Physics 570

Homework #11

Due Thursday, 19 April, 2007

1. All stationary solutions with cylindrical symmetry can be characterized as having the two commuting Killing vectors, ∂_ϕ and ∂_t. Weyl and Papapetrou showed that the most general such solution of the vacuum field equations may be written in the following form utilizing the usual notions of cylindrical coordinates, $\{\rho, \phi, z, t\}$:

$$\mathbf{g} \;=\; e^{-2U}\left[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2\, d\phi^2\right] - e^{+2U}(dt - \omega\, d\phi)^2 \;,$$

where U, ω, and γ are (real-valued) functions of (only) ρ and z, subject to being solutions of some partial differential equations.

The simplest case is obviously when the solution is actually static, so that $\omega = 0$. In that case the constraining equations are the following:

$$0 = \nabla^2 U \equiv \left\{\partial_\rho^2 + \tfrac{1}{\rho}\partial_\rho + \partial_z^2\right\} U \;,$$
$$(\partial_\rho + i\partial_z)\gamma = \rho[(\partial_\rho + i\partial_z)U]^2 \;.$$

The equation for U is just the ordinary, azimuthally-symmetric, Laplace equation, and is **linear**. Once U is chosen, then the equations for γ is also linear, and the integrability condition for them to have a common solution is simply the requirement already given on U. Therefore, one may find many interesting solutions of the field equations in this way; the difficulty is mostly arranging that the solution should be finite and well-behaved on the $\hat{z}$-axis, i.e., when $\rho = 0$. A standard method to resolve these equations even more goes by the name of the Ernst equation, which can also be applied to various sorts of non-vacuum solutions; when ω vanishes, the (in general complex-valued) Ernst function—the solution of the Ernst equation—is given by e^{+2U}.

As the Schwarzschild solution has these symmetries, it must be capable of being written in terms of these coordinates. Please show that this can indeed be done, using the following coordinate transformations:

$$\rho = \sqrt{(r-m)^2 - m^2}\;\sin\theta \;, \quad z = (r-m)\,\cos\theta \;, \quad \phi = \varphi \;,$$

where the appropriate solution for the Laplace equation is given by

$$e^{+2U} = 1 - \frac{2m}{r} \;.$$

In the process, you will need to determine the function γ, which is most easily written out as a function of r and θ, rather than ρ and z. [It is not required that you show that it satisfies the differential equations given above.]

Then show that the Schwarzschild horizon, at any fixed value of t, corresponds to that part of the z-axis between $z = \pm m$, i.e., when $\rho = 0$ and z varies between m and $-m$. Therefore, in these coordinates the horizon looks rather like a "short rod" along the axis, which is singular. This sort of a singularity is often referred to as a "strut," since in more complicated solutions it amounts to some sort of "extra force" holding two or more objects apart that would otherwise move toward (or away from) each other due to gravitational attractions.

2. We have "talked around" the (principal null) eigenvectors for the Riemann tensor several times in class. Let's see if we can't put together some more coherent "discussion" in the form of this problem, even though it will begin with a few things you may have to take on faith, from the spinor approach to spacetime physics.

Truthfully they are eigenvectors for the conformal (or Weyl) tensor; however, as we have been (almost) only considering vacuum solutions of the Einstein equations, where the Ricci tensor vanishes, then it didn't matter. Therefore, we concentrate on the Weyl tensor, which, you recall, has 10 independent components. We then divide that tensor into its self-dual and anti-self-dual parts via the following (obvious) identities:

$$C^{\mu\nu}{}_{\lambda\eta} = \frac{1}{2}\left\{C^{\mu\nu}{}_{\lambda\eta} + {}^*C^{\mu\nu}{}_{\lambda\eta}\right\} + \frac{1}{2}\left\{C^{\mu\nu}{}_{\lambda\eta} - {}^*C^{\mu\nu}{}_{\lambda\eta}\right\} \equiv \underset{SD}{C}{}^{\mu\nu}{}_{\lambda\eta} + \underset{ASD}{C}{}^{\mu\nu}{}_{\lambda\eta}\ ,$$
$${}^*C^{\mu\nu}{}_{\lambda\eta} \equiv \tfrac{i}{2}\eta^{\mu\nu\sigma\rho}g_{\sigma\alpha}g_{\rho\beta}C^{\alpha\beta}{}_{\lambda\eta}\ ,$$

