Physics 570

Homework #1

Due Thursday, 1 February, 2007

Solutions

1. Please make a good drawing of a (standard, only 2-dimensional) Minkowski diagram, as seen by an observer $\mathcal{O}$, with his x and t-axes shown at 90° to one another. [Recall that you are using $c = 1$.] Create the diagram either on some good graph paper or, preferably, on a computer inside an computer-mathematics program such as Maple.

On this diagram, please show the following things:

a.) the world line of a clock that $\mathcal{O}$ maintains at the location $x = 1$ m;

b.) the trajectories of two light rays sent out by $\mathcal{O}$ at his time $t = 0$, into the $+\hat{x}$-direction and the $-\hat{x}$-direction;

c.) the world line of a distinct inertial observer, $\mathcal{O}'$, that $\mathcal{O}$ measures to be moving with speed $v = 0.25$, and which was coincident with $\mathcal{O}$ at their mutual values of $t = 0 = t'$;

d.) the line of events that $\mathcal{O}'$ measures to be simultaneously occurring at her time $t' = 3$ m;

e.) the world line of an observer at rest with respect to $\mathcal{O}'$but who was at the location $x = -2$ m at the time $t = 0$ m;

f.) the locus of all events which, relative to the origin, have $(\Delta s)^2 = -9$ m^2;

g.) the locus of all events which, relative to the origin, have $(\Delta s)^2 = +4$ and which also occur at $t \geq 0$.

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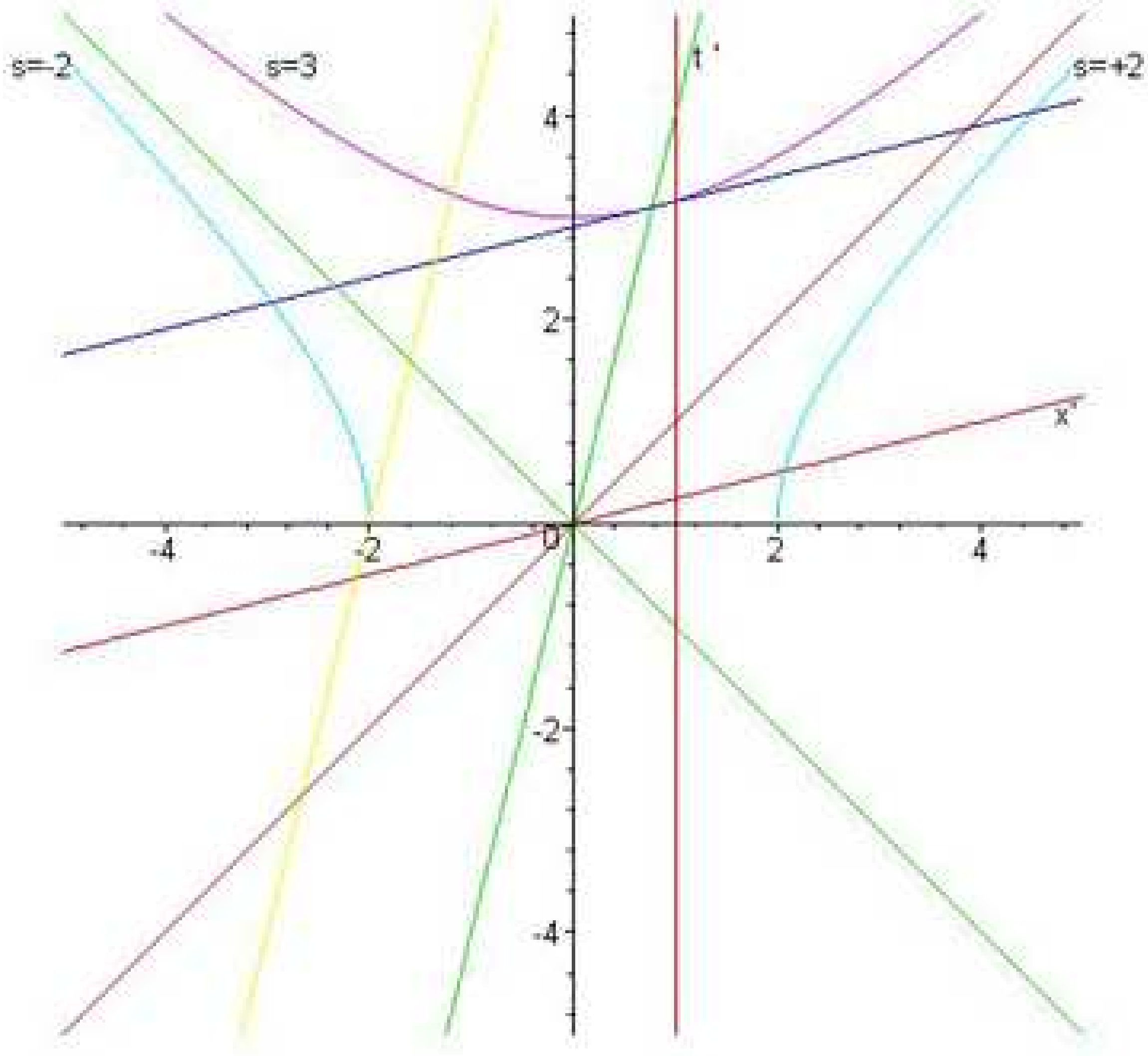

