

# Physics 570

## Exam I

1 March, 2007

### Solutions

There are 5 questions. Please answer only FOUR, including both No. 1 and No. 2. You must OMIT any one of questions 3, 4, or 5, only. All questions have equal weight.

1. A certain manifold may locally be described in terms of coordinates  $\{x^\alpha | r, \theta, \varphi, t\}$ , where the metric is defined by the statement that the following is an orthonormal basis set:

$$\varpi^{\hat{r}} = \frac{dr}{\sqrt{H(r)}}, \quad \varpi^{\hat{\theta}} = r d\theta, \quad \varpi^{\hat{\varphi}} = r \sin \theta d\varphi, \quad \varpi^{\hat{t}} = \sqrt{H(r)} dt, \quad \mathbf{g} = \eta_{\hat{\mu}\hat{\nu}} \varpi^{\hat{\mu}} \otimes \varpi^{\hat{\nu}}.$$

Suppose that  $\widetilde{W}$  is a particular vector field defined on this manifold, which may be written as  $\widetilde{W} \equiv \widetilde{e}_{\hat{t}}$ ; i.e.,  $W^{\hat{\mu}} = \delta_{\hat{t}}^{\hat{\mu}}$ . Please calculate all the components of its covariant derivative along itself, i.e., calculate

$$\nabla_{\widetilde{W}} \widetilde{W} = (W^\mu W^\nu{}_{;\mu}) \widetilde{e}_\nu.$$

As a help I note that the connections may be written as

$$\begin{aligned} \Gamma_{\hat{r}\hat{\theta}} &= -\frac{\sqrt{H}}{r} \varpi^{\hat{\theta}}, & \Gamma_{\hat{r}\hat{\varphi}} &= -\frac{\sqrt{H}}{r} \varpi^{\hat{\varphi}}, & \Gamma_{\hat{r}\hat{t}} &= \frac{H'}{2\sqrt{H}} \varpi^{\hat{t}}, \\ \Gamma_{\hat{\theta}\hat{\varphi}} &= -\frac{\cot \theta}{r} \varpi^{\hat{\varphi}}, & \Gamma_{\hat{\theta}\hat{t}} &= 0, & \Gamma_{\hat{\varphi}\hat{t}} &= 0, \end{aligned}$$

where the prime is used to indicate the derivative with respect to  $r$ .

.....

The calculation is fairly straightforward:

$$\begin{aligned} \nabla_{\widetilde{W}} \widetilde{W} &= \widetilde{e}_\mu W^\lambda W^\mu{}_{;\lambda} = \widetilde{e}_\mu W^\mu{}_{;\hat{t}} = \widetilde{e}_\mu \{W^\mu{}_{,\hat{t}} + \Gamma^\mu{}_{\eta\hat{t}} W^\eta\} = \widetilde{e}_\mu \Gamma^\mu{}_{\hat{t}\hat{t}} = \widetilde{e}_{\hat{r}} \Gamma_{\hat{r}\hat{t}\hat{t}} \\ &= \left\{ \frac{H'}{2\sqrt{H}} \right\} \widetilde{e}_{\hat{r}} = \frac{1}{2} H' \partial_r. \end{aligned}$$

2. For the manifold in the previous problem, the *principal null geodesics* are defined as a particular pair of null geodesics, one outgoing and one incoming, which describe null rays which move “as directly radial as possible.” Therefore, these geodesics will have only radial and temporal components.

- a. To determine this pair of geodesics in this geometry, please first write down tangent vectors for **null** geodesics, spanned just on the basis elements  $\widetilde{e}_{\hat{r}}$  and  $\widetilde{e}_{\hat{t}}$ , where these are the reciprocal basis vectors to the set of basis 1-forms given in the problem above.

- b. Next, write out sufficiently many of the geodesic equations to show explicitly that the quantity  $\sqrt{H(r)} u^{\hat{t}}$  is a constant of the motion, i.e., that its derivative with respect to  $\lambda$  vanishes, where  $\lambda$  is an arbitrary choice of *affine* parameter along the curves to which these are tangent vectors, and then write out the tangent vector explicitly,
- c. Using the statement that  $H(r) = 1 - 2m/r$ , you now have enough information to integrate the geodesic equations to provide equations for the curves (paths) to which they are tangent. Give those equations in the form of equations for  $t = t(r)$ , for each of the two directions.

.....

