Physics 570

Homework #10

Due Thursday, 12 April, 2007

Solutions

1. Consider the manifold corresponding to the spherically-symmetric, interior Schwarzschild solution, corresponding to a fluid with constant density, and look at the 3-dimensional slices of constant time, where we know the metric is given by

$$\underset{3}{\mathbf{g}} = \frac{dr^2}{1 - 2\frac{\mathcal{M}(r)}{r}} + r^2\, d\Omega^2 = \frac{dr^2}{1 - 2m\frac{r^2}{R^3}} + r^2\, d\Omega^2 \; .$$

Please calculate the conformal tensor for this 3-dimensional manifold, and show that it vanishes. Such a space is referred to as conformally flat. As well show that the remaining ingredients in the curvature are simply constant, so that this is indeed a 3-manifold of constant curvature.

There is a coordinate transformation that preserves spherical symmetry and is such that the new (3-dimensional) metric in those coordinates is proportional to

$$\underset{3}{\mathbf{g}} \propto d\chi^2 + \sin^2\chi\, d\Omega^2 \; ,$$

which is proportional to the metric of a (closed) Robertson-Walker cosmology at a constant time slice. Can you find this transformation between the two?

. . . .

a. We first use some method to determine the connections and the curvature for this metric. I chose the obvious orthonormal basis:

$$\omega^{\hat{r}} \equiv \frac{dr}{\sqrt{1 - 2\frac{mr^2}{R^3}}} \; , \quad \omega^{\hat{t}} \equiv r\, d\theta \; , \quad \omega^{\hat{\varphi}} \equiv r \sin\theta\, d\varphi \; ,$$

and then asked GRTensorII to determine the connections and curvatures. It told me that

$$\underset{\sim}{\Gamma}_{\hat{r}\hat{\theta}\hat{\theta}} = -\frac{1}{r}\sqrt{1 - \frac{2mr^2}{R^3}} = \underset{\sim}{\Gamma}_{\hat{r}\hat{\varphi}\hat{\varphi}} \; , \quad \underset{\sim}{\Gamma}_{\hat{\theta}\hat{\varphi}\hat{\varphi}} = -\frac{\cot\theta}{r} \; ,$$

and that

$$R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}} = R_{\hat{r}\hat{\varphi}\hat{r}\hat{\varphi}} = R_{\hat{\theta}\hat{\varphi}\hat{\theta}\hat{\varphi}} = \frac{2m}{R^3} \equiv Z \; ,$$

all others not connected by a symmetry transformation of course being zero. It is intriguing that this is a (3-dimensional) *"space of constant curvature"* and isotropic in the sense that all the non-zero components of the curvature are the same. I then also asked it for the Weyl tensor and it told me that it vanished completely. However, perhaps you also want to recall how to perform this calculation by hand. I retreat to the N-dimensional definition of the

Weyl tensor in terms of the Riemann tensor, as applied to $N=3$ [Eq.(3.147) of Carroll, for example]:

$$C_{abcd} = R_{abcd} - g_{a[c}\mathcal{R}_{bd]} - g_{b[d}\mathcal{R}_{ac]} + \tfrac{1}{2} g_{a[c}g_{bd]}\mathcal{R} \, .$$

In our problem we then first need to calculate the Ricci tensor, and then its trace, noting that it is obviously diagonal:

$$\begin{aligned}
\mathcal{R}_{\hat{r}\hat{r}} &= R^{a}{}_{\hat{r}a\hat{r}} = R_{\hat{\theta}\hat{r}\hat{\theta}\hat{r}} + R_{\hat{\varphi}\hat{r}\hat{\varphi}\hat{r}} = 2Z \, , \\
\mathcal{R}_{\hat{\theta}\hat{\theta}} &= R^{a}{}_{\hat{\theta}a\hat{\theta}} = R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}} + R_{\hat{\varphi}\hat{\theta}\hat{\varphi}\hat{\theta}} = 2Z \, , \\
\mathcal{R}_{\hat{\varphi}\hat{\varphi}} &= R^{a}{}_{\hat{\varphi}a\hat{\varphi}} = R_{\hat{r}\hat{\varphi}\hat{r}\hat{\varphi}} + R_{\hat{\theta}\hat{\varphi}\hat{\theta}\hat{\varphi}} = 2Z \, , \\
\Longrightarrow \quad & \mathcal{R} = \mathcal{R}^{a}{}_{a} = 6Z \, .
\end{aligned}$$

Please do recall—although NOT required—that the Ricci tensor needs to be proportional to the energy-momentum tensor, and that in its rest-frame, the spatial components of that tensor are simply the pressures. As the situation is isotropic, all these pressures are the same. This is the reason all the components of the Ricci tensor above are non-zero and the same, and of course why it is diagonal.

