Physics 570

Bonus Exam

26 April, 2007: TakeHome and return tomorrow, Friday, 27 April, 2007, by 4 pm

Solutions

There are two BONUS questions, for BONUS credit on the course, if you choose, for 25 pts each. This is an examination as usual; therefore you may consult any papers or books that you have brought with you **tonight** to the second exam, BUT NO others! You also may NOT discuss it with other persons; you do, however, have most of tomorrow to complete the answers to the questions, if you choose to do so.

BONUS QUESTIONS: optional, but due at 4pm tomorrow.

I. Suppose that two such black holes of equal mass, but of exactly opposite rotation, collide, with the result that there is, afterward, a single, non-rotating black hole, of some mass M. The surface area of the pair of rotating (Kerr) black hole is given by $A = 16\pi m[m + \sqrt{m^2 - a^2}]$.

a. What is the minimum allowed value for M, which of course depends on a?

b. What is the value of a for which this minimum has its own smallest value?

[Do note that the number originally given in the problem was intended for both black holes, but obviously did not say so. I have now changed the wording to what it should have been; however, both possibilities received full credit.]

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Although it was not required, nor requested, we nonetheless go back and derive the formula that was given for the area. To do that we begin with the notion that we need to determine the differential surface area for a small piece of the horizon, at a fixed time, and then to integrate over the entire area. That small piece will be the square root of the determinant of the metric evaluated for $dr = 0 = dt$, and evaluated at $r = r_+$:

$$\underset{2}{\mathbf{g}} = \Sigma\, d\theta^2 + \frac{(r_+^2 + a^2)^2}{\Sigma}\, d\varphi^2 \ ,$$

$$\Longrightarrow \qquad \sqrt{\det \underset{2}{\mathbf{g}}} = (r_+^2 + a^2)\, d\theta\, d\varphi \ ,$$

$$\Longrightarrow \qquad A = \int (r_+^2 + a^2)\, d\theta\, d\varphi = 4\pi(r_+^2 + a^2) \ .$$

We now recall that for the Kerr metric, $r_+ = m + \sqrt{m^2 - a^2}$, so that

$$A = 4\pi\left[m^2 + 2m\sqrt{m^2 - a^2} + m^2 - a^2 + a^2\right] = 8\pi m\left[m + \sqrt{m^2 - a^2}\right] \ .$$

In this problem we have two such Kerr black holes, with identical values of m and a^2; therefore, the total initial area is twice as much as above, namely it is given by

$$A_{\rm initial} = 16\pi m\left[m + \sqrt{m^2 - a^2}\right] \ .$$

We now go on with the problem itself. When the two black holes collide, the result is given—very reasonably since the rotations are in opposite directions—and end up with $a_{\text{final}} = 0$ and the mass as M, so that the final area is just $4\pi(2M)^2$. The theorem that says that the total surface area of a system of black holes must never decrease gives us the identity, which determines the lower bound on the mass of the final black hole:

$$M^2 \geq m^2 + m\sqrt{m^2 - a^2} \quad \text{or} \quad M \geq m\sqrt{1 + \sqrt{1 - (a/m)^2}} \geq m; .$$

As this bound is a function of the initial rotation a, we see that it has its own minimum value, as a function of a, when $a = m$:

$$\min_a = M\Big|_{a=m} = m \, .$$

Therefore, when they are spinning as rapidly as possible the collision that brings them to rest radiates away half of the original mass!

II. A massive object flies by a stationary, co-moving observer in a Robertson Walker universe, who measures the object to have velocity V. It continues on its course for some distance, as the universe expands by a factor of $1/(1+Z)$. At that point it comes to the attention of another stationary, co-moving observer, who measures its velocity to be W. Please find W as a function of V and Z. Since $\gamma_V V$ is a component of a 4-vector, the simplest way to describe this relation is to determine the ratio $\gamma_V V/\gamma_W W$ as a function of Z.

Hint: The most straightforward approach is the following:

i. first, suppose that it has only moved some infinitesimal proper distance further away, determined by a change $d\chi$, say,

ii. then determine the velocity with which that new observer is moving relative to the original observer,

iii. then use that information and the standard (relativistic) velocity-addition theorem (only to first order of course) to determine the velocity that new observer will measure our object to be moving, which is of course only infinitesimally different.

iv. Then, having an infinitesimal change in the velocity caused by an infinitesimal change in $dR(t)$—proportional to the original change $d\chi$—one can write a differential equation for dV/dR, which can then be integrated, to determine W!

. . . .

It is useful to first recall that if we establish ourselves as at the value $\chi = 0$, then another stationary observer at some non-zero value χ seems to us to be moving because his metric distance is given by

$$\ell = R\chi \quad \Longrightarrow \quad v \equiv \frac{d\ell}{dt} = \frac{dR}{dt}\chi = \frac{\dot{R}}{R}R\chi = \frac{\dot{R}}{R}\ell = H\,\ell \, .$$

Following the suggested infinitesimal approach, we apply that to a second stationary observer only some small metric difference from our own position, of amount $d\ell$, so that his velocity is given by $dv = H\,d\ell$. However, as the bullet passed by us with proper velocity V, it will have travelled that extra metric distance in a time dt, so that we may write $d\ell = V\,dt$, where it is reasonable to do this in this simple-minded way since only a very small distance has been traversed. However, we can put all this together in the form

$$dv = H\,d\ell = H\,V\,dt = \frac{dR/dt}{R}\,V\,dt = \frac{dR}{R}\,V\ .$$

Using the usual addition velocity theorem from special relativity, we may determine the nearby observer's measurement of the object's velocity, which we approximate to lowest order, since dv is small:

$$V' = \frac{V - dv}{1 - V\,dv} = (V - dv)(1 - V\,dv)^{-1} = (V - dv)[1 + V\,dv + O^2(dv)] = V - (1 - V^2)\,dv + O^2(dv)\ .$$

Therefore the change in velocity is given by

$$dV = V' - V = -(1 - V^2)\,dv = -(1 - V^2)\,V\frac{dR}{R} \quad \text{or} \quad \frac{dV/V}{1 - V^2} = -\frac{dR}{R}\ .$$

We can integrate this equation, via partial fractions most easily:

$$+\log C/R = -\int \frac{dR}{R} = \int \frac{dV/V}{1 - V^2} = \int \frac{dV}{V} + \int \frac{V\,dV}{1 - V^2} = \log V - \tfrac{1}{2}\log(1 - V^2) = \log(\gamma_{\scriptscriptstyle V} V)\ ,$$
$$\Longrightarrow \qquad R\gamma_{\scriptscriptstyle V} V = C\ , \quad \text{a constant.}$$

We may then apply this to our problem by using this invariant quantity. It gives us

$$R_0\gamma_{V_0}V_0 = R\gamma_{\scriptscriptstyle V}V \quad \Longrightarrow \quad \frac{\gamma_W W}{\gamma_{\scriptscriptstyle V} V} = \frac{R(t_0)}{R(t_W)} = \frac{1}{1 + Z}\ .$$

Notice that this is just a statement about the ratio of the magnitude of the 3-momentum measured at the two places, which scales like the red shift, as one would/should expect.
