dimensional, magnetic-field vector, $\vec{B}$, into a 4-dimensional spacetime, and make appropriate its relationship with the 3-dimensional electric-field vector, $\vec{E}$, we must first take its dual, making it part of a 2-form, instead of a 1-form. [This is what is sometimes stated as saying that $\vec{B}$ is a different sort of vector than $\vec{E}$. In particular, when one considers their behavior under a parity transformation, $\vec{E}$ changes sign, but $\vec{B}$ does not.] It is of course also true that this particular way of uniting the two quantities that one thought were both 3-vectors, back in 3-dimensional space, causes the join to transform in a simple, tensorial, way in the entire spacetime.