Physics 570

Homework #4

Due Thursday, 15 February, 2007

Solutions

1. On a 2-dimensional manifold, using coordinates $\{x, y\}$, we are given the local vector field $\widetilde{V}$, with components, relative to the obvious coordinate basis, given by

$$V^x = xy\,, \quad V^y = -y^2\,.$$

a. Please determine the curve $\Gamma(\eta)$ for which this vector field is the tangent vector; normalize things so that for $\eta = 0$ the curve is at the point on the manifold which has coordinates $x = 2$, $y = 1$, and produce a sketch of this curve, and a few other adjacent curves, i.e., ones that pass through nearby points.

b. Next do the same thing for the vector field $\widetilde{C}$, which has components, again relative to the coordinate basis, given by

$$C^x = y\,,\ C^y = -x\,.$$

. . . .

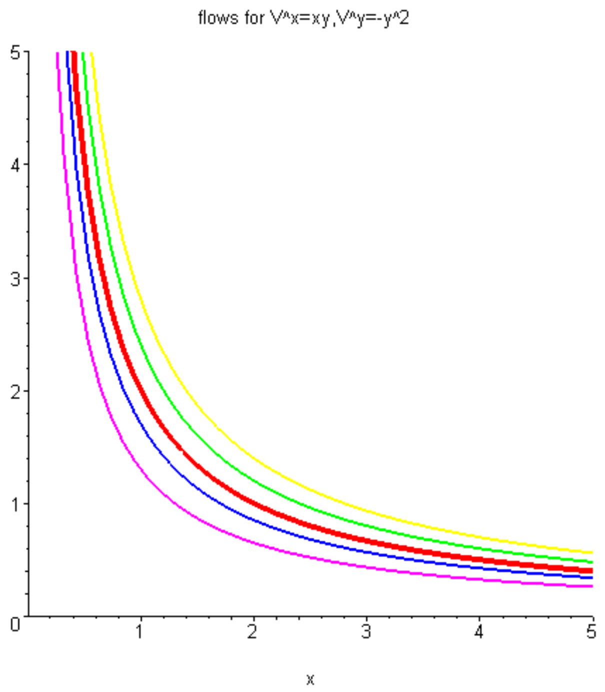
Beginning with $\widetilde{V}$, we have the ordinary differential equations to determine the curve::

$$\frac{dx}{d\eta} = V^x = xy\ , \qquad \frac{dy}{d\eta} = V^y = -y^2\ .$$

The second equation easily integrates to give $y = 1/(\eta - c)$, for c a constant of integration. Inserting this information into the first equation, it can be integrated to give $x = \alpha(\eta - c)$. Then, requiring that this curve pass through the point with coordinates $(2, 1)$, gives us the values $c = -1$ and $\alpha = 2$, so that the curve is described by

$$x\{\Gamma(\eta)\} = 2(\eta + 1)\ , \quad y\{\Gamma(\eta)\} = 1/(\eta + 1)\ .$$

These are *rectangular hyperbolae*, as shown below:



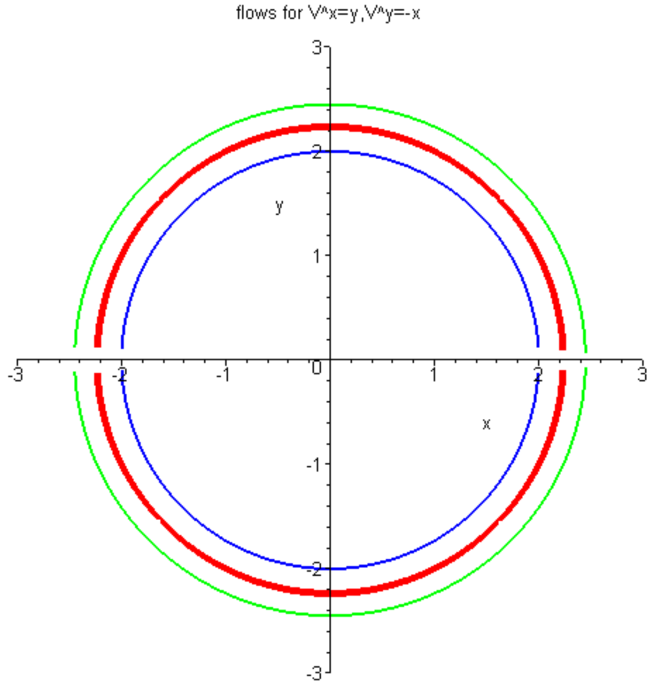
We now proceed to the second one:

$$\frac{dx}{d\eta} = C^x = y \ , \qquad \frac{dy}{d\eta} = C^y = -x \ .$$

This coupled set is most easily integrated by putting in terms of $z \equiv x + iy$, which gives the following, where we label the curve by $\Delta(\eta)$, and, again, insist that it pass through the point with $x = 2$ and $y = 1$:

$$\begin{gathered}\frac{dz}{d\eta} = y - ix = -i(x+iy) = -iz \implies \quad z = (a+ib)e^{-i\eta} \\ \implies x = a\,\cos\eta + b\,\sin\eta\ , \quad y = b\,\cos\eta - a\,\sin\eta\ ; \\ x\{\Delta(\eta)\} = 2\,\cos\eta + \sin\eta\ , \quad y\{\Delta(\eta)\} = \cos\eta - 2\,\sin\eta\ .\end{gathered}$$

These are *circles*, as shown below:



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2. Because the earth has a gravitational field, the equivalence principle tells us that we may replace considerations of that field by thinking of the earth as at the center of some non-flat manifold which may well be described in spherical coordinates so that it would have a singularity at $r = 0$. [Of course this is only actually true exterior to the physical earth, so that this singularity of the coordinates does not really occur.]

A good approximation to the metric for that curved space, outside the surface of the earth, ignoring its rotation, is provided by

$$\mathbf{g} = ds^2 = \frac{1}{J^2}\,dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\varphi^2) - J^2\,dt^2\,, \quad J \equiv \sqrt{1+2\Phi}\,, \Phi \equiv -\frac{M}{r}\,,$$

where one can use the equivalence principle to interpret Φ as the potential for the earth's gravitational field.

a. Using this metric, what is the value of the proper time elapsed on a stationary clock at a distance R from the earth's center, as a function of the coordinate time t? If we compare two such clocks, at different distances from the earth's center, which one runs faster?

b. Please write down a (non-holonomic) basis for 1-forms for which the metric is simply

$$ds^2 = (\underset{\sim}{\omega}^{\hat{r}})^2 + (\underset{\sim}{\omega}^{\hat{\theta}})^2 + (\underset{\sim}{\omega}^{\hat{\varphi}})^2 - (\underset{\sim}{\omega}^{\hat{t}})^2\,.$$

Then use the exterior differentials of your chosen basis set to determine the connection 1-forms, $\underset{\sim}{\Gamma}^{\mu}{}_{\nu}$, remembering that they are determined via

$$d\underset{\sim}{\omega}^{\mu} \equiv \underset{\sim}{\omega}^{\nu} \wedge \underset{\sim}{\Gamma}^{\mu}{}_{\nu}\,.$$

As those 1-forms are simply proportional to the gradients of the coordinates, and the metric coefficients are constant, the "guess" method should work.

c. In terms of the 4-velocity $\widetilde{u}$, use these connection 1-forms to write out the four geodesic equations,

$$\frac{du^\gamma}{d\tau} + u^\alpha \Gamma^\gamma{}_{\alpha\beta} u^\beta = 0 \,,$$

for a timelike observer moving under the action of no outside forces. Now apply those equations to the situation of an observer who simply orbits the earth at the equator, i.e., moves so that r and $\theta = \pi/2$ are constant. Therefore one has the initial conditions for the pde's that $u^{\hat{r}}\Big|_{\tau=0} = 0$ and $u^{\hat{\theta}}\Big|_{\tau=0} = 0$. First show that the equations allow these initial conditions to be maintained at all later proper times. Next use the remaining equations to determine the angular frequency of rotation of our observer in his circular orbit. Are you surprised by the result?