where the things at the end of the first line simply give us names for the self-dual and anti-self-dual parts, while the second line reminds us how to effect the dual transform on the components of a 2-form. [Of course since this tensor actually has 2 pairs of two skew-symmetric indices, while a simple 2-form would only have one such pair, we can take the left dual or the right dual. So long as we consistently make the same choice, of left or right, it won't matter which one we choose.] As both the self-dual part and the anti-self-dual parts are complex-valued, and the anti-self-dual part is simply the complex conjugate of the self-dual part, all the information is contained in, for instance, the self-dual part, which then contains only 5 complex-valued linearly-independent components. Therefore, the desired "eigenvectors" would in fact appear to be 2-forms—or second-rank, skew-symmetric tensors, $\Lambda^{\mu\nu}$:

$$\underset{SD}{C}{}^{\mu\nu}{}_{\lambda\eta}\,\Lambda^{\lambda\eta} = \alpha\,\Lambda^{\mu\nu}\ .$$

However, as the eigen-2-form is complex-valued, it turns out that it may indeed be shown to specify a single null tangent vector in a way that is much clearer if we first re-describe everything in terms of spinors. At that point, we will describe the 5 independent (complex) components of the self-dual part of the conformal tensor in terms of the 5 independent components of a **fourth-rank, totally symmetric** spinor, C_{ABCD}. Therefore, I will next append into this already-rather-lengthy problem description a quick, 2-page summary of the most important parts of an understanding of multiple-rank spinors that we will need to effect this.

We recently showed that there were two (inequivalent), complex-valued, 2×2 matrix representations of the Lorentz group. Matrices of course act as operators on an associated vector space, also referred to as a *carrier space* for these representations; therefore, associated with each of these representations is a

2 complex-dimension vector space. The elements of these vector spaces are called *spinors.* We refer to the carrier space for the representation $D^{(0,\frac{1}{2})}$ by the symbol V^2, and we denote the components of an arbitrary element of this space by a symbol such as $\{\xi^A \mid A = 1,2\}$. Remembering that the representation $D^{(\frac{1}{2},0)}$ is the "complex conjugate" of the other representation, there is a second space, which we refer to as the "complex-conjugated space," $\overline{V}^2$, with elements labelled by "dotted" indices: $\{\psi^{\dot{A}} \mid A = 1,2\}$. The operation of complex-conjugation is an anti-isomorphism between these two different spaces; it can be easily effected by making a constraint on the choices of bases in these two spaces, so that the corresponding spinors in the two spaces will have components which are in fact simply complex conjugates of one another: for $\xi^A \in V^2$, we describe its anti-isomorphic image as having components which are simply its complex conjugates: $\overline{\xi^A} = \xi^{\dot{A}} \in \overline{V}^2$, where the overline means complex conjugation as usual.

As vector spaces each of these has associated with it a dual space; we label them V^2_* and $\overline{V}_*^2$. To preserve the Einstein summation convention we use the notation ξ_A and $\xi_{\dot{A}}$ to refer to elements of these spaces. Again considering (linear) transformations of our spinors, if we make a transformation in V^2, we would expect that the spinors in V_2 would transform by the inverse transformation, so that the scalar made from the vector and its dual would remain invariant under this transformation:

$$\xi'^{\,R} = A^R{}_C\,\xi^C \quad \text{and} \quad \zeta'_{\,S} = (A^{-1})^B{}_S\,\zeta_B \iff \xi'^{\,R}\zeta'_{\,R} = \xi^C\zeta_C \,.$$

As carrier spaces for representations, not only of the Lorentz group but also of $\boldsymbol{SL}(2,\mathbb{C})$, the Levi-Civita symbol that creates determinants is an invariant tensor there. Since, in our spaces with just 2 dimensions, the Levi-Civita symbol, ϵ_{AB} has just 2 indices, it can also be used as a mapping sending ξ^A into ξ_A; i.e., we can use ϵ_{AB} to lower indices. Therefore, in each of our four (2-dimensional) vector spaces, we have a numerical, index quantity, a Levi-Civita symbol, with exactly the same numerical values for their components, namely $+1$ when the components take the (ordered) values $1,2$, the value -1 for the values of the indices as $2,1$, and 0 otherwise, i.e., 0 along the diagonal:

$$\epsilon_{AB} \equiv \epsilon_{\dot{A}\dot{B}} \Longrightarrow \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} \Longleftarrow \epsilon^{AB} \equiv \epsilon^{\dot{A}\dot{B}} \,,$$

$$\Longrightarrow \quad \epsilon_{BC}\,\epsilon^{AC} = \delta^A_B \Longrightarrow (\epsilon_{AB})^{-1} = -\epsilon^{AB} \,.$$