- a. A null geodesic has zero length; therefore, in our orthonormal basis set, we know that we must have the following,  $u^{\hat{r}} = \pm u^{\hat{t}}$ , modulo perhaps some constant proportionality on both sides, which we can safely ignore, by proper choice of affine parameter,  $\lambda$ . As well, perhaps, one should note that we also want  $u^{\hat{\theta}} = 0 = u^{\hat{\phi}}$ .
- b. Now we must write down the geodesic equations, noticing that we should really only need the ones for radial and temporal directions:

$$\frac{d}{d\lambda} u^{\hat{r}} = -\frac{H'}{2\sqrt{H}} (u^{\hat{t}})^2, \quad \frac{d}{d\lambda} u^{\hat{t}} = -\frac{H'}{2\sqrt{H}} u^{\hat{r}} u^{\hat{t}}.$$

Multiplying the second of these equations by  $\sqrt{H}$ , we may use integration by parts to re-write it as

$$\frac{d}{d\lambda} \left( \sqrt{H} u^{\hat{t}} \right) = \frac{1}{2} \frac{H'}{\sqrt{H}} u^{\hat{t}} \left\{ \frac{dr}{d\lambda} - \sqrt{H} u^{\hat{r}} \right\}.$$

However, of course the radial component of the tangent vector to the curve is just given by  $u^{\hat{r}} = (dr/d\lambda)/\sqrt{H}$ , so that the expression on the righthand side vanishes, telling us that the expression of the lefthand side is constant. Naming that constant  $A$  we have the desired expression

$$\sqrt{H} u^{\hat{t}} = A.$$

Therefore, taking  $\lambda$  as an arbitrary affine parameter along our null ray, we may write immediately that

$$\sqrt{H} \frac{dt}{d\lambda} = u^{\hat{t}} = \frac{A}{\sqrt{H}} = \pm u^{\hat{r}} = \pm \frac{1}{\sqrt{H}} \frac{dr}{d\lambda}, \quad \implies \quad \tilde{u} = \frac{A}{\sqrt{H}} (\tilde{e}_{\hat{t}} \pm \tilde{e}_{\hat{r}}).$$

- c. We may now go on from there to find the equations for the paths:

$$\frac{dr}{d\lambda} = \pm A, \quad \frac{dt}{d\lambda} = \frac{A}{H} \quad \implies \quad \frac{dt}{dr} = \frac{dt/d\lambda}{dr/d\lambda} = \pm \frac{1}{1 - 2m/r}.$$

This is an easy integral, which gives us

$$\begin{aligned} t &= t_0 + r + 2m \log(r - 2m), & \text{for the outgoing ray,} \\ t &= t_0 - r - 2m \log(r - 2m), & \text{for the incoming ray,} \end{aligned}$$

It is perhaps worth noting that since  $dr/d\lambda$  is a constant, one could have chosen  $r$  itself as an affine parameter along this null curve. I also note that these are sometimes called tortoise coordinates because the times go to  $\pm\infty$  as  $r$  approaches  $2m$ .

3. Using the definitions of the covariant and the Lie derivatives, show explicitly that the components of the Lie derivative of a 1-form may be written in terms of either ordinary or covariant derivatives; i.e., show that

$$\mathcal{L}_{\xi} \alpha \equiv \omega^{\mu} \{ \xi^{\nu} \alpha_{\mu;\nu} + \xi^{\nu}{}_{;\mu} \alpha_{\nu} \} = \omega^{\mu} \{ \xi^{\nu} \alpha_{\mu;\nu} + \xi^{\nu}{}_{;\mu} \alpha_{\nu} \} .$$

.....

We simply concern ourselves with the components of this vector, and begin with the form involving covariant derivatives, AND restrict ourselves to the case where we have chosen the basis vectors for 1-forms, and tangent vectors, as coordinate bases:

$$\left( \mathcal{L}_{\xi} \alpha \right)_{\mu} = \xi^{\nu} \alpha_{\mu;\nu} + \xi^{\nu}{}_{;\mu} \alpha_{\nu} = \xi^{\nu} [ \alpha_{\mu;\nu} - \Gamma^{\eta}{}_{\mu\nu} \alpha_{\eta} ] + [ \xi^{\nu}{}_{;\mu} + \Gamma^{\nu}{}_{\eta\mu} \xi^{\eta} ] \alpha_{\nu} = \xi^{\nu} \alpha_{\mu;\nu} + \xi^{\nu}{}_{;\mu} \alpha_{\nu} ,$$

where the last equality shows the desired result, namely that the terms involving the connections should cancel, but of course uses the fact that the connection is symmetric in the last two indices when one uses a coordinate basis.

4. The following  $4 \times 4$  matrix is a generator for a particular Lorentz transformation:

$$Q^{\mu}{}_{\nu} = \begin{pmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} ,$$

where the basis is the usual,  $\{\hat{x}, \hat{y}, \hat{z}, \hat{t}\}$ .

