Physics 570

Exam II

26 April, 2007

Solutions

There are 4 questions, each with equal weight of $33\frac{1}{3}$ pts each, BUT you may provide answers to only **any THREE** of them.

1. A very brave physicist has been boosted into a directly radial, infall-trajectory for a nearby black hole, and is now moving on a radial geodesic in the equatorial plane. His 4-velocity is given by

$$\widetilde{u} = \frac{1}{\sqrt{1-2m/r}}\left[-\sqrt{2m/r}\,\widetilde{e}_{\hat{r}} + \widetilde{e}_{\hat{t}}\right] .$$

a. As he passes the location with $r = 200m$, what is the magnitude of his 3-velocity?

He makes measurements of local phenomena as he passes, using his own, local tetrad of basis vectors, which of course are **orthonormal and parallel transported** along his worldline.

b. His local time is of course provided by his own (accurate) clock that he carries with him, so that his timelike tetrad vector, $\widetilde{f}_{\hat{t}}$, is just $\partial/\partial\tau$. However, in our coordinate system and tetrad, how do we describe this (timelike) unit vector?

c. As well he needs three vectors to form his own spatial triad. First, please show that it is allowed for him to choose the two of those triad vectors that are perpendicular to his own motion as the following:

$$\widetilde{f}_{\hat{\theta}} = \tfrac{1}{r}\partial_\theta = \widetilde{e}_{\hat{\theta}}\ , \quad \widetilde{f}_{\hat{\varphi}} = \tfrac{1}{r}\partial_\varphi = \widetilde{e}_{\hat{\varphi}}\Big|_{\theta=\pi/2} .$$

Secondly, determine a proper form for the third of his triad vectors, i.e., the one that he would use to consider nearby objects with components along his direction of motion. This 4-vector must of course be parallelly transported and normal to **all** the others.

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a. Using the given 4-velocity, his (locally-measured but in our coordinates and [orthonormal] tetrad) is given by the following which we display, and then evaluate at $r = 200m$:

$$\vec{v} = \frac{\vec{u}}{u^{\hat{t}}} = -\sqrt{2m/r}\,\widetilde{e}_{\hat{r}} \quad \underset{r/m=200}{\longrightarrow} \quad -\sqrt{1/100}\,\widetilde{e}_{\hat{r}} = -0.1\,\widetilde{e}_{\hat{r}} .$$

The magnitude of this vector is of course just 0.1, i.e., one tenth of the speed of light!

b. Of course, he measures his own 4-velocity as just what it would be if he is at rest; i.e., he uses $\widetilde{u}$ as his timelike tetrad member, so we have

$$\widetilde{f}_{\hat{t}} = \widetilde{u} .$$

As an aside, it is worth noting that he is freely-falling, so that his worldline is a geodesic; therefore, it is being parallelly-propagated along his worldline, as is required for his tetrad vectors.

c. We must now first show that the two vectors given to us, for his two orthogonal triad members, are being parallelly-propagated. The general equation is then

$$0 = \nabla_{\underset{\sim}{u}}\widetilde{v} = \widetilde{e}_{\hat\alpha} u^{\hat\beta} v^{\hat\alpha}{}_{;\hat\beta} = \widetilde{e}_{\hat\alpha} u^{\hat\beta}\left\{ v^{\hat\alpha}{}_{,\hat\beta} + \Gamma^{\hat\alpha}{}_{\hat\zeta\hat\beta}\, v^{\hat\zeta}\right\} \ .$$

Of course our curve has only two non-zero components in its tangent vector, so we may look for the four equations above with that constraint filled in, for each required value of α:

$$0 = u^{\hat r}\left\{ v^{\hat\alpha}{}_{,\hat r} + \Gamma^{\hat\alpha}{}_{\hat\zeta\hat r} v^{\hat\zeta}\right\} + u^{\hat t}\left\{ v^{\hat\alpha}{}_{,\hat t} + \Gamma^{\hat\alpha}{}_{\hat\zeta\hat t} v^{\hat\zeta}\right\} \ .$$

We want to show that the given vectors, $\widetilde{e}_{\hat\theta}$ and $\widetilde{e}_{\hat\varphi}$ satisfy this equation, where $v^{\hat\alpha}$ is a constant for each of them, so that the ordinary derivative term vanishes; we look first at $\widetilde{e}_{\hat\theta}$:

$$u^{\hat r}\Gamma^{\hat\alpha}{}_{\hat\theta\hat r} + u^{\hat t}\Gamma^{\hat\alpha}{}_{\hat\theta\hat t} \ .$$

