The Kerr Metric—for a Rotating Black Hole

In Boyer-Lindquist coordinates the Kerr metric may be written in the following form,

$$
\begin{aligned}
ds^2 &= \Sigma\left(\frac{dr^2}{\Delta}+d\theta^2\right)+(r^2+a^2)\sin^2\theta\,d\varphi^2-dt^2+\frac{2mr}{\Sigma}(a\sin^2\theta d\varphi-dt)^2\,,\\
&=\Sigma\left(\frac{dr^2}{\Delta}+d\theta^2\right)+\frac{A}{\Sigma}\sin^2\theta\,d\varphi^2-2\frac{2mar}{\Sigma}\sin^2\theta\,d\varphi\,dt-\left(1-\frac{2mr}{\Sigma}\right)dt^2\,.
\end{aligned}
\tag{1}
$$

which describes the gravitational field exterior to a rotating black hole of mass m and angular momentum per unit mass of amount a, pointing along the positive $\hat{z}$-direction.

It is reasonable to describe this system via a "locally, non-rotating" tetrad, i.e., a LNRF, of the following form:

$$
\begin{aligned}
\underset{\sim}{\omega}^r &= \sqrt{\frac{\Sigma}{\Delta}}dr\,,\quad \underset{\sim}{\omega}^\theta=\sqrt{\Sigma}\,d\theta\,,\quad \underset{\sim}{\omega}^t=\sqrt{\frac{\Sigma\Delta}{A}}\,dt\,,\\
\underset{\sim}{\omega}^\varphi &= \sqrt{\frac{A}{\Sigma}}\sin\theta\,d\varphi-\frac{2mar\sin\theta}{\sqrt{\Sigma A}}\,dt=\sqrt{\frac{A}{\Sigma}}\sin\theta\,(d\varphi-\omega\,dt)\,,\\
&\text{where}\ \begin{cases}\Sigma\equiv r^2+(a\cos\theta)^2\,,\\ \Delta\equiv r^2+a^2-2mr\,,\quad \omega\equiv\dfrac{2mar}{A}\,,\\ A\equiv\ (r^2+a^2)^2-a^2\Delta\,\sin^2\theta=(r^2+a^2)\Sigma+2ma^2r\sin^2\theta\,.\end{cases}
\end{aligned}
\tag{2}
$$

On the other hand the associated, reciprocal basis of tangent vectors is

$$
\begin{aligned}
\widetilde{e}_r=\sqrt{\frac{\Delta}{\Sigma}}\,\partial_r\,,\ \widetilde{e}_\theta=\frac{1}{\sqrt{\Sigma}}\,\partial_\theta\,,\ \widetilde{e}_\varphi=\sqrt{\frac{\Sigma}{A}}\frac{1}{\sin\theta}\,\partial_\varphi\,,\\
\widetilde{e}_t=\sqrt{\frac{A}{\Sigma\Delta}}\,(\partial_t+\omega\,\partial_\varphi)\ .
\end{aligned}
\tag{3}
$$

After some considerable calculation, one finds that, relative to this choice of LNRF, it is sufficient to have 8 functions, of r and θ, as well as m and a, in order to describe the connections. We find that

$$
\begin{pmatrix} \underset{\sim}{\Gamma}_{r\theta} \\ \underset{\sim}{\Gamma}_{r\varphi} \\ \underset{\sim}{\Gamma}_{rt} \\ \underset{\sim}{\Gamma}_{\theta\varphi} \\ \underset{\sim}{\Gamma}_{\theta t} \\ \underset{\sim}{\Gamma}_{\varphi t} \end{pmatrix} = \begin{pmatrix} B & C & 0 & 0 \\ 0 & 0 & D & -F \\ 0 & 0 & -F & \frac{r-m}{\sqrt{\Sigma\Delta}} + D \\ 0 & 0 & -\frac{\cot\theta}{\sqrt{\Sigma}} - G & -H \\ 0 & 0 & -H & -G \\ -F & -H & 0 & 0 \end{pmatrix} \begin{pmatrix} \underset{\approx}{\omega}^r \\ \underset{\approx}{\omega}^\theta \\ \underset{\approx}{\omega}^\varphi \\ \underset{\approx}{\omega}^t \end{pmatrix}
$$

$$
\text{where} \quad \begin{cases} B \equiv -\dfrac{1}{\sqrt{\Sigma}}(\log\sqrt{\Sigma})_\theta\,, \quad C \equiv -\sqrt{\dfrac{\Delta}{\Sigma}}(\log\sqrt{\Sigma})_r\,, \\ D \equiv -\sqrt{\dfrac{\Delta}{\Sigma}}(\log\sqrt{A/\Sigma})_r\,, \quad \dfrac{r-m}{\sqrt{\Sigma\Delta}} + D = \sqrt{\dfrac{\Delta}{\Sigma}}(\log\sqrt{\Sigma\Delta/A})_r\,, \\ G \equiv -\dfrac{1}{\sqrt{\Sigma}}(\log\sqrt{\Sigma/A})_\theta\,, \quad \dfrac{\cot\theta}{\sqrt{\Sigma}} + G = \dfrac{1}{\sqrt{\Sigma}}\left(\log(\sqrt{A/\Sigma}\sin\theta)\right)_\theta\,, \\ F \equiv -\dfrac{A\sin\theta}{2\Sigma^{3/2}}\omega_r\,, \quad H = \dfrac{A\sin\theta}{2\Delta^{1/2}\Sigma^{3/2}}\omega_\theta\,. \end{cases} \tag{4}
$$