We use these matrices as a metric in the sense that they map spinors to the corresponding dual spinor, and vice versa. However, because this symbol is skew-symmetric, rather than symmetric, it is important to be very careful as to the order of the indices when using it to "raise" and "lower" indices; following von Neumann, Debever, and Plebański, the convention with regard to order that I use is the following one:

$$\xi_A \equiv \epsilon_{AB}\,\xi^B \,, \quad \xi^C \equiv \xi_D\,\epsilon^{DC} \,,$$

$$\xi^A = \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix} \Longrightarrow \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \xi_A = \begin{pmatrix} \xi^2 \\ -\xi^1 \end{pmatrix} \,,$$

along with the same rules for "dotted indices." Because of the skew symmetry of the Levi-Civita symbol, it is always true that the invariant sum of the components of a spinor and its dual, i.e., $F_A\,F^A$, is always identically zero, while for two, non-parallel spinors we have the following (doubtless unexpected) phenomenon:

$$F_A\,G^A = -F^A\,G_A \,.$$

As the first two vector spaces carry two different representations of $\boldsymbol{SL}(2,\mathbb{C})$, we should then wonder about the representations carried by these dual spaces: of course, as dual spaces it must be that they carry the inverse representations. We can see this in detail if we want, using the properties of the determinant to prove consistency:

$$\begin{aligned} \epsilon_{RS}\, A^R{}_E\, A^S{}_F = \epsilon_{EF}(\det A) \quad &\Longrightarrow \quad \epsilon_{RS}\, A^S{}_F = (\det A)\epsilon_{BF}\,(A^{-1})^B{}_R \\ \Longrightarrow \quad \epsilon_{RS}\, A^S{}_F\, \epsilon^{BF} &= (\det A)\,(A^{-1})^B{}_R = (A^{-1})^B{}_R\ ; \\ \Longrightarrow \quad (A^{-1})^B{}_R\, \xi_B = \xi'{}_R &= \epsilon_{RS}\, \xi'^{\,S} = \epsilon_{RS}\, A^S{}_C\, \xi^C = \epsilon_{RS}\, A^S{}_C\, \epsilon^{BC}\, \xi_B\ . \end{aligned}$$

All the same sorts of statements are also true for the "dotted" spinors, using the "dotted" transformation equations:

$$\xi'^{\,\dot R} = A^{\dot R}{}_{\dot C}\, \xi^{\dot C}\ , \quad \xi'{}_{\dot R} = (A^{-1})^{\dot C}{}_{\dot R}\, \xi_{\dot C} \quad .$$

Having such an abundance of vector spaces, we may now easily consider all manner of tensor spaces defined over them, having both contravariant and covariant, dotted and un-dotted indices. The first one to consider rests on the fact that we have already shown the equivalence between $D^{(0,\frac12)} \otimes D^{(\frac12,0)}$ and the original Lorentz transformations acting on 4-vectors; therefore, there must be a mapping between the carrier spaces as well: a mapping between (sums of) spinors of the form $k^A\ell^{\dot B}$ and 4-vectors. In fact that mapping is between Hermitian, 2nd-rank spinors and 4-vectors, or 1-forms, in spacetime. An Hermitian tensor $X^{A\dot B}$ is an element of the tensor product space $V^2 \otimes \overline{V}^2$, with the additional constraint that the process of complex conjugation and transposition brings us back to where we began:

$$\overline{X^{A\dot B}} = X^{B\dot A}\ .$$

A standard way to create the desired mapping is via the use of a set of 4 (generalized) Pauli matrices, appropriate for 4-dimensional spacetime, which we display, along with the relationship of the invariant scalar products for each mode of presentation:

$$\sigma^\mu \equiv \begin{pmatrix} \vec\sigma \\ I_2 \end{pmatrix} \equiv \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ I_2 \end{pmatrix}\ , \quad \mathbf{X} \equiv \widetilde{x}\cdot\widetilde{\boldsymbol\sigma} = x^\mu\sigma_\mu = \begin{pmatrix} z-t & x-iy \\ x+iy & -z-t \end{pmatrix}\ ,$$

$$\sigma_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\ , \quad \sigma_y \equiv \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}\ , \quad \sigma_z \equiv \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}\ , \quad \mathbf{I}_2 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\ ;$$

$$x^T H x = x^\mu\, \eta_{\mu\nu}\, x^\nu \equiv \boldsymbol\eta(\widetilde x,\widetilde x) \ \equiv \widetilde x\cdot\widetilde x = -\det\mathbf{X}\ .$$