In fact both terms vanish, as desired, because we have

$$\underset{\sim}{\Gamma}{}^{\hat r}{}_{\hat\theta} \propto \underset{\approx}{\omega}{}^{\hat\theta} \ , \quad \underset{\sim}{\Gamma}{}^{\hat\varphi}{}_{\hat\theta} \propto \underset{\approx}{\omega}{}^{\hat\varphi} \ , \qquad \underset{\sim}{\Gamma}{}^{\hat\theta}{}_{\hat\theta} = 0 = \underset{\sim}{\Gamma}{}^{\hat t}{}_{\hat\theta} \ .$$

Then, for the second such vector, $\widetilde{e}_{\hat\varphi}$, the completely analogous equation is

$$u^{\hat r}\Gamma^{\hat\alpha}{}_{\hat\varphi\hat r} + u^{\hat t}\Gamma^{\hat\alpha}{}_{\hat\varphi\hat t} \ .$$

In fact both terms again vanish, as desired, because we have

$$\underset{\sim}{\Gamma}{}^{\hat r}{}_{\hat\varphi} \propto \underset{\approx}{\omega}{}^{\hat\varphi} \ , \quad \underset{\sim}{\Gamma}{}^{\hat\theta}{}_{\hat\varphi} \propto \underset{\approx}{\omega}{}^{\hat\varphi} \ , \qquad \underset{\sim}{\Gamma}{}^{\hat\varphi}{}_{\hat\varphi} = 0 = \underset{\sim}{\Gamma}{}^{\hat t}{}_{\hat\varphi} \ .$$

Therefore, indeed both these vectors are parallelly transported along his worldline. It is of course quite obvious that they are each normal and orthogonal both to each other and to the worldline in question.

The only remaining requirement is the properly-normalized, parallelly-transported, spacelike vector he should choose. It is clear that it should have components both along the radial and temporal directions; as it must also be orthogonal to the worldline, we simply "switch" the components, suggesting that the correct answer is

$$\widetilde{f}_{\hat r} = \frac{1}{\sqrt{1-2m/r}}\left[\widetilde{e}_{\hat r} - \sqrt{2m/r}\,\widetilde{e}_{\hat t}\right] \ .$$

It is straightforward to calculate the desired orthonormality conditions, although the notation is simpler if we first denote new symbols

$$\beta \equiv \sqrt{2m/r} \quad h \equiv \sqrt{1-2m/r} \qquad \widetilde{u} = (-\beta\,\widetilde{e}_{\hat r} + \widetilde{e}_{\hat t})/h \ , \quad \widetilde{f}_{\hat r} = (\widetilde{e}_{\hat r} - \beta\widetilde{e}_{\hat t})/h \ ,$$
$$\widetilde{f}_{\hat r}\cdot\widetilde{f}_{\hat r} = (1-\beta^2)/h^2 = +1 \ , \quad \widetilde{f}_{\hat r}\cdot\widetilde{f}_{\hat t} = -\beta + \beta = 0 \ , \quad \widetilde{f}_{\hat t}\cdot\widetilde{f}_{\hat t} = (\beta^2-1)/h^2 = -1 \ .$$

However, we should also still show that this vector is parallelly-transported. Therefore we first re-write the parallel-transport equations above for this case, and using these abbreviations, noting as well the important fact that $\underset{\sim}{\Gamma}_{\hat{r}\hat{t}} \propto \underset{\sim}{\omega}^{\hat{t}}$, so that $\Gamma_{\hat{r}\hat{t}\hat{r}} = 0$, and also note that none of our quantities depends on t:

$$\begin{aligned}
& 0 = -\beta \left\{ h\, \partial_r v^{\hat{\alpha}} + \Gamma^{\hat{\alpha}}{}_{\hat{\zeta}\hat{r}}\, v^{\hat{\zeta}} \right\} + \Gamma^{\hat{\alpha}}{}_{\hat{\zeta}\hat{t}}\, v^{\hat{\zeta}} \; ; \\
\alpha = \hat{r} : \quad & -\alpha\, h\, \partial_r \frac{1}{h} - \frac{dH/dr}{2h}\frac{\alpha}{h} = 0 \; , \\
\alpha = \hat{\theta} : \quad & -\alpha \Gamma_{\hat{\theta}\hat{\zeta}\hat{r}} v^{\hat{\zeta}} + \Gamma_{\hat{\theta}\hat{\zeta}\hat{t}} v^{\hat{\zeta}} = 0 \; , \\
\alpha = \hat{\varphi} : \quad & -\alpha \Gamma_{\hat{\varphi}\hat{\zeta}\hat{r}} v^{\hat{\zeta}} + \Gamma_{\hat{\varphi}\hat{\zeta}\hat{t}} v^{\hat{\zeta}} = 0 \; , \\
\alpha = \hat{t} : \quad & +\alpha\, h \partial_r \frac{\alpha}{h} - \Gamma_{\hat{t}\hat{r}\hat{t}} \frac{1}{h} = -\frac{\alpha^4}{4mh^2} + \frac{m/r^2}{h^2} = 0 \; .
\end{aligned}$$

We see therefore that this (desirable) spacelike triad vector is also parallelly propagated, and so does constitute the last of his tetrad vectors.