We may then determine the Weyl tensor explicitly, assuming, again and reasonably, that it will have the same "diagonal" form as does the Riemann tensor:

$$\begin{aligned}
C_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}} &= R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}} - \mathcal{R}_{\hat{\theta}\hat{\theta}} - \mathcal{R}_{\hat{r}\hat{r}} + \tfrac{1}{2}\mathcal{R} = Z - 2Z - 2Z + 3Z = 0 \, , \\
C_{\hat{r}\hat{\varphi}\hat{r}\hat{\varphi}} &= R_{\hat{r}\hat{\varphi}\hat{r}\hat{\varphi}} - \mathcal{R}_{\hat{\varphi}\hat{\varphi}} - \mathcal{R}_{\hat{r}\hat{r}} + \tfrac{1}{2}\mathcal{R} = Z - 2Z - 2Z + 3Z = 0 \, , \\
C_{\hat{\theta}\hat{\varphi}\hat{\theta}\hat{\varphi}} &= R_{\hat{\theta}\hat{\varphi}\hat{\theta}\hat{\varphi}} - \mathcal{R}_{\hat{\theta}\hat{\theta}} - \mathcal{R}_{\hat{\varphi}\hat{\varphi}} + \tfrac{1}{2}\mathcal{R} = Z - 2Z - 2Z + 3Z = 0 \, ,
\end{aligned}$$

b. Defining the new coordinate χ by $r = \alpha \sin\chi$, we may differentiate this and then insert all this into the 3-metric above and obtain

$$\underset{3}{\mathbf{g}} = \alpha^2 \left\{ \frac{\cos^2\chi}{1 - \frac{2m}{R^3}\alpha^2 \sin^2\chi}(d\chi)^2 + \sin^2\chi \, d\Omega^2 \right\} \, .$$

We easily see that if we choose $\alpha^2 = R^3/2m$ the denominator above will just be $\cos^2\chi$ and we will have the desired form:

$$\underset{3}{\mathbf{g}} = \frac{R^3}{2m}\left\{(d\chi)^2 + \sin^2\chi \, d\Omega^2\right\} \, .$$

As the conformal tensor is independent of an overall, multiplicative factor in the metric, we now learn that this metric, also, is conformally flat; i.e., its Weyl (conformal) tensor will also vanish, so that it will be determined ONLY by its matter tensor (and possible cosmological constant).

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2. One can write various sorts of divergences of tensorial quantities in a simplified way which does not appear to need any covariant derivatives. To see this, show that for an arbitrary tangent vector $\widetilde{T}$ with components relative to some **coordinate basis** as T^α, show that the following equality is true:

$$T^\alpha{}_{;\alpha} = \frac{1}{\sqrt{-g}} \left(\sqrt{-g} T^\alpha\right)_{,\alpha} \ ,$$

where $-g$ is the negative of the determinant of the metric, and therefore a positive quantity.

. . . .

The basic key to the entire problem is to show the relationship between a partially-summed, coordinate-based connection and the square root of the determinant of the metric, in that same coordinate basis. Therefore we begin with the formula relating the connection components in a coordinate-basis to the first derivatives of the metric components:

$$\begin{aligned} \left\{ \begin{matrix} \mu \\ \lambda\eta \end{matrix} \right\} &= \tfrac{1}{2} g^{\mu\nu} \left(-g_{\lambda\eta,\nu} + g_{(\lambda\nu,\eta)}\right) \\ \Longrightarrow \quad \left\{ \begin{matrix} \mu \\ \lambda\mu \end{matrix} \right\} &= \tfrac{1}{2} g^{\mu\nu} \left(-g_{\lambda\mu,\nu} + g_{\lambda\nu,\mu} + g_{\mu\nu,\lambda}\right) = \tfrac{1}{2} g^{\mu\nu} g_{\mu\nu,\lambda} \ . \end{aligned}$$

At this point we revert to the relationship between the components of matrices and their determinants. This is given on pp. 162-3 of Carroll's text, but I repeat it here, as he does, first for an arbitrary matrix M, and then for the (desired) special case of M being the metric, and δ indicating some arbitrary variation—such as a partial derivative:

$$\begin{aligned} \log(\det M) = \operatorname{trace}(\log M) \quad &\Longleftrightarrow \quad \delta \log(\det M) = \operatorname{trace}(M^{-1}\delta(M)) \ , \\ &\Longrightarrow \quad \delta \log(g) = g^{\mu\nu} \delta g_{\nu\mu} \ . \end{aligned}$$

We then insert this into our desired connection sum:

$$\left\{ \begin{matrix} \mu \\ \lambda\mu \end{matrix} \right\} = \tfrac{1}{2} g^{\mu\nu} g_{\mu\nu,\lambda} = \tfrac{1}{2} \left(\log(g)\right)_{,\lambda} = \left(\log(\sqrt{-g})\right)_{,\lambda} = \frac{1}{\sqrt{-g}} \left(\sqrt{-g}\right)_{,\lambda} \ .$$

Lastly we may now begin to insert this into the formulae for covariant derivatives. The total 4-divergence of a 4-vector is then given by

$$T^\alpha{}_{;\alpha} = T^\alpha{}_{,\alpha} + \left\{ \begin{matrix} \alpha \\ \beta\alpha \end{matrix} \right\} T^\beta = T^\alpha{}_{\alpha} + \frac{1}{\sqrt{-g}} \left(\sqrt{-g}\right)_{,\beta} T^\beta = \frac{1}{\sqrt{-g}} \left(\sqrt{-g}\, T^\beta\right)_{,\beta} \ .$$
