Physics 570

Homework #2

Due Thursday, 1 February, 2007

Solutions

1. In the usual 4-dimensional spacetime, in Cartesian coordinates, as already noted, the components of the electromagnetic field tensor can be presented in the following way:

$$F^{\mu\nu} = \begin{pmatrix} 0 & B^z & -B^y & -E^x \\ -B^z & 0 & B^x & -E^y \\ B^y & -B^x & 0 & -E^z \\ E^x & E^y & E^z & 0 \end{pmatrix} .$$

a. Please use this to determine the following two 4-vectors, and to present them in terms of their 3-vector content and their fourth component, which is a 3-scalar, as shown below

$$\begin{pmatrix} \vec{W} \\ W^4 \end{pmatrix} = W^\mu \equiv F^{\mu\nu} J_\nu \qquad \text{and} \qquad \begin{pmatrix} \vec{X} \\ X^4 \end{pmatrix} = X^\nu \equiv \partial_\mu F^{\mu\nu} ,$$

where the two additional 4-vectors are

$$\text{the 4-current density: } J^\nu = \begin{pmatrix} \vec{J} \\ +\rho \end{pmatrix} , \quad \text{the 4-gradient operator: } \partial_\mu = \begin{pmatrix} \nabla \\ +\partial/\partial t \end{pmatrix} .$$

Please use your knowledge of regular electromagnetism to identify these 4 quantities: the two 3-vectors and the two 3-scalars.

b. Now use the metric to lower the indices and show that the associated Faraday 2-form may be written as follows:

$$\underset{\sim}{F} = \tfrac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu = B^z dx \wedge dy + B^y dz \wedge dx + B^x dy \wedge dz + (E^x dx + E^y dy + E^z dz) \wedge dt .$$

Picking out the portion of this form that's concerned only with the magnetic field, we can see that could be considered as a 2-form over only 3-dimensional space. Let's name that portion $*\underset{\sim}{B}$ and then please calculate its 3-dimensional Hodge dual, to determine the one-form associated with it:

$$\underset{\sim}{B} = *\{*\underset{\sim}{B}\} ,$$

remembering the definition of the Hodge dual for 3-dimensional, Cartesian coordinates, $\{dx, dy, dz\} = \{\underset{\sim}{\omega}^a\}_1^3$, as given in the appropriate handout, namely the following, where we also recall that the dual of the dual is the identity operation:

$$\Lambda^1 \leftrightarrow \Lambda^2 : \; * \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = - \begin{pmatrix} dy \wedge dz \\ dz \wedge dx \\ dx \wedge dy \end{pmatrix} , \qquad \Lambda^0 \leftrightarrow \Lambda^3 : \; *1 = dx \wedge dy \wedge dz .$$

. . . .

a. The first calculation asks for a sum over the second index of F and the only index of J; therefore, for a matrix presentation we take J^μ in the form of a column vector, so that its index labels rows, and matrix multiplication is appropriate with it coming second in the order. However, it is also true that this index on J is shown as "downstairs," i.e., covariant, while the form we are given for J is contravariant. This means that we must first "lower that index." Since this is just the usual Minkowski spacetime, this only requires that the sign of the 4th component should change, giving us the following form for calculation:

$$\widetilde{W} = \begin{pmatrix} \vec{W} \\ W^4 \end{pmatrix} = W^\mu = F^{\mu\nu} J_\nu = \begin{pmatrix} 0 & B^z & -B^y & -E^x \\ -B^z & 0 & B^x & -E^y \\ B^y & -B^x & 0 & -E^z \\ E^x & E^y & E^z & 0 \end{pmatrix} \begin{pmatrix} J^x \\ J^y \\ J^z \\ -\rho \end{pmatrix}$$

$$= \begin{pmatrix} B^z J^y - B^y J^z + E^x \rho \\ -B^z J^x + B^x J^z + E^y \rho \\ B^y J^x - B^x J^y + E^z \rho \\ E^x J^x + E^y J^y + E^z J^z \end{pmatrix} = \begin{pmatrix} \rho \vec{E} + \vec{J} \times \vec{B} \\ \\ \vec{E} \cdot \vec{J} \end{pmatrix} .$$

We surely recognize the 3-vector part of this as the Lorentz force equation, per unit volume. The 3-scalar portion—the fourth component of the 4-vector—is the power generated per unit volume, so that the entire 4-vector can be seen as the 4-force per unit volume. [See the notes on the relativistic 4-force, and its components, if necessary.]