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2. Show that the existence of a covariantly-constant vector field on a manifold puts constraints on the allowed possible curvatures. Alternatively, but equivalently, show that a manifold with an arbitrary non-zero Riemann curvature tensor will not admit a covariantly-constant vector field. Next, what if, instead, the covariant derivative of that vector field is not zero, but just has its symmetric part zero. Again, does this put any constraints on the possible curvatures?

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a. If a vector field, $\widetilde{k}$ is covariantly-constant, this tells us that

$$k^{\mu}{}_{;\nu} = 0 \; , \qquad \forall \mu, \nu = 1, 2, 3, 4 \; .$$

Therefore, we may ask—if you like that approach—what are the integrability conditions for this; alternatively, we may say that if this statement is true in some neighborhood—as would be necessary to have it true for a vector **field**—then we may differentiate the equation. Therefore we have the following, which generates a constraint on the Riemann curvature tensor:

$$0 = k^{\mu}{}_{;[\nu\lambda]} = R^{\mu}{}_{\eta\nu\lambda} k^{\eta} \; .$$

That is to say the existence of this covariantly-constant vector field requires that it be an eigenvector for the curvature tensor, with zero eigenvalue!

b. For the second case, it is convenient to first use the metric to lower the index on the components of the vector field, so that we are actually dealing with its associated 1-form field—a co-vector field. Then we are told that

$$k_{\mu;\nu} + k_{\nu;\mu} = 0 \; .$$

These are actually Killing's equation! Therefore the manifold that allows—even in some neighborhood only, perhaps—such a vector field must have a symmetry, and is therefore clearly constrained away from the most general manifold, which will certainly not have any symmetries at all. This is sufficient answer; on the other hand if more explicit detail is wanted we may refer to the (known) integrability conditions for a Killing vector, as it affects the curvature tensor, which are the following two requirements:

$$R^{\mu}{}_{\eta\lambda\nu}k_{\mu} = k_{\nu;\eta\lambda} \ , \qquad \mathcal{L}_{\widetilde{k}} R^{\mu}_{\eta\lambda\nu} = 0 \ .$$

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3. Using the geodesic equations, determine the angular frequency, $\Omega \equiv d\varphi/dt$, of a circular geodesic in the equatorial plane in the Reissner-Nordström metric. This is then the generalization of the usual Kepler relation, that $\Omega^2 = m/r^3$, to this case for a charged central mass.

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Using the vector field, $\widetilde{v}$ for the tangent vector to this geodesic, a circular geodesic clearly needs to have $v^{\hat{r}} = 0$; therefore, it is important that the geodesic equation which says that $v^{\hat{r}}{}_{;\nu} = 0$ should be satisfied. Using the general, spherically-symmetric connections, from the handout, for the special case that $JH = 1$, we have this important requirement in detail:

$$0 = \frac{d}{d\tau} v^{\hat{r}} = \frac{\sqrt{H}}{r}\left[(v^{\hat{\theta}})^2 + (v^{\hat{\varphi}})^2\right] - \frac{dH/dr}{2\sqrt{H}}(v^{\hat{t}})^2 \ .$$

However, we know that this geodesic stays in the equatorial plane, so that we should take $v^{\hat{\theta}} = 0$. As well we know the value of H for this manifold and these coordinates:

$$H = 1 - \frac{2m}{r} + \left(\frac{q}{r}\right)^2 \quad \Longrightarrow \quad \frac{dH}{dr} = \frac{2m}{r^2} - 2\frac{q^2}{r^3} \ .$$

Therefore, our geodesic equation gives us a relationship between $v^{\hat{\varphi}}$ and $v^{\hat{t}}$:

$$(v^{\hat{\varphi}})^2 = \frac{r\, dH/dr}{2H}(v^{\hat{t}})^2$$
$$\Longrightarrow \Omega^2 \equiv \left(\frac{d\varphi}{dt}\right)^2 = \left(\frac{\frac{1}{r}v^{\hat{\varphi}}}{v^{\hat{t}}/\sqrt{H}}\right)^2 = \frac{dH/dr}{2r} = \frac{m}{r^3} - \frac{q^2}{r^4} \ .$$