- a. Please calculate the Lorentz transformation in question that has parameter  $a$ , i.e., determine the matrix

$$L^{\mu}{}_{\nu} = e^{a Q^{\mu}{}_{\nu}} .$$

- b. Then show that it is indeed a Lorentz transformation by proving that it leaves the usual Lorentz metric,  $\eta_{\mu\nu}$  invariant. Also please give “qualitative” arguments, rather than extensive calculations, to show that it is **not** a pure rotation, **nor** is it a pure boost.
- c. If I begin with the 4-velocity for a particle at rest, and allow this transformation matrix to act upon it, what will be the magnitude of the 3-velocity of that particle then?

.....

- a. In principle we must determine the exponential of  $aQ$  by summing the series for the exponential that involves all of its powers; however, we first calculate a few of them, and determine that already its third power vanishes, therefore rendering the sum quite simple:

$$Q^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -a^2 & 0 & +a^2 \\ 0 & 0 & 0 & 0 \\ 0 & -a^2 & 0 & +a^2 \end{pmatrix}, \quad Q^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore the desired transformation matrix is simply

$$L \equiv e^{aQ} = I + aQ + \frac{1}{2}a^2Q^2 = \begin{pmatrix} 1 & a & 0 & -a \\ -a & 1 - a^2/2 & 0 & a^2/2 \\ 0 & 0 & 1 & 0 \\ -a & -a^2/2 & 0 & 1 + a^2/2 \end{pmatrix}.$$

- b. To show that it is a Lorentz transformation we effect the transformation that it makes on the Lorentz metric,  $\eta_{\alpha\beta}$ :

$$\begin{aligned} H' = L^T H L &= \begin{pmatrix} 1 & -a & 0 & -a \\ a & 1 - a^2/2 & 0 & -a^2/2 \\ 0 & 0 & 1 & 0 \\ -a & a^2/2 & 0 & 1 + a^2/2 \end{pmatrix} \begin{pmatrix} 1 & a & 0 & -a \\ -a & 1 - a^2/2 & 0 & a^2/2 \\ 0 & 0 & 1 & 0 \\ a & a^2/2 & 0 & -1 - a^2/2 \end{pmatrix} \\ &= \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = H, \end{aligned}$$

as desired. As well, it cannot be a rotation for it mixes the spatial and temporal components of vectors, while it cannot be a pure boost since it is not a symmetric matrix.

- c. Acting on a particle originally at rest, and therefore, with 4-velocity,  $\tilde{u} = \tilde{e}_{\hat{t}}$ , we acquire

$$\tilde{u} = \begin{pmatrix} 1 & a & 0 & -a \\ -a & 1 - a^2/2 & 0 & a^2/2 \\ 0 & 0 & 1 & 0 \\ -a & -a^2/2 & 0 & 1 + a^2/2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -a \\ a^2/2 \\ 0 \\ 1 + a^2/2 \end{pmatrix}.$$

As we know the 4-velocity should have the generic form

$$\tilde{u} = \gamma_v \begin{pmatrix} v^x \\ v^y \\ v^z \\ 1 \end{pmatrix} \implies \vec{v} = \frac{-a\hat{x} + (a^2/2)\hat{z}}{1 + a^2/2} \implies v = |\vec{v}| = a \frac{\sqrt{1 + a^2/4}}{1 + a^2/2}.$$

Note that such a generator, which vanishes completely after only finitely powers is referred to as having generated a *parabolic transformation*. Note also that the velocity in question varies from 0 to 1 as the parameter  $a$  varies from 0 to  $\infty$ .

- 
5. We have studied the Brinkman metric a little bit. Recall that it is a solution of the Einstein vacuum field equations that describes plane gravitational waves: in terms of coordinates  $\{a, b, u, v\}$ , it may be written in terms of those coordinates and one arbitrary function of one variable,  $h = h(u)$ :

$$\mathbf{g} \equiv ds^2 = 2 da db + a^2 h(u) du^2 - du dv .$$

We used the following non-holonomic basis for 1-forms:

$$\varpi^{\hat{a}} \equiv da , \quad \varpi^{\hat{b}} \equiv db , \quad \varpi^{\hat{u}} \equiv du , \quad \varpi^{\hat{v}} \equiv \frac{1}{2}(a^2 h du - dv) , \quad \mathbf{g} \equiv g_{\mu\nu} \varpi^\mu \otimes \varpi^\nu ,$$

and found that the only non-zero connection 1-forms were

$$\mathfrak{L}^{\hat{u}\hat{a}} = ah \varpi^{\hat{u}} = -\mathfrak{L}^{\hat{a}\hat{u}} .$$