b. The second calculation asks for $\partial_\mu F^{\mu\nu}$. Since the μ-index of F denotes a row index of the corresponding matrix, it will be convenient to treat ∂_ν as having an index for a matrix which we consider a column; then this is a sum over a column of the first matrix and the row of the second, which is matrix multiplication, and we use an upper dot to indicate a derivative with respect to time, and when it becomes appropriate we will use the standard 3-vector forms of Maxwell's equations to simplify the results:

$$\widetilde{X} = \begin{pmatrix} \vec{X} \\ X^4 \end{pmatrix} = X^\nu = \partial_\mu F^{\mu\nu} = \begin{pmatrix} \partial_x & \partial_y & \partial_z & \partial_t \end{pmatrix} \begin{pmatrix} 0 & B^z & -B^y & -E^x \\ -B^z & 0 & B^x & -E^y \\ B^y & -B^x & 0 & -E^z \\ E^x & E^y & E^z & 0 \end{pmatrix}$$

$$= \Big(-\partial_y B^z + \partial_z B^y + \dot{E}^x \,, \quad \partial_x B^z - \partial_z B^x + \dot{E}^y \,, \quad -\partial_x B^y + \partial_y B^x + \dot{E}^z \,, \\ -\left(\partial_x E^x + \partial_y E^y + \partial_z E^z\right) \Big)$$

$$= \Big(-\nabla \times \vec{B} + \dot{\vec{E}} \,, \quad -\nabla \cdot \vec{E} \Big) = \left(-\mu_0 \vec{J} \,, \quad -\rho/\epsilon_0 \right)$$

$$= -\mu_0 \left(\vec{J} \,, \quad \rho \right) = -\mu_0 \begin{pmatrix} \vec{J} \\ \rho \end{pmatrix}^T = -\mu_0 \, J^\mu = -\mu_0 \widetilde{J} \,,$$

where it was necessary to remember our convention on units so that $1 = 1/c^2 = \mu_0 \epsilon_0$, so that $1/\epsilon_0 = \mu_0$. This calculation identifies the previously-unknown 4-vector $\widetilde{X}$ with $-\mu_0 \widetilde{J}$, a constant times the 4-current. However, as we had to use two (or four) of Maxwell's equations in order to do this, we have also "discovered" the Lorentz-covariant form of those equations as follows:

$$\partial_\nu F^{\nu\mu} = -\mu_0 J^\mu \qquad \text{or} \qquad \underset{\sim}{\partial} \cdot \widetilde{F} = -\mu_0 \widetilde{J} \ .$$

c. I first lower **both indices** on $F^{\mu\nu}$ to obtain the desired coefficients $F_{\mu\nu}$. We know that all this does is change the signs on all entries in both the 4th column and 4th row; in our case, this simply means that we write down the same matrix as before, but change, everywhere, the sign of the electric field. Then we write out the entire 16 terms of the sum, and then, carefully ignoring those that are zero, add up the ones which are identical, remembering that $da \wedge db = -db \wedge da$:

$$\begin{aligned}\underset{\sim}{F} = \tfrac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu = \tfrac{1}{2}\Big\{ & F_{11}dx \wedge dx + F_{12}dx \wedge dy + F_{13}dx \wedge dz + F_{14}dx \wedge dt \\ & + F_{21}dy \wedge dx + F_{23}dy \wedge dz + F_{24}dy \wedge dt \\ & + F_{31}dz \wedge dx + F_{32}dz \wedge dy + F_{34}dz \wedge dt \\ & + F_{41}dt \wedge dx + F_{42}dt \wedge dy + F_{43}dt \wedge dz \Big\} \\ = \ & B^z dx \wedge dy + B^y dz \wedge dx + B^x dy \wedge dz + (E^x dx + E^y dy + E^z dz) \wedge dt \ ,\end{aligned}$$

where I have used the fact that every non-zero term occurs exactly twice to cancel the factor of a half that was previously in front of the entire expression.
We now pull out the magnetic part, and apply the rules for the 3-dimension, Cartesian basis Hodge dual, which say that $*(dx \wedge dy) = -dz$, $*(dy \wedge dz) = -dx$, and $*(dz \wedge dx) = -dy$:

$$\underset{\sim}{B} \equiv *\Big(B^z dx \wedge dy + B^y dz \wedge dx + B^x dy \wedge dz\Big) = -(B^z dz + B^y dy + B^z dz) = -\vec{B} \ ,$$

where in the last equals we are simply ignoring the difference between forms and vectors, not unreasonable since we are in a Cartesian basis in flat 3-space.