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4. A family of nearby observers are ZAMO's in **circular** orbit about a Kerr black hole, all at the same radius, in the equatorial plane. They each use our standard orthonormal set of basis 1-forms for their own measurements—as given, for example, in the Kerr metric handout. Therefore, their worldlines all have tangent vector $u^{\hat\beta}\,\widetilde{e}_{\hat\beta} = \widetilde{u} = \widetilde{e}_{\hat t}$ for a fixed value of r, with of course slightly different initial values of φ_0.

a. **Explain why** this 4-velocity is a timelike Killing vector.

b. Show that this worldline has zero twist. Recall that the simplest way to define this condition of a (congruence of nearby) worldlines is to look at the associated (dual) 1-form, $\underset{\sim}{u} = g_{\hat\alpha\hat\beta}u^{\hat\beta}\,\underset{\sim}{\omega}^{\hat\alpha}$, and show that it is surface forming: i.e., that $\underset{\sim}{u} \wedge d\underset{\sim}{u} = 0$.

c. Even though the 4-velocity in question is non-twisting, nonetheless it is not freely-falling; i.e., these are not inertial observers since they do not travel along geodesics. Show that this is true. However, it is only required to demonstrate that the acceleration is non-zero; it is not requested that you calculate it explicitly!

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a. From the Kerr metric handout we see that the worldline for the ZAMO is just

$$\widetilde{u} = \widetilde{e}_{\hat t} = \sqrt{\frac{A}{\Sigma\Delta}}\,(\partial_t + \omega\,\partial_\varphi) \ ,$$

where of course all the functions there are evaluated at constant values of $\theta = \pi/2$ and r, so that A, $\Sigma = r^2$, Δ, and ω are all just constants for our assembly of ZAMO's. However, we also know that ∂_t and ∂_φ are Killing vectors. Therefore, $\widetilde{u}$ is simply a sum of two Killing vectors multiplied by various constants; such a thing is surely also a Killing vector, so we are done!

b. While it is true that $\widetilde{u} = \widetilde{e}_{\hat t}$ is a Killing vector for this case, it is NOT true that it is a geodesic. Therefore, we now want to show that this is true. However, we are first asked to calculate its twist, and even given a hint. The associated 1-form to $\widetilde{e}_{\hat t}$ is of course $\underset{\sim}{\omega}^{\hat t}$. We now calculate its twist—that calculation being very much easier when done with the 1-forms:

$$\begin{aligned}
\underset{\sim}{\omega}^{\hat t} &= H\,dt\ , \qquad H^2 \equiv \frac{\Sigma\Delta}{A}\Big|_{\theta=\pi/2}\ ,\\
d\underset{\sim}{\omega}^{\hat t} &= \frac{\partial H}{\partial\theta}\,d\theta\wedge dt + \frac{\partial H}{\partial r}\,dr\wedge dt\ ,\\
\underset{\sim}{\omega}^{\hat t}\wedge d\underset{\sim}{\omega}^{\hat t} &= H\,dt\wedge\left(\frac{\partial H}{\partial\theta}\,d\theta\wedge dt + \frac{\partial H}{\partial r}\,dr\wedge dt\right) = 0\ .
\end{aligned}$$

c. Having shown that it has zero twist, we nonetheless need to show that not all of the covariant derivative is zero. Therefore I note that an appropriate way to write out the geodesic equation is

$$\nabla_{\widetilde{u}}\widetilde{u} = \left\{\widetilde{u}(u^{\hat\lambda} + \underset{\sim}{\Gamma}^{\hat\lambda}{}_{\hat\nu}(\widetilde{u})u^{\hat\nu}\right\}\widetilde{e}_{\hat\lambda}\ .$$

In our case, we have $u^{\hat\nu} = \delta^{\nu}_{\hat t}$, which is constant, so this equation says that

$$\left\{\nabla_{\widetilde{u}}\widetilde{u}\right\}^{\hat\lambda} = \Gamma^{\hat\lambda}{}_{\hat t\hat t}\ .$$

Consulting the table of connections for the Kerr metric, we see quickly that this quantity is non-zero for $\hat{\lambda} = \hat{r}$ and also for $\hat{\lambda} = \hat{\theta}$. Therefore the non-zero values of each of these makes it quite clear that there are accelerations in each of those directions; i.e., this is not a congruence of geodesics.