The relationship of the representations then is as follows, where $A = D^{(0,\frac12)}(L)$:

$$\widetilde x\,' = L\widetilde x \quad \Longrightarrow \quad \mathbf{X}' = A\mathbf{X}A^\dagger\ , \quad \text{and} \quad \det A = +1\ .$$

Using this approach we may easily begin with some single, individual spinor, say k^A, and create an Hermitian spinor which will represent a real null vector in Minkowski space. We do this by using also the corresponding complex-conjugated, or “dotted,” spinor, $k^{\dot A} \equiv \overline{k^A} \in \overline{V}^2$:

$$K^{A\dot B} \equiv k^A k^{\dot B} = k^A \overline{k^B} = \overline{\overline{k^{\dot A}} k^B} = \overline{k^B k^{\dot A}} = \overline{K^{B\dot A}} \ ,$$
$$\Longrightarrow \quad \det K = \epsilon_{AC} K^{A\dot 1} K^{C\dot 2} \ = \epsilon_{AC} k^A k^{\dot 1} k^C k^{\dot 2} = 0 \ .$$

The creation of more general sorts of Hermitian spinors from 1-index spinors requires beginning with two of them that are not parallel; however, that is not where I want to go at the moment. Instead, let's consider a second-rank tensor in $V^2 \otimes V^2$. Beginning from two initial spinors, k^A and ℓ^B, we may consider their tensor product:

$$k^A \ell^B = \tfrac{1}{2}\left[k^A \ell^B + k^B \ell^A\right] + \tfrac{1}{2}\left[k^A \ell^B - k^B \ell^A\right] \ .$$

However, the second pair of terms above is skew symmetric. In two dimensions, if X^{AB} is skew symmetric, then $X^{00} = 0 = X^{11}$ while $X^{01} = -X^{10}$, so that it is clear that it is just proportional to the Levi-Civita symbol, which is an invariant. One can easily check that in fact we have

$$k^A \ell^B - k^B \ell^A = \epsilon^{AB}\left(k^C \ell_C\right) \ .$$

This says that the skew-symmetric piece of this tensor space is not very interesting, so that we concentrate **only** on the symmetric part. The symmetric part has 3 independent components, and in fact describes the self-dual part of a 2-form in our original 4-dimensional spacetime. This more lengthy approach is simply the same as an earlier statement about representations:

$$D^{(0,\frac{1}{2})} \otimes D^{(0,\frac{1}{2})} = D^{(0,1)} \oplus D^{(0,0)} \ ,$$

which explains the decomposition of our second-rank spinors as explained above. To see the statement that these are the self-dual 2-forms, we then look at the decomposition of the second-rank, 4-vector representation:

$$D^{(\frac{1}{2},\frac{1}{2})} \otimes D^{(\frac{1}{2},\frac{1}{2})} = D^{(1,1)} \oplus \left\{D^{(0,1)} \oplus D^{(1,0)}\right\} \oplus D^{(0,0)} \ ,$$

where the first entry on the right-hand side, with 9 components, is the symmetric, traceless, second-rank 4-tensors, the last entry is the associated trace, and the pair in the middle, inside the braces, is the skew-symmetric second-rank 4-tensors. Provided one is willing to introduce the complex numbers into their decomposition, as is necessary for the idea of self-duality, then those skew-symmetric objects, i.e., 2-forms, are divided further into their self-dual parts and their anti-self-dual parts. From a different point of view, however, what we can see is that, given an original first-rank spinor, such as k^A, we now know how to create from it a null 4-vector, i.e., $k^A k^{\dot B}$, and a self-dual 2-form, $k^A k^B$, which shows an important relationship between those two objects: a non-obvious additional benefit granted by looking at things through the “spyglass” of spinors.

Since a self-dual, skew-symmetric, second-rank 4-tensor can be described via a symmetric, second-rank, 2-spinor, it should, hopefully not be too small a “jump” to believe that a fourth-rank 4-tensor that has two pair of skew-symmetric indices should be described by a fourth-rank, totally symmetric spinor [which transforms under the representation $D^{(0,2)}$], namely C_{ABCD}. So how does one create such a symmetric spinor; the (obvious) answer is that you find 4 different simple spinors, multiply them together in all (24) possible orders and add them:

$$C_{ABCD} = b_{(A}\ell_B h_C f_{D)} \ ,$$

where (this time) the symmetrizing parentheses suggest the symmetrization over all 4 indices contained between them, and dividing the total by 4! (the number of distinct terms involved in the symmetric sum).