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2. Using the 3-dimensional Hodge dual, as quoted, for example, in the previous problem, please determine the following p-forms, where $\underset{\sim}{A}$ is an arbitrary 1-form in 3-space, presented in a Cartesian basis set:

$$*d*\underset{\sim}{A} \ , \quad *d\underset{\sim}{A} \ , \quad d*d*\underset{\sim}{A} \ , \quad *d*d\underset{\sim}{A} \ , \quad d(d\underset{\sim}{A}) \ .$$

Identifying the components of $\underset{\sim}{A}$ as if they constituted the components of an "normal" 3-vector $\vec{A}$ in that 3-space, then identify the usual way of writing each of the 4 quantities above, using the differential operator ∇ and the dot or cross product as appropriate.

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To establish notation we suppose as given that

$$\underset{\sim}{A} = A_x dx + A_y dy + A_z dz \ .$$

Then we may calculate the two 2-forms which are its dual and its exterior derivative:

$$d\underset{\sim}{A} = (dA_x)\wedge dx + (dA_y)\wedge dy + (dA_z)\wedge dz = \partial_y A_x\, dy\wedge dx + \partial_z A_x\, dz\wedge dx$$
$$+\,\partial_x A_y\, dx\wedge dy + \partial_z A_y\, dz\wedge dy + \partial_x A_z dx\wedge dz + \partial_y A_z dy\wedge dz$$
$$= (\partial_x A_y - \partial_y A_x)dx\wedge dy + (\partial_y A_z - \partial_z A_y)dy\wedge dz + (\partial_z A_x - \partial_x A_z)dz\wedge dx\ ;$$
$$*\underset{\sim}{A} = -\Big\{A_x dy\wedge dz + A_y dz\wedge dx + A_z dx\wedge dy\Big\}\ .$$

Having those we may calculate the dual of the top one and the exterior derivative of the bottom one:

$$*d\underset{\sim}{A} = -\Big\{(\partial_x A_y - \partial_y A_x)dz + (\partial_y A_z - \partial_z A_y)dx + (\partial_z A_x - \partial_x A_z)dy\Big\}\ ,$$
$$d*\underset{\sim}{A} = -\Big\{\partial_x A_x + \partial_y A_y + \partial_z A_z\Big\}dx\wedge dy\wedge dz\ ,$$
$$\Longrightarrow *d*\underset{\sim}{A} = -\left(\partial_x A_x + \partial_y A_y + \partial_z A_z\right) = -\nabla\cdot\vec{A}\ .$$

These calculations allow us to see at least some of the desired relationships, where we again blithely use $\vec{A}$ to mean the tangent vector with the same Cartesian components as $\underset{\sim}{A}$:

$$*d*\underset{\sim}{A} = -\nabla\cdot\vec{A}\ ,\quad *d\underset{\sim}{A} = -\nabla\times\vec{A}\ .$$

Continuing onward, however, we are now asked to compute three different second derivatives. We begin by calculating d on $*d*\underset{\sim}{A}$ with the input from the equations above:

$$d(*d*\underset{\sim}{A}) = -\left\{dx\partial_x + dy\partial_y + dz\partial_z\right\}(\partial_x A_x + \partial_y A_y + \partial_z A_z)$$
$$= \Big(-\nabla\cdot\vec{A}\Big) = -\nabla(\nabla\cdot\vec{A})\ .$$

As $*d*\underset{\sim}{A}$ is a simple scalar, we see that the action of d on it is just the same as would be the action of the gradient operator on a scalar, again under this "identification" of tangent vectors and 1-forms.

Next we turn our attention to the action of $*d$ on $*d\underset{\sim}{A}$, which will result, as the previous one, in a 1-form:

$$d(*d\underset{\sim}{A}) = -\,d\Big\{(\partial_x A_y - \partial_y A_x)dz + (\partial_y A_z - \partial_z A_y)dx + (\partial_z A_x - \partial_x A_z)dy\Big\}$$
$$= -\Big\{\partial_x(\partial_x A_y - \partial_y A_x)dx\wedge dz + \partial_y(\partial_x A_y - \partial_y A_x)dy\wedge dz + \partial_y(\partial_y A_z - \partial_z A_y)dy\wedge dx$$
$$+\,\partial_z(\partial_y A_z - \partial_z A_y)dz\wedge dx + \partial_x(\partial_z A_x - \partial_x A_z)dx\wedge dy + \partial_z(\partial_z A_x - \partial_x A_z)dz\wedge dy\Big\}$$
$$= +\Big\{\Big((\partial_y^2 + \partial_z^2)A_x - \partial_x(\partial_y A_y + \partial_z A_z)\Big)dy\wedge dz$$
$$+\Big((\partial_y^2 + \partial_x^2)A_z - \partial_z(\partial_y A_y + \partial_x A_x)\Big)dx\wedge dy$$
$$+\Big((\partial_x^2 + \partial_z^2)A_y - \partial_y(\partial_x A_x + \partial_z A_z)\Big)dz\wedge dz\Big\}$$
$$= +\Big\{\Big((\partial_y^2 + \partial_z^2 + \partial_x^2)A_x - \partial_x(\partial_y A_y + \partial_z A_z + \partial_x A_x)\Big)dy\wedge dz$$
$$+\Big((\partial_y^2 + \partial_x^2 + \partial_z^2)A_z - \partial_z(\partial_y A_y + \partial_x A_x + \partial_z A_z)\Big)dx\wedge dy$$
$$+\Big((\partial_x^2 + \partial_z^2 + \partial_y^2)A_y - \partial_y(\partial_x A_x + \partial_z A_z + \partial_y A_y)\Big)dz\wedge dz\Big\}$$