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2. A timelike worldline is one such that the displacement vector between any two points is timelike; all worldlines belonging to any physical observer, or indeed any physical object, must be everywhere timelike.
For a timelike worldline we may define the 4-velocity, $\widetilde{u}$, in terms of the displacement between two infinitesimally-near events (on that worldline), with spatial displacement $d\vec{r}$ and temporal displacement dt, and the proper time, $d\tau$ that separates the events belonging to the motion of the object in question:

$$\widetilde{u} \equiv \begin{pmatrix} \frac{d\vec{r}}{d\tau} \\ \frac{dt}{d\tau} \end{pmatrix} = \frac{dt}{d\tau}\begin{pmatrix} \vec{v} \\ 1 \end{pmatrix} \equiv \gamma_v \begin{pmatrix} \vec{v} \\ 1 \end{pmatrix} , \qquad \gamma_v \equiv \frac{1}{\sqrt{1-v^2}} .$$

a. Using the usual definition of the 3-vector acceleration, $\vec{a} \equiv d\vec{v}/dt$, determine the form of the

4-acceleration

$$\widetilde{a} \equiv \frac{d}{d\tau}\widetilde{u} \ ,$$

in terms of $\vec{a}$, $\vec{v}$, and γ_v.

b. Determine the Lorentz invariant associated with this 4-vector, i.e., the value of $\widetilde{a} \cdot \widetilde{a} = \eta_{\mu\nu} a^\mu a^\nu$. Verify that this quantity is never negative, thus allowing us to say that this vector is *spacelike*. What is its value in the so-called *instantaneous rest frame*, i.e., an inertial frame moving in such a way that at the moment in question it measures the object to momentarily at rest?

c. Divide the 3 spatial portions of the 4-acceleration vector into those parts that are parallel to its velocity and those that are perpendicular to its velocity, in as simple a way as possible. In each of these there is an overall factor of γ_v to some power; what is that power in each of these two cases?

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a. The calculation is straightforward, if perhaps slightly involved. It is worthwhile, first, to determine the time-derivative of the usual γ_v factor, where we remember the following:

$$v^2 = \vec{v}^2 = \vec{v} \cdot \vec{v} \quad \text{and} \quad \frac{d}{dt}\vec{v} = \vec{a} \ ,$$

$$\frac{d}{dt}\gamma_v = \frac{d}{dt}\frac{1}{\sqrt{1-v^2}} = -\frac{1}{2}\frac{-2\vec{v}\cdot\vec{a}}{[1-v^2]^{3/2}} = \gamma_v^3(\vec{v}\cdot\vec{a}) \ .$$

With this information we have

$$\widetilde{a} \equiv \frac{d}{d\tau}\widetilde{u} = \frac{dt}{d\tau}\frac{d}{dt}\widetilde{u} = \gamma_v \frac{d}{dt}\gamma_v \begin{pmatrix} \vec{v} \\ 1 \end{pmatrix} = \gamma_v^2 \begin{pmatrix} \vec{a} \\ 0 \end{pmatrix} + \gamma_v^4(\vec{v}\cdot\vec{a}) \begin{pmatrix} \vec{v} \\ 1 \end{pmatrix} = \gamma_v^2 \begin{pmatrix} \vec{a} + \gamma_v^2(\vec{v}\cdot\vec{a})\vec{v} \\ \gamma_v^2(\vec{v}\cdot\vec{a}) \end{pmatrix} \ ,$$

$$\equiv \begin{pmatrix} \vec{\alpha} \\ a^4 \end{pmatrix} \ ,$$

where it was convenient, although obviously not essential, to give names to the spatial and the temporal components of our 4-vector. (We cannot simply use $\vec{a}$ for its spatial part because that symbol already is in use to mean the usual 3-acceleration.)

