# Physics 570

### Spherically-symmetric, Collapsing Dust Models

One would like to consider the collapse of an originally spherically-symmetric distribution of matter, such as a star. Unfortunately the only model for this which can be obtained without "great mathematical complexity" is one where there is no pressure, but (only) a cloud of dust, i.e., particles with non-zero energy density, $\epsilon$, but zero pressure. If they started to collapse, then those at a given radius would all have some sort of uniform notion of time—the proper time along their trajectories. It is worthwhile to point out that this statement assumes they will stay together, which surely assumes they are indeed freely falling; i.e., it assumes they are travelling along geodesics. *It turns out* that the dust version of a perfect fluid indeed always does travel on geodesics. To see this we recall that the matter-energy tensor is divergenceless; beginning with a perfect fluid described by $\epsilon$ and $P$, but with $P = 0$, this condition gives us 4 equations:

$$(\epsilon u^\nu) = 0 \;, \quad \epsilon u^\mu \, u^\nu{}_{;\mu} = 0 \;. \tag{1.1}$$

This tells us that, yes, the worldline along which the dust travels is a congruence of geodesics, all with the same tangent vector; therefore, there is an associated proper time, which we will call $\tau$, and write the metric, and an orthonormal tetrad, in the form

$$\begin{gathered} \mathbf{g} = e^{2\lambda(\rho,\tau)} d\rho^2 + r^2(\rho,\tau)\, d\Omega^2 - d\tau^2 \;, \\ \underset{\sim}{\omega}^{\hat\rho} \equiv e^\lambda \, d\rho \equiv L\, d\rho \;, \quad \underset{\sim}{\omega}^{\hat\theta} \equiv r\, d\theta \;, \quad \underset{\sim}{\omega}^{\hat\varphi} \equiv r \sin\theta \, d\varphi \;, \quad \underset{\sim}{\omega}^{\hat\tau} \equiv d\tau \;. \end{gathered} \tag{1.2}$$

As well we suppose the matter tensor to be given simply by

$$T^{\mu\nu} = \epsilon \, u^\mu u^\nu \;. \tag{1.3}$$

Of course in these co-moving coordinates only $T^{\hat\tau\hat\tau} = \epsilon$ will be non-zero.

We then determine the connections:

$$\begin{gathered} \underset{\sim}{\Gamma}_{\hat\rho\hat\theta} = -\frac{r^{\blacktriangleright}}{rL}\underset{\sim}{\omega}^{\hat\theta} \;, \quad \underset{\sim}{\Gamma}_{\hat\rho\hat\varphi} = -\frac{r^{\blacktriangleright}}{rL}\underset{\sim}{\omega}^{\hat\varphi} \;, \quad \underset{\sim}{\Gamma}_{\hat\theta\hat\varphi} = -\frac{\cot\theta}{r}\,\underset{\sim}{\omega}^{\hat\varphi} \;, \\ \underset{\sim}{\Gamma}_{\hat\rho\tau} = -\frac{\dot L}{L}\underset{\sim}{\omega}^{\hat\rho} \;, \quad \underset{\sim}{\Gamma}_{\hat\theta\tau} = -\frac{\dot r}{r}\underset{\sim}{\omega}^{\hat\theta} \;, \quad \underset{\sim}{\Gamma}_{\hat\varphi\tau} = -\frac{\dot r}{r}\underset{\sim}{\omega}^{\hat\varphi} \;, \end{gathered} \tag{1.4}$$

where we are using $\dot r$ to mean $\partial r/\partial\tau$ and $r^{\blacktriangleright}$ to mean $\partial r/\partial\rho$.
The curvature tensor is reasonably complicated, but we will display the Einstein tensor:

$$\begin{aligned} G_{\hat\rho\hat\rho} &= -\frac{1}{(rL)^2}\left[L^2 - (r^{\blacktriangleright})^2 + (\dot r)^2 L^2 + 2rL^2\ddot r\right] \;, \\ G_{\hat\rho\hat\tau} &= \frac{2}{rL^2}\left[L(\dot r)^{\blacktriangleright} - (r^{\blacktriangleright})\dot L\right] \;, \\ G_{\hat\theta\hat\theta} = G_{\hat\varphi\hat\varphi} &= \frac{1}{rL^3}\left[Lr^{\blacktriangleright\blacktriangleright} - r^{\blacktriangleright}L^{\blacktriangleright} - \dot r\dot L L^2 - \ddot r L^3 - rL^2\ddot L\right] \;, \\ G_{\hat\tau\hat\tau} &= -\frac{1}{r^2L^3}\left[2rLr^{\blacktriangleright\blacktriangleright} - 2rr^{\blacktriangleright}L^{\blacktriangleright} - 2rL^2\dot L\dot r + L(r^{\blacktriangleright})^2 - L^3 - (\dot r)^2L^3\right] = -\kappa\epsilon \;, \end{aligned} \tag{1.5}$$

where the very last equality comes from the density of dust term in the matter-energy tensor.