We should now, finally, take the dual of this rather long expression, which will reduce it back to a 1-form:

$$*d*d\underset{\sim}{A} = -\nabla^2(A_z dz + A_y dy + A_x dx) + d(*d*\underset{\sim}{A})\ .$$

This generates what was of course the intended, final expression, namely that

$$\nabla^2 \underset{\sim}{A} = d*d*\underset{\sim}{A} - *d*d\underset{\sim}{A}\ .$$

As a concluding calculation we show that $d(d\underset{\sim}{A})$ is just zero, as expected:

$$\begin{aligned} d(d\underset{\sim}{A}) &= d\Big\{(\partial_x A_y - \partial_y A_x)dx\wedge dy + (\partial_y A_z - \partial_z A_y)dy\wedge dz + (\partial_z A_x - \partial_x A_z)dz\wedge dx\Big\} \\ &= \Big\{\partial_z(\partial_x A_y - \partial_y A_x) + \partial_x(\partial_y A_z - \partial_z A_y) + \partial_y(\partial_z A_x - \partial_x A_z)\Big\}dx\wedge dy\wedge dz \\ &= \Big\{\partial_z\partial_x A_y - \partial_z\partial_y A_x + \partial_x\partial_y A_z - \partial_x\partial_z A_y + \partial_y\partial_z A_x - \partial_y\partial_x A_z\Big\}dx\wedge dy\wedge dz = 0\ . \end{aligned}$$

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3. Still working in 3-dimensional space, we know from Pythagoras that we can measure "infinitesimal" distances via

$$ds^2 \equiv dx^2 + dy^2 + dz^2 \equiv \delta_{ij}dx^i\,dx^j\ ,\quad dx^i = \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}\ .$$

Although in Cartesian coordinates the components of this tensor are just δ_{ij}, in other coordinates they will be more complicated; therefore, we will refer to this, in general, as a symmetric tensor of type (0,2), and use the symbol $\mathbf{g}$ for it. Therefore, we could do the algebra which would tell us, in spherical polar coordinates, for instance, that

$$ds^2 = \mathbf{g} = g_{ij}dr^i\otimes dr^j = dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\varphi^2\ ,\quad dr^i = \begin{pmatrix} dr \\ d\theta \\ d\varphi \end{pmatrix}\ .$$

In this basis, please write out the presentation of the components of $\mathbf{g}$, i.e., the coefficients g_{ij}, as a 3×3 matrix. Then determine the matrix inverse to that one. Viewed as components, that matrix determines a (2,0) tensor; we will label its components as g^{ij}. Define a new basis for 1-forms, $\{\underset{\sim}{\omega}^a\}_1^3$, and re-write the metric tensor in this basis, giving us the components, g_{ab}, in this new basis, which you should determine and present, again, in the form of a 3×3 matrix:

$$\underset{\sim}{\omega}^a = \begin{pmatrix} dr \\ r\,d\theta \\ r\,\sin\theta\,d\varphi \end{pmatrix}\ ,\qquad \mathbf{g} = g_{ab}\underset{\sim}{\omega}^a\otimes\underset{\sim}{\omega}^b\ .$$

With this same idea, then, determine a (new) basis for tangent vectors which is ***dual*** to this basis of 1-forms; i.e., determine a set of basis vectors $\{\widetilde{e}_b\}_1^3$ such that the action on them of the new basis of 1-forms is "reciprocal": $\underset{\sim}{\omega}^a(\widetilde{e}_b) = \delta^a_b$. In this basis what form does the following (2,0) tensor have:

$$\mathbf{g}^* \equiv g^{ab}\widetilde{e}_a \otimes \widetilde{e}_b \; .$$

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a. Comparing the definition of g_{ij} in spherical polar coordinates with the equation for ds^2, we immediately write down the (diagonal) matrix that presents the components of this tensor in this basis:

$$g_{ij} \Longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & (r\sin\theta)^2 \end{pmatrix} .$$