We may then go further, yet, and describe the curvature itself in terms of only 4 distinct functions:

$$
\begin{aligned}
&R_{r\theta r\theta} = -R_{t\varphi t\varphi} = -Q_1\,, \quad R_{r\theta\varphi t} = +Q_2\,, \\
&R_{rtrt} = -R_{\theta\varphi\theta\varphi} = -Q_1\frac{2+z}{1-z}\,, \quad R_{rt\theta t} = R_{r\varphi\theta\varphi} = S\,Q_2\,, \\
&R_{rtr\varphi} = -R_{\theta t\theta\varphi} = S\,Q_1\,, \quad R_{rt\theta\varphi} = -Q_2\frac{2+z}{1-z}\,, \\
&R_{\theta t\theta t} = -R_{r\varphi r\varphi} = Q_1\frac{1+2z}{1-z}\,, \quad R_{r\varphi\theta t} = -Q_2\frac{1+2z}{1-z}\,.
\end{aligned} \tag{5}
$$

$$
\text{with} \quad \begin{cases} Q_1 \equiv = mr(r^2 - 3a^2\cos^2\theta)/\Sigma^3\,, \quad Q_2 \equiv Ma\cos\theta(3r^2 - a^2\cos^2\theta)/\Sigma^3\,, \\ S \equiv 3a\sin\theta\sqrt{\Delta}\,(r^2+a^2)/A\,, \quad z \equiv \Delta\left(\dfrac{a\sin\theta}{r^2+a^2}\right)^2\,. \end{cases}
$$

Geodesic Trajectories:

If $\widetilde{u}$ is the tangent vector field for a geodesic curve, then it may be related to the proper-time derivative of the coordinates and to the standard notions for 3-velocity and γ factor as follows:

$$\gamma \equiv u^{\hat{t}} = \sqrt{\frac{\Sigma\Delta}{A}}\,\frac{dt}{d\tau}\;, \qquad v^{i} \equiv \frac{u^{\hat{i}}}{u^{\hat{t}}} \Longrightarrow \begin{pmatrix} \frac{\sqrt{A}}{\Delta}\frac{dr}{dt} \\ \sqrt{\frac{A}{\Delta}}\,\frac{d\theta}{dt} \\ \frac{A\,\sin\theta}{\Sigma\sqrt{\Delta}}\left(\frac{d\varphi}{dt}-\omega\right) \end{pmatrix} . \tag{6}$$

then the coordinates that describe that trajectory satisfy the following equations:

$$\begin{aligned} \Sigma\,\frac{dr}{d\tau} &= \pm\sqrt{V_r} \equiv \pm\sqrt{T^2-\Delta[(\mu r)^2+(L-aE)^2+Q]}\;, \\ \Sigma\,\frac{d\theta}{d\tau} &= \pm\sqrt{V_\theta} \equiv \pm\sqrt{Q-[a^2(\mu^2-E^2)+L^2/(\sin^2\theta)]\cos^2\theta}\;, \\ \Sigma\frac{d\varphi}{d\tau} &= -[aE-L/(\sin^2\theta)]+aT/\Delta\;, \\ \Sigma\frac{dt}{d\tau} &= -a(aE\sin^2\theta-L)+(r^2+a^2)T/\Delta\;, \end{aligned} \tag{7}$$

$$T \equiv E(r^2+a^2)-La\;, \qquad E=-p_t\;,\; L=p_\varphi\;,$$

$$Q = p_\theta^2+[a^2(\mu^2-p_t^2)+p_\varphi^2/(\sin^2\theta)]\cos^2\theta\;.$$

Here the quantities E, L, Q, and $\mu = 1$ or 0 are constants of the motion; the first two are caused by the obvious Killing vectors, ∂_t and ∂_φ while the third is caused by a non-obvious Killing tensor and the fourth is simply the value that describes if the geodesic is timelike or null.

Interesting Surfaces

There is an outer, and an inner, horizon, where g_{rr} becomes infinite, at

$$r_\pm = m \;\pm\; \sqrt{m^2-a^2}\;. \tag{8}$$

Between these two horizons ∂_r is timelike. However, there is also a surface of static limit, at which g_{tt} vanishes, so that ∂_t becomes null at that point. The volume between this surface and the outer horizon is referred to as the *ergosphere*. Within the ergosphere, there are no timelike curves that do not rotate along with the star. The inner horizon is a Cauchy horizon, and within it, ∂_r again becomes spacelike. Lastly, there is a true curvature singularity, at the boundary of the disc defined by $r^2 + a^2 \cos^2 \theta = 0$. With proper acceleration one can avoid this singularity on a timelike path.