b. To determine the squared invariant for this vector, we simply proceed as follows:

$$\widetilde{a} \cdot \widetilde{a} \equiv \vec{\alpha} \cdot \vec{\alpha} - (a^4)^2 = \gamma^4 \left\{ \vec{a}\cdot\vec{a} + 2\gamma^2(\vec{v}\cdot\vec{a})^2 + \gamma^4 v^2 (\vec{v}\cdot\vec{a})^2 - \gamma^4(\vec{v}\cdot\vec{a})^2 \right\}$$
$$\gamma^4 \left\{ a^2 + \gamma^2(\vec{v}\cdot\vec{a})^2 \right\} = \gamma^4 a^2 [1 + \gamma^2 v^2 \cos^2\theta] = \gamma^6 a^2 (1 - v^2 \sin^2\theta) \ .$$

Note to the grader: presumably ANY of the equivalent expressions on the bottom line are sufficient to the intent of the problem, namely to calculate the square of the acceleration 4-vector!

c. To pick out those parts of $\vec{\alpha}$ which are parallel to, and perpendicular to, the velocity vector, $\vec{v}$, we first pull out the easy part, the perpendicular part:

$$\vec{\alpha}_\perp = \gamma^2 \vec{a}_\perp \ ,$$

which simply has a factor of γ_v squared multiplying the perpendicular part of the usual 3-acceleration.

For the parallel part, the simplest thing to do is to take account of the fact that the vector that is the part of that vector parallel to $\vec{v}$ may be written in the following way:

$$\vec{a}_{\|} \equiv (\vec{a}\cdot\hat{v})\hat{v}\ ,$$
$$\vec{\alpha}_{\|} = \gamma^2(\vec{a}\cdot\hat{v})\left\{1+\gamma^2 v^2\right\}\hat{v} = \gamma^2(\vec{a}\cdot\hat{v})\gamma_v^2\hat{v} = \gamma^4\vec{a}_{\|}\ .$$

We see that the two different portions of the 4-acceleration are proportional to rather different powers of γ_v, making it quite impossible to, for instance, re-define the concept of mass, m, in order to allow some statement suggesting that the vectors $\vec{F}$ and $\vec{a}$ would be proportional in a relativistically-correct statement.

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3. In the usual Euclidean 3-space, a one-dimensional curve may be specified by giving each of the three Euclidean coordinates as functions of some single parameter. Below we give such a description of each of two curves, the first parametrized by λ and the second by μ, where both of these parameters vary, at least in principle, from $-\infty$ to $+\infty$:

$$\text{curve 1:}\quad x=\lambda\,,\quad y=(\lambda-1)^2\,,\quad z=-\lambda\,,$$
$$\text{curve 2:}\quad x=\cos\mu\,,\quad y=\sin\mu\,,\quad z=\mu-1\,,$$

a. Please calculate the components of the tangent vector of each of these curves.

b. Let a particular function, f, be defined on this 3-space:

$$f = f(x,y,z) \equiv x^2+y^2-yz\ .$$

As this function varies along each of these two curves, please calculate its rate of change; i.e., calculate $df/d\lambda$ and $df/d\mu$.

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We first recall that the tangent vector to a curve is a vector with components $t^i = dx^i/\lambda$, where λ is the parameter the runs along it; therefore, we may write the tangent vectors to these two curves in either of the following two ways, where in one presentation we give the vector itself in terms of our choice of basis vectors, and in the other presentation we simply present the components of that vector, relative to the same choice of basis vectors, as the elements of a column vector, or single-column matrix:

$$\text{for curve \#1:}\quad \vec{t}_1 = \hat{x}_i\frac{dx^i}{d\lambda} = \begin{cases}\begin{pmatrix}1\\ 2(\lambda-1)\\ -1\end{pmatrix}\,,\\ \hat{x}+2(\lambda-1)\hat{y}-\hat{z}\ .\end{cases}$$

$$\text{for curve \#2:}\quad \vec{t}_2 = \hat{x}_i\frac{dx^i}{d\lambda} = \begin{cases}\begin{pmatrix}-\sin\mu\\ \cos\mu\\ 1\end{pmatrix}\,,\\ -\sin\mu\,\hat{x}+\cos\mu\,\hat{y}+\hat{z}\ .\end{cases}$$

To determine the rates of change of our given function, $f(x, y, z)$, as it is moved along either one of these curves, we basically need the directional derivatives, evaluated in terms of the curve parameter:

$$\frac{df}{d\lambda} = \frac{d\vec{x}}{d\lambda} \cdot \nabla f = \frac{dx^i}{d\lambda}\frac{\partial}{\partial x^i} f = t^i \frac{\partial}{\partial x^i} f = 1\frac{\partial f}{\partial x} + 2(\lambda - 1)\frac{\partial f}{\partial y} - 1\frac{\partial f}{\partial z} = 2x + 2(\lambda - 1)(2y - z) + y \; .$$