Using the standard transformation formula for the affine connections please determine all the non-zero Christöfel symbols,  $\left\{ \begin{smallmatrix} \alpha \\ \beta \gamma \end{smallmatrix} \right\} = \left\{ \begin{smallmatrix} \alpha \\ \gamma \beta \end{smallmatrix} \right\}$ , for this metric, where they determine the connection in this choice of basis via

$$\mathfrak{L}'^\alpha{}_\beta = \left\{ \begin{smallmatrix} \alpha \\ \beta \gamma \end{smallmatrix} \right\} dx^\gamma .$$

We are treating the relationship between the original non-holonomic basis and the standard coordinate basis (for 1-forms) as a transformation from the non-holonomic basis to the coordinate basis:

$$\begin{aligned} \text{When } dx^\alpha \equiv \varpi'^\alpha &\equiv X^\alpha{}_\mu \varpi^\mu , & \partial_{x^\beta} \equiv \tilde{e}'_\beta &\equiv Y^\nu{}_\beta \tilde{e}_\nu , & Y^\nu{}_\beta X^\beta{}_\mu &= \delta_\mu^\nu , \\ \text{then } \mathfrak{L}'^\alpha{}_\beta &= X^\alpha{}_\mu Y^\nu{}_\beta \mathfrak{L}^\mu{}_\nu + X^\alpha{}_\mu dY^\mu{}_\beta . \end{aligned}$$

.....

We begin by writing down the matrices  $X$  and  $Y$  for these basis vectors:

$$X^\alpha{}_\mu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & a^2 h & -2 \end{pmatrix} , \quad Y^\nu{}_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2}a^2 h & -\frac{1}{2} \end{pmatrix} .$$

With those matrices in hand, it is probably worthwhile to simply write down the transformation equation as if it were a matrix equation—some of the matrices of course having entries which are 1-forms—with  $\alpha$  the row index and  $\beta$  the column index:

$$\mathfrak{L}' = X \mathfrak{L} Y + X dY .$$

Since only one entry in the matrix  $Y$  is not constant, we find that the “extra” term in the transformation equation, i.e., the second term on the right hand side above, is just

$$X^\alpha{}_\mu dY^\mu{}_\beta = -(2ah da + a^2 h' du) \delta_v^\alpha \delta_\beta^u ,$$

which then will contribute only to  $\underline{\Gamma}'^v{}_u$ ; i.e., this is a matrix—with rows labelled by  $\alpha$  and columns labelled by  $\beta$ , which has only one non-zero entry. To proceed with the tensorial part of the transformation equation it is most convenient to first raise the first index on the connections that we have been given in the non-holonomic basis, so that they can be inserted into the matrix equations:

$$ah du = ah \varpi^{\hat{u}} = \underline{\Gamma}_{\hat{u}\hat{a}} = \underline{\Gamma}^{\hat{v}}{}_{\hat{a}}, \quad -ah du = -ah \varpi^{\hat{u}} = \underline{\Gamma}_{\hat{a}\hat{u}} = \underline{\Gamma}^{\hat{b}}{}_{\hat{u}}.$$

At this point, one could multiply the 3 matrices directly, as the tensorial part of the transformation, which I will do below. However, as the matrices are mostly zeros and ones, I can also simply work them out by hand, using the appropriate pieces of the matrices  $X$  and  $Y$ , which give me non-zero values only for  $\underline{\Gamma}^v{}_a$  and  $\underline{\Gamma}^b{}_u$ . As these have no overlap with our first determination, from the 'extra' term, using it as well we may now write down the entire set, which includes 3 1-forms:

$$\underline{\Gamma}'^v{}_a = -2ah du, \quad \underline{\Gamma}'^b{}_u = -ah du, \quad \underline{\Gamma}'^v{}_u = -(2ah da + a^2 h' du).$$

We now re-write this content in terms of the Christöffel symbols themselves, displaying the symmetry that they have on the lower two indices:

$$\left\{ \begin{matrix} v \\ au \end{matrix} \right\} = -2ah, \quad \left\{ \begin{matrix} b \\ uu \end{matrix} \right\} = -ah, \quad \left\{ \begin{matrix} v \\ ua \end{matrix} \right\} = -2ah, \quad \left\{ \begin{matrix} v \\ uu \end{matrix} \right\} = -a^2 h'.$$

However, as noted above, the calculation may certainly be done as a matrix equation. We first re-write the different values of the connection in the form of a square matrix:

$$\Gamma' = ah du \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

which then gives us

$$\begin{aligned} X \Gamma Y &= ah du \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & a^2 h & -2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & a^2 h/2 & -1/2 \end{pmatrix} \\ &= ah du \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{pmatrix} \\ \implies \Gamma' &= X \Gamma Y + X dY = ah du \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{pmatrix} - d(a^2 h) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \end{aligned}$$

which is completely consistent with the results calculated above.