However, depending on the particular manifold, it may be that not all 4 of these different spinors are linearly independent. Therefore, the homework problem associated with all this discussion is the following:

Consider an arbitrary 1-spinor,

$$\zeta^A = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \ , \quad \text{and consider the equation } C_{ABCD}\zeta^A\zeta^B\zeta^C\zeta^D = 0 \ ,$$

which we treat as a polynomial equation in a single variable, say β/α. Show that the number of distinct roots of this polynomial equation is the same as the number of distinct, i.e., non-proportional spinors needed to create the given conformal spinor via the sum described above, according to the scheme shown just below.

The so-called Petrov type of the conformal tensor is determined via the multiplicity of these roots:

$$\textbf{Petrov Type:} \qquad \begin{cases} I: \ C_{ABCD} &= b_{(A}\ell_B h_C f_{D)} \ , \quad \text{4 distinct roots;} \\ II: \ C_{ABCD} &= b_{(A}b_B h_C f_{D)} \ , \quad \text{3 distinct roots, 1 double;} \\ D: \ C_{ABCD} &= b_{(A}b_B h_C h_{D)} \ , \quad \text{2 double roots;} \\ III: \ C_{ABCD} &= b_{(A}b_B b_C f_{D)} \ , \quad \text{2 distinct roots, 1 triple;} \\ N: \ C_{ABCD} &= b_{(A}b_B b_C b_{D)} \ , \quad \text{1 quadruple root.} \end{cases}$$

In these various cases the specific 1-spinors needed to create the sum may be used to

a.) determine the various independent eigen-2-forms for the tensor form of the conformal tensor, and

b.) create the various so-called principal null directions, which are null vectors that "characterize" the form of the conformal curvature.

If, instead, one writes the 5 independent (complex) terms in the self-dual part of the curvature tensor in terms of a symmetric, traceless, 3×3 complex matrix, P—see the handout on spherically-symmetric, static metrics—then these different Petrov Types have different "normal forms," in terms of their eigenvalues, which may be shown via this subdivision:

$$\textbf{Petrov Type:}\qquad \begin{cases} I:\ P = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & -\lambda_1-\lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} , \\ II:\ P = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & -2\lambda \end{pmatrix} , \\ D:\ P = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & -2\lambda \end{pmatrix} , \\ III:\ P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} , \\ N:\ P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \end{cases}$$

but you are not being asked to show these last results.

HOWEVER, do please do the following, for the Schwarzschild metric with which you are now intimately familiar: First create a null-basis for 1-forms for that metric via

$$\nu^1 \equiv \frac{1}{\sqrt{2}}\left(\underset{\approx}{\omega}^{\hat\theta} + i\underset{\approx}{\omega}^{\hat\varphi}\right) , \quad \nu^2 \equiv \frac{1}{\sqrt{2}}\left(\underset{\approx}{\omega}^{\hat\theta} - i\underset{\approx}{\omega}^{\hat\varphi}\right) ,$$
$$\nu^3 \equiv \frac{1}{\sqrt{2}}\left(\underset{\approx}{\omega}^{\hat r} + \underset{\approx}{\omega}^{\hat t}\right) , \quad \nu^4 \equiv \frac{1}{\sqrt{2}}\left(\underset{\approx}{\omega}^{\hat r} - \underset{\approx}{\omega}^{\hat t}\right) ,$$

and then determine the components of our conformal spinor via

$$\begin{aligned} C_{0000} &= R_{3131} , \\ C_{0001} &= \tfrac{1}{2}\left(R_{1231} + R_{3431}\right) , \\ C_{0011} &= R_{4231} + \mathcal{R}/12 , \\ C_{0111} &= \tfrac{1}{2}\left(R_{1242} + R_{3442}\right) , \\ C_{1111} &= R_{4242} , \end{aligned}$$

Then determine the two independent spinors k^A and ℓ^B that factorize this system, and determine the associated null vectors and self-dual 2-forms for each one. Note that the determination of these two spinors is obviously indeterminate modulo multiplying one by some factor and dividing the other by the same factor, so try to make a reasonable, symmetric choice of their factors.