While the above is true, as we really want to know how it varies along the curve, it is best to also insert the dependence on the parameter λ of the coordinates of the points on the curve:

$$\frac{d}{d\lambda} f = 2(1) + 2(\lambda - 1)[2(\lambda - 1)^2 + \lambda] + (\lambda - 1)^2 = 4\lambda^3 - 9\lambda^2 + 10\lambda - 3 \; .$$

For the second curve, the method is the same:

$$\frac{d}{d\mu} f = \vec{t}_2 \cdot \nabla f = -2x \, \sin\mu + (2y - z) \, \cos\mu - y = (1 - \mu)\cos\mu - \sin\mu \; .$$

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4. Take a standard (flat) 2-dimensional plane, with origin, and treat it as a vector space. For this vector space choose (arbitrarily) some vector of unit length and call it $\hat{\xi}_1$. Then choose another vector of unit length, which makes an angle of α relative to the first one, and call it $\hat{\xi}_2$. For simplicity choose $\alpha \le \pi/2$.

We want to use these two vectors as basis vectors for our vector space; note that they are not orthogonal. Therefore, now choose a special vector, called $\vec{x}$. The purpose of this problem is to determine the components of $\vec{x}$ relative to our choice of basis vectors. There are two ways to do this, both of which I am asking you to calculate, as functions of the arbitrary angle α. It is easiest to visualize this process by making yourself a drawing on a piece of paper:

a. As one learned in freshman physics when decomposing vectors, we suppose that there are two numbers, x^1 and x^2—collectively referred to as $\{x^i\}_1^2$—which are such that

$$\vec{x} = x^1\,\hat{\xi}_1 + x^2\,\hat{\xi}_2 \ .$$

Obviously these numbers are uniquely determined, and they uniquely determine our vector.

b. Since we understand angles in this plane, we may also suppose that we draw from the tip of the vector $\vec{x}$ a line perpendicular to $\hat{\xi}_1$, which then meets $\hat{\xi}_1$, perhaps extended along its own direction if necessary, at some distance from the origin; this is the perpendicular projection of $\vec{x}$ onto the direction established by $\hat{\xi}_1$. Refer to this number as x_1. Now do the same thing to determine x_2 by projecting perpendicular to $\hat{\xi}_2$. Using the usual Euclidean scalar product, which we denote by a central "dot" as usual, we could write these quantities according to the formula:

$$x_i = \vec{x}\cdot\hat{\xi}_i \ , \qquad i = 1, 2 \ .$$

It is again obvious that these numbers are uniquely determined by the vector $\vec{x}$, and vice versa. Therefore, it is clear that either pair may legitimately be considered as "the components" of the vector $\vec{x}$ relative to this non-orthogonal choice of basis vectors.

Please determine, in as geometrical a way as possible, and in terms of α, the relationship between the two pairs of components of $\vec{x}$.

c. Now calculate the matrix which is the scalar product of the two basis vectors with themselves and each other, i.e.,

$$g_{ij} \equiv \hat{\xi}_i\cdot\hat{\xi}_j \ , \qquad i, j = 1, 2 \ .$$

Then show that in fact the two sets of components are related via

$$x_i = g_{ij}x^j \ , \qquad i, j = 1, 2 \ ,$$

remembering that the Einstein summation convention reminds us that there is a sum in the expression on the right-hand side, over all values of j.

d. Lastly calculate the quadratic expression

$$L^2 \equiv g_{ij}x^i x^j$$

and use the Law of Cosines, for instance, to show that L is in fact the length of the vector $\vec{x}$.