To resolve these equations we begin with the off-diagonal one, which tells us

$$\frac{\dot{L}}{L} = \frac{d}{d\tau}\log(r^{\blacktriangleright}) \quad \Longrightarrow \quad L = \frac{r^{\blacktriangleright}}{g(\rho)} \ , \tag{1.6}$$

where $g = g(\rho)$ is currently an arbitrary function, being simply the "constant of integration" for that equation. Substitution of this form for $L$ into the equation for $G_{\hat{\rho}\hat{\rho}}$ then gives us the following differential equation for $r$ as a function of time, with the following next integral, where, again, $F = F(\rho)$ is a constant of integration:

$$2r\ddot{r} + (\dot{r})^2 = g^2(\rho) - 1 \ . \tag{1.7}$$

This nonlinear equation requires some care to integrate. The simplest approach seems to be to first change the independent variables, from $(\rho, \tau)$ to $(\rho, r)$, which reduces the equation to a linear one. After integrating that linear equation, and changing back to the earlier independent variables, we have a constraining equation for the $\tau$-dependence of $r$:

$$(\dot{r})^2 - F(\rho)/r = g^2(\rho) - 1 \ . \tag{1.8}$$

One next resolves this equation for $g(\rho)$ and inserts it back into the equation for $L$, and inserts these into the equation for $G_{\hat{\theta}\hat{\theta}}$, which turns out to then be an identity, after a considerable bit of algebra. This leaves one equation remaining, the one for $G_{\hat{\tau}\hat{\tau}} = -\kappa\epsilon$, which **eventually** can be resolved to give

$$F^{\blacktriangleright} = \kappa\,\epsilon\, r^2\, r^{\blacktriangleright} \ . \tag{1.9}$$

We must now make consistent this last equation for the $\rho$-dependence of $r(\rho, \tau)$ with the $\tau$-dependence given above. The most straightforward approach to doing that is to next integrate that equation, parametrically. That equation is essentially the equation to determine the trajectory of the falling dust; i.e., it describes the behavior of $dr/d\tau$. Therefore, we look at this equation again, and see that we may interpret it in the usual way in classical mechanics; i.e., it has a kinetic energy term, a potential energy term, the sum of which is equal to an energy term. That right-hand side, the energy term, is constant from the point of view of the integration, which is to determine the $\tau$-dependence. Therefore, physically we may see that there are actually 3 rather different possibilities for falling dust. It can have an energy that is greater than zero, equal to zero, or less than zero. These three options correspond to whether the dust is falling so that its energy at infinity was positive, zero, or negative in the sense that it did not have enough energy to even get to infinity. When those radial infall equations are solved in either the case of ordinary classical Newtonian mechanics, or in the Schwarzschild metric, one finds that those equations need to integrated in a somewhat different way in each of these three cases; therefore, it is useful now to define a three-case parameter, $k = 0, \pm 1$, that will distinguish these three cases in a straightforward way, and a new function $f(\rho)$ such that

$$(\dot{r})^2 - F(\rho)/r = g^2(\rho) - 1 \equiv -k\, f^2(\rho)\ . \tag{1.10}$$

Then, following the same scheme as in Newtonian mechanics, we introduce an "arc"-parameter, $\eta$, such that

$$\frac{d\eta}{d\tau} = \frac{f(\rho)}{r(\rho,\tau)}\ , \quad \Longrightarrow \quad \left(\frac{dr}{d\eta}\right)^2 = \frac{F}{f^2} r - k\, r^2\ . \tag{1.11}$$

For $k \neq 0$, these equations are then integrated as follows:

$$\left.\begin{aligned} r = &\ \tfrac{1}{2}\frac{F}{f^2}\frac{d}{d\eta}h_k(\eta)\ , \\ \tau - \tau_0(\rho) = &\ \pm\tfrac{1}{2}\frac{F}{f^3}h_k(\eta)\ , \end{aligned}\right\} \qquad h_k(\eta) = \begin{cases} \ \ \eta - \sin\eta\ ,\ k = +1\ , \\ \sinh\eta - \eta\ ,\ k = -1\ , \end{cases} \tag{1.12}$$

while for $k = 0$—analogous to falling originally from rest from infinity—the rather simpler equation:

$$\tau - \tau_0(\rho) = \pm\tfrac{2}{3} r \sqrt{\frac{r}{F(\rho)}}\ , \quad k = 0\ . \tag{1.13}$$

This is sufficient to give the metric in the following form—inserting $L$:

$$\mathbf{g} = \left(\frac{\partial r}{\partial \rho}\right)^2 \frac{d\rho^2}{1 - k\, f^2(\rho)} + r^2\, d\Omega^2 - d\tau^2\ , \tag{1.14}$$

with the density, $\epsilon(\rho,\tau)$, determining $\partial r/\partial\rho$ via: $\kappa r^2 \dfrac{\partial r}{\partial \rho}\epsilon(\rho,\tau) = \dfrac{dF}{d\rho}(\rho)\ ,$

subject of course still to the $\tau$-dependence of $r$ given by the appropriate one of the equations above. We can always re-scale the coordinate $\rho$ to remove one of the arbitrary parameters; however, it is not so simple as that since it may well be that the different "layers" of dust can cross through one another, as they fall; this will cause coordinate singularities. Therefore, one wants to maintain the various degrees of generality until a given density of dust is considered.

To apply this to a collapsing star, we choose some boundary $\rho_0$ for the star, and prescribe that the density vanishes for all $\rho > \rho_0$. Since the manifold where the density vanishes is a vacuum, spherically-symmetric one, it must be the Schwarzschild solution, of course in some non-standard coordinates, which makes sense since we are in a *co-moving* frame. Therefore, we need an interior solution, with non-zero (energy) density, but zero pressure, and the exterior solution which is Schwarzschild, and we must join them up "nicely" at the boundary described by $\rho = \rho_0$.