Physics 570

Homework #2

Due Thursday, 1 February, 2007

1. In the usual 4-dimensional spacetime, in Cartesian coordinates, as already noted, the components of the electromagnetic field tensor can be presented in the following way:

$$F^{\mu\nu} = \begin{pmatrix} 0 & B^z & -B^y & -E^x \\ -B^z & 0 & B^x & -E^y \\ B^y & -B^x & 0 & -E^z \\ E^x & E^y & E^z & 0 \end{pmatrix} .$$

a. Please use this to determine the following two 4-vectors, and to present them in terms of their 3-vector content and their fourth component, which is a 3-scalar, as shown below

$$\begin{pmatrix} \vec{W} \\ W^4 \end{pmatrix} = W^\mu \equiv F^{\mu\nu} J_\nu \qquad \text{and} \qquad \begin{pmatrix} \vec{X} \\ X^4 \end{pmatrix} = X^\nu \equiv \partial_\mu F^{\mu\nu} ,$$

where the two additional 4-vectors are

$$\text{the 4-current density: } J^\nu = \begin{pmatrix} \vec{J} \\ +\rho \end{pmatrix} , \quad \text{the 4-gradient operator: } \partial_\mu = \begin{pmatrix} \nabla \\ +\partial/\partial t \end{pmatrix} .$$

Please use your knowledge of regular electromagnetism to identify these 4 quantities: the two 3-vectors and the two 3-scalars.

b. Now use the metric to lower the indices and show that the associated Faraday 2-form may be written as follows:

$$\underset{\sim}{F} = \tfrac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu = B^z dx \wedge dy + B^y dz \wedge dx + B^x dy \wedge dz + (E^x dx + E^y dy + E^z dz) \wedge dt .$$

Picking out the portion of this form that's concerned only with the magnetic field, we can see that could be considered as a 2-form over only 3-dimensional space. Let's name that portion $*\underset{\sim}{B}$ and then please calculate its 3-dimensional Hodge dual, to determine the one-form associated with it:

$$\underset{\sim}{B} = *\{*\underset{\sim}{B}\} ,$$

remembering the definition of the Hodge dual for 3-dimensional, Cartesian coordinates, $\{dx, dy, dz\} = \{\underset{\sim}{\boldsymbol{\omega}}^a\}_1^3$, as given in the appropriate handout, namely the following, where we also recall that the dual of the dual is the identity operation:

$$\Lambda^1 \leftrightarrow \Lambda^2 : \; * \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = - \begin{pmatrix} dy \wedge dz \\ dz \wedge dx \\ dx \wedge dy \end{pmatrix} \quad , \qquad \Lambda^0 \leftrightarrow \Lambda^3 : \; *1 = dx \wedge dy \wedge dz \qquad .$$

2. Using the 3-dimensional Hodge dual, as quoted, for example, in the previous problem, please determine the following p-forms, where $\underset{\sim}{A}$ is an arbitrary 1-form in 3-space, presented in a Cartesian basis set:

$$*d * \underset{\sim}{A} , \quad *d\underset{\sim}{A} , \quad d * d * \underset{\sim}{A} , \quad *d * d\underset{\sim}{A} , \quad d(d\underset{\sim}{A}) .$$

Identifying the components of $\underset{\sim}{A}$ as if they constituted the components of an "normal" 3-vector $\vec{A}$ in that 3-space, then identify the usual way of writing each of the 4 quantities above, using the differential operator ∇ and the dot or cross product as appropriate.

3. Still working in 3-dimensional space, we know from Pythagoras that we can measure "infinitesimal" distances via

$$ds^2 \equiv dx^2 + dy^2 + dz^2 \equiv \delta_{ij} dx^i \, dx^j \ , \quad dx^i = \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} \ .$$

Although in Cartesian coordinates the components of this tensor are just δ_{ij}, in other coordinates they will be more complicated; therefore, we will refer to this, in general, as a symmetric tensor of type (0,2), and use the symbol $\mathbf{g}$ for it. Therefore, we could do the algebra which would tell us, in spherical polar coordinates, for instance, that

$$ds^2 = \mathbf{g} = g_{ij} dr^i \otimes dr^j = dr^2 + r^2 \, d\theta^2 + r^2 \sin^2\theta \, d\varphi^2 \ , \quad dr^i = \begin{pmatrix} dr \\ d\theta \\ d\varphi \end{pmatrix} \ .$$

In this basis, please write out the presentation of the components of $\mathbf{g}$, i.e., the coefficients g_{ij}, as a 3×3 matrix. Then determine the matrix inverse to that one. Viewed as components, that matrix determines a (2,0) tensor; we will label its components as g^{ij}. Define a new basis for 1-forms, $\{\underset{\sim}{\omega}^a\}_1^3$, and re-write the metric tensor in this basis, giving us the components, g_{ab}, in this new basis, which you should determine and present, again, in the form of a 3×3 matrix:

$$\underset{\sim}{\omega}^a = \begin{pmatrix} dr \\ r \, d\theta \\ r \, \sin\theta \, d\varphi \end{pmatrix} \ , \qquad \mathbf{g} = g_{ab} \underset{\sim}{\omega}^a \otimes \underset{\sim}{\omega}^b \ .$$

With this same idea, then, determine a (new) basis for tangent vectors which is ***dual*** to this basis of 1-forms; i.e., determine a set of basis vectors $\{\widetilde{e}_b\}_1^3$ such that the action on them of the new basis of 1-forms is "reciprocal": $\underset{\sim}{\omega}^a(\widetilde{e}_b) = \delta^a_b$. In this basis what form does the following (2,0) tensor have:

$$\mathbf{g}^* \equiv g^{ab} \widetilde{e}_a \otimes \widetilde{e}_b \ .$$