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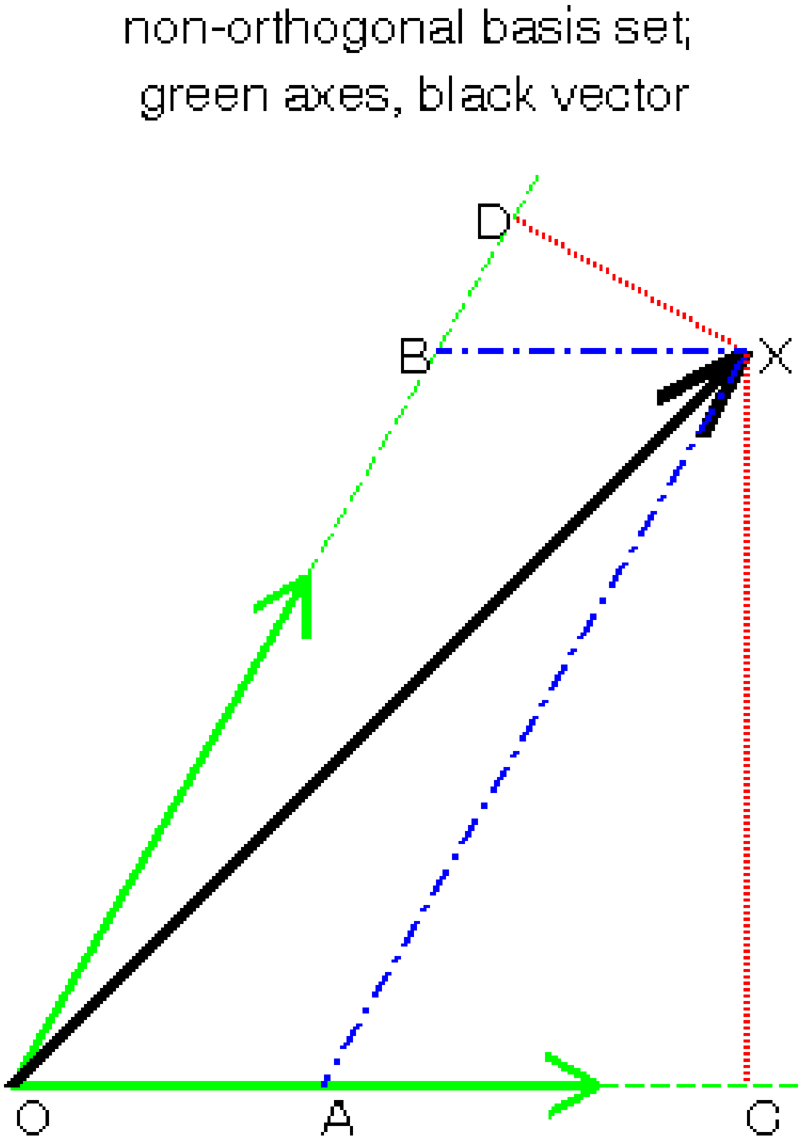

We show above the figure of the situation as described, for our vector $\vec{X}$, (in black), and the two basis vectors, $\vec{\xi}_1$, shown in green and horizontal, and $\vec{\xi}_2$, shown in green and at a 60° angle relative to the first one.
The contravariant coordinates, $\{x^i\}_1^2$, are the distances $x^1 = OA$ and $x^2 = OB$, while the covariant coordinates, $\{x_i\}_1^2$, are the distances $x_1 = OC$ and $x_2 = OD$. If we name the angle between the two basis vectors as α, then we can write the following simply by perusal of the figure and some little trigonometry:

$$x_1 = OC = OA + AC = x^1 + AX \cos\alpha = x^1 + x^2 \cos\alpha \,,$$
$$x_2 = OD = OB + BD = x^2 + OA \cos\alpha = x^2 + x^1 \cos\alpha \,.$$

Putting these two relationships in matrix form gives us

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & \cos\alpha \\ \cos\alpha & 1 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} \,.$$

c. Given that the vectors are of unit length, and that the angle between the two basis vectors is α, it follows immediately that the matrix containing their scalar products is

$$((\vec{\xi}_i \cdot \vec{\xi}_j)) = \begin{pmatrix} 1 & \cos\alpha \\ \cos\alpha & 1 \end{pmatrix} \equiv g_{ij} \ .$$

From the geometrical relationship obtained above, it has then already been shown that the matrix with elements g_{ij} is the one that produces the covariant components when it multiplies, on the left, the contravariant components.

d. To apply the Law of Cosines, we take as our triangle the one which has OA, AX, and OX, the vector itself, as sides. The angle one needs is the angle equal to α which is the angle that AC makes with AX. The Law of Cosines then tells us that the length of the line OX is given by

$$\begin{aligned} L^2 = (OX)^2 = (OA)^2 \ + (AX)^2 + 2(OA)(AX)\cos\alpha &= (OA)^2 + (OB)^2 + 2(OA)(OB)\cos\alpha \\ = \begin{pmatrix} x^1 & x^2 \end{pmatrix} \begin{pmatrix} 1 & \cos\alpha \\ \cos\alpha & 1 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} &= x^a \, g_{ab} \, x^b \ , \end{aligned}$$

which is what was desired, verifying the role of g_{ij} as the quantity that determines lengths, and therefore usually referred to as **the metric relative to this choice of basis.**
