Physics 570

Homework #1

Due Thursday, 1 February, 2007

1. Please make a good drawing of a (standard, only 2-dimensional) Minkowski diagram, as seen by an observer $\mathcal{O}$, with his x and t-axes shown at 90° to one another. [Recall that you are using $c = 1$.] Create the diagram either on some good graph paper or, preferably, on a computer inside an computer-mathematics program such as Maple.
On this diagram, please show the following things:

a.) the world line of a clock that $\mathcal{O}$ maintains at the location $x = 1$ m;

b.) the trajectories of two light rays sent out by $\mathcal{O}$ at his time $t = 0$, into the $+\hat{x}$-direction and the $-\hat{x}$-direction;

c.) the world line of a distinct inertial observer, $\mathcal{O}'$, that $\mathcal{O}$ measures to be moving with speed $v = 0.25$, and which was coincident with $\mathcal{O}$ at their mutual values of $t = 0 = t'$;

d.) the line of events that $\mathcal{O}'$ measures to be simultaneously occurring at her time $t' = 3$ m;

e.) the world line of an observer at rest with respect to $\mathcal{O}'$but who was at the location $x = -2$ m at the time $t = 0$ m;

f.) the locus of all events which, relative to the origin, have $(\Delta s)^2 = -9\ \mathrm{m}^2$;

g.) the locus of all events which, relative to the origin, have $(\Delta s)^2 = +4$ and which also occur at $t \geq 0$.

2. A timelike worldline is one such that the displacement vector between any two points is timelike; all worldlines belonging to any physical observer, or indeed any physical object, must be everywhere timelike.
For a timelike worldline we may define the 4-velocity, $\widetilde{u}$, in terms of the displacement between two infinitesimally-near events (on that worldline), with spatial displacement $d\vec{r}$ and temporal displacement dt, and the proper time, $d\tau$ that separates the events belonging to the motion of the object in question:

$$\widetilde{u} \equiv \begin{pmatrix} \frac{d\vec{r}}{d\tau} \\ \frac{dt}{d\tau} \end{pmatrix} = \frac{dt}{d\tau} \begin{pmatrix} \vec{v} \\ 1 \end{pmatrix} \equiv \gamma_v \begin{pmatrix} \vec{v} \\ 1 \end{pmatrix} , \qquad \gamma_v \equiv \frac{1}{\sqrt{1 - v^2}} .$$

a. Using the usual definition of the 3-vector acceleration, $\vec{a} \equiv d\vec{v}/dt$, determine the form of the 4-acceleration

$$\widetilde{a} \equiv \frac{d}{d\tau} \widetilde{u} ,$$

in terms of $\vec{a}$, $\vec{v}$, and γ_v.

b. Determine the Lorentz invariant associated with this 4-vector, i.e., the value of $\widetilde{a} \cdot \widetilde{a} = \eta_{\mu\nu} a^\mu a^\nu$. Verify that this quantity is never negative, thus allowing us to say that this vector is *spacelike*. What is its value in the so-called *instantaneous rest frame*, i.e., an inertial frame moving in such a way that at the moment in question it measures the object to momentarily at rest?

c. Divide the 3 spatial portions of the 4-acceleration vector into those parts that are parallel to its velocity and those that are perpendicular to its velocity, in as simple a way as possible. In each of these there is an overall factor of γ_v to some power; what is that power in each of these two cases?

3. In the usual Euclidean 3-space, a one-dimensional curve may be specified by giving each of the three Euclidean coordinates as functions of some single parameter. Below we give such a description of each of two curves, the first parametrized by λ and the second by μ, where both of these parameters vary, at least in principle, from $-\infty$ to $+\infty$:

$$\begin{array}{llll} \text{curve 1:} & x = \lambda\,, & y = (\lambda - 1)^2\,, & z = -\lambda\,, \\ \text{curve 2:} & x = \cos\mu\,, & y = \sin\mu\,, & z = \mu - 1\,, \end{array}$$

a. Please calculate the components of the tangent vector of each of these curves.

b. Let a particular function, f, be defined on this 3-space:

$$f = f(x, y, z) \equiv x^2 + y^2 - yz\ .$$

As this function varies along each of these two curves, please calculate its rate of change; i.e., calculate $df/d\lambda$ and $df/d\mu$.

4. Take a standard (flat) 2-dimensional plane, with origin, and treat it as a vector space. For this vector space choose (arbitrarily) some vector of unit length and call it $\hat{\xi}_1$. Then choose another vector of unit length, which makes an angle of α relative to the first one, and call it $\hat{\xi}_2$. For simplicity choose $\alpha \le \pi/2$.

We want to use these two vectors as basis vectors for our vector space; note that they are not orthogonal. Therefore, now choose a special vector, called $\vec{x}$. The purpose of this problem is to determine the components of $\vec{x}$ relative to our choice of basis vectors. There are two ways to do this, both of which I am asking you to calculate, as functions of the arbitrary angle α. It is easiest to visualize this process by making yourself a drawing on a piece of paper:

a. As one learned in freshman physics when decomposing vectors, we suppose that there are two numbers, x^1 and x^2—collectively referred to as $\{x^i\}_1^2$—which are such that

$$\vec{x} = x^1\,\hat{\xi}_1 + x^2\,\hat{\xi}_2\ .$$

Obviously these numbers are uniquely determined, and they uniquely determine our vector.

b. Since we understand angles in this plane, we may also suppose that we draw from the tip of the vector $\vec{x}$ a line perpendicular to $\hat{\xi}_1$, which then meets $\hat{\xi}_1$, perhaps extended along its own direction if necessary, at some distance from the origin; this is the perpendicular projection of $\vec{x}$ onto the direction established by $\hat{\xi}_1$. Refer to this number as x_1. Now do the same thing to determine x_2 by projecting perpendicular to $\hat{\xi}_2$. Using the usual Euclidean scalar product, which we denote by a central "dot" as usual, we could write these quantities according to the formula:

$$x_i = \vec{x}\cdot\hat{\xi}_i\ , \qquad i = 1, 2\ .$$

It is again obvious that these numbers are uniquely determined by the vector $\vec{x}$, and vice versa. Therefore, it is clear that either pair may legitimately be considered as "the components" of the vector $\vec{x}$ relative to this non-orthogonal choice of basis vectors.

Please determine, in as geometrical a way as possible, and in terms of α, the relationship between the two pairs of components of $\vec{x}$.

c. Now calculate the matrix which is the scalar product of the two basis vectors with themselves and each other, i.e.,

$$g_{ij} \equiv \hat{\xi}_i \cdot \hat{\xi}_j \ , \qquad i, j = 1, 2 \ .$$

Then show that in fact the two sets of components are related via

$$x_i = g_{ij} x^j \ , \qquad i, j = 1, 2 \ ,$$

remembering that the Einstein summation convention reminds us that there is a sum in the expression on the right-hand side, over all values of j.

d. Lastly calculate the quadratic expression

$$L^2 \equiv g_{ij} x^i x^j$$

and use the Law of Cosines, for instance, to show that L is in fact the length of the vector $\vec{x}$.