As it is diagonal, its inverse is easily computed:

$$g^{ij} \Longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & \frac{1}{(r\sin\theta)^2} \end{pmatrix} .$$

b. Again, writing out in detail the definition, assuming that the result will be a symmetrical tensor, we have

$$\begin{aligned} ds^2 &= dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\varphi^2 = \mathbf{g} \\ &= g_{ab}\underset{\sim}{\omega}^a \otimes \underset{\sim}{\omega}^b = g_{11}(dr)^2 + 2g_{12}(dr)(r\,d\theta) + 2g_{13}(dr)(r\sin\theta\,d\varphi) + g_{22}(r\,d\theta)^2 \\ &\qquad + 2g_{23}(r\,d\theta)(r\sin\theta\,d\varphi) + g_{33}(r\sin\theta\,d\varphi)^2 \; . \end{aligned}$$

Comparing we see that in this basis the matrix presentation of the components of the metric is just **the identity matrix**. That tells us that this choice of basis was made just so that we may claim that **it is an orthonormal basis,** which we take as meaning that all the basis elements are of unit length and orthogonal.

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c. Now, the inverse matrix to the identity is of course the identity. Therefore, we want a basis for tangent vectors, at least somewhat different from the usual one, $\{\partial_{x^i}\}_1^3$, such that the components of the identity will give the inverse metric tensor. Therefore we should probably retreat slightly and calculate that tensor in the (more usual) coordinate basis, using the other inverse matrix that we determined in part (a):

$$\mathbf{g}^* = g^{ij}\partial_{x^i}\partial_{x^j} = \partial_r^2 + \frac{1}{r^2}\partial_\theta^2 + \frac{1}{r^2\sin^2\theta}\partial_\varphi^2 \; .$$

In the statement of the problem, the reciprocal (dual) basis of tangent vectors, for which we are hunting, is symbolized by $\{\widetilde{e}_b\}_1^3$, so we must then have the following requirement:

$$g^{ab}\widetilde{e}_a \otimes \widetilde{e}_b = \widetilde{e}_1^2 + \widetilde{e}_2^2 + \widetilde{e}_3^2 = \partial_r^2 + \frac{1}{r^2}\partial_\theta^2 + \frac{1}{r^2\sin^2\theta}\partial_\varphi^2 \; ,$$

from which we easily pick out their presentations:

$$\widetilde{e}_1 = \partial_r \ , \quad \widetilde{e}_2 = \frac{1}{r}\partial_\theta \ , \quad \widetilde{e}_3 = \frac{1}{r\sin\theta}\partial_\varphi \ .$$

To verify that this is indeed true, we need to calculate the actions of our basis of 1-forms on our basis of 1-vectors, which of course will constitute a 3×3 matrix, the components of which "should" come out to be the identity, if we have chosen correctly. When performing these calculations we of course use the fact that the coordinate bases are reciprocal, i.e., $\partial r^i/\partial r^j = \delta^i_j$, where r^i means $r^1 = r$, $r^2 = \theta$, and $r^3 = \varphi$:

$$\begin{aligned}
\underset{\sim}{\omega}^a(\widetilde{e}_b) &= \begin{pmatrix} dr(\partial_r) & dr(\frac{1}{r}\partial_\theta) & dr(\frac{1}{r\sin\theta}\partial_\varphi) \\ r\,d\theta(\partial_r) & r\,d\theta(\frac{1}{r}\partial_\theta) & r\,d\theta(\frac{1}{r\sin\theta}\partial_\varphi) \\ r\sin\theta\,d\varphi(\partial_r) & r\sin\theta\,d\varphi(\frac{1}{r}\partial_\theta) & r\sin\theta\,d\varphi(\frac{1}{r\sin\theta}\partial_\varphi) \end{pmatrix} \\
&= \begin{pmatrix} dr(\partial_r) & \frac{1}{r}\,dr(\partial_\theta) & \frac{1}{r\sin\theta}\,dr(\partial_\varphi) \\ r\,d\theta(\partial_r) & d\theta(\partial_\theta) & \frac{1}{\sin\theta}\,d\theta(\partial_\varphi) \\ r\sin\theta\,d\varphi(\partial_r) & \sin\theta\,d\varphi(\partial_\theta) & d\varphi(\partial_\varphi) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \ ,
\end{aligned}$$

as was desired!