. . . .

a. For a stationary clock, $dx^i = 0$, so that we have simply

$$d\tau^2 \equiv -ds^2 = (1 - 2M/r)\, dt^2 \quad \Longrightarrow \quad d\tau = J\, dt = \sqrt{1 - 2M/r}\, dt \,.$$

Comparing two clocks at different distances, the one at the larger distance from the center of the earth will have larger r, so that $2M/r$ will be smaller, so that $1 - 2M/r$ will be larger, saying that

> for equal values of dt, the value of $d\tau$ will be larger on a clock at larger distance from the center of the earth; therefore, the clock at the higher altitude runs faster than the one at the lower altitude, or, if you prefer, the "person" at the higher altitude ages faster than the one at a lower altitude.

b. A (non-holonomic) basis for 1-forms for which the metric is simply

$$ds^2 = (\underset{\approx}{\omega}^{\hat{r}})^2 + (\underset{\approx}{\omega}^{\hat{\theta}})^2 + (\underset{\approx}{\omega}^{\hat{\varphi}})^2 - (\underset{\approx}{\omega}^{\hat{t}})^2 \,.$$

is given by the following set of 1-forms:

$$\underset{\approx}{\omega}^{\hat{r}} \equiv \frac{1}{J} dr \,, \quad \underset{\approx}{\omega}^{\hat{\theta}} \equiv r\, d\theta \,, \quad \underset{\approx}{\omega}^{\hat{\varphi}} \equiv r \sin\theta\, d\varphi \,, \quad \underset{\approx}{\omega}^{\hat{t}} \equiv J\, dt \,, \quad J \equiv \sqrt{1 + 2\Phi(r)} \,.$$

We use the exterior differentials of the above basis set to determine the connection 1-forms. As those 1-forms are simply proportional to the gradients of the coordinates, and the metric coefficients are constant, the "guess" method should work:

$$\begin{aligned}
d\underset{\approx}{\omega}^{\hat{r}} &= 0 \,, && \Longrightarrow \quad \underset{\sim}{\Gamma}_{\hat{r}\hat{\theta}} \propto \underset{\approx}{\omega}^{\hat{\theta}} \,,\ \underset{\sim}{\Gamma}_{\hat{r}\hat{\varphi}} \propto \underset{\approx}{\omega}^{\hat{\varphi}} \,,\ \underset{\sim}{\Gamma}_{\hat{r}\hat{t}} \propto \underset{\approx}{\omega}^{\hat{t}} \,; \\
d\underset{\approx}{\omega}^{\hat{\theta}} &= dr \wedge d\theta = \frac{J}{r} \underset{\approx}{\omega}^{\hat{r}} \wedge \underset{\approx}{\omega}^{\hat{\theta}} \,, && \Longrightarrow \quad \underset{\sim}{\Gamma}_{\hat{r}\hat{\theta}} = -\frac{J}{r} \underset{\approx}{\omega}^{\hat{\theta}} \,,\ \underset{\sim}{\Gamma}_{\hat{\theta}\hat{\varphi}} \propto \underset{\approx}{\omega}^{\hat{\varphi}} \,,\ \underset{\sim}{\Gamma}_{\hat{\theta}\hat{t}} \propto \underset{\approx}{\omega}^{\hat{t}} \,; \\
d\underset{\approx}{\omega}^{\hat{\varphi}} &= \frac{J}{r} \underset{\approx}{\omega}^{\hat{r}} \wedge \underset{\approx}{\omega}^{\hat{\varphi}} + \frac{\cot\theta}{r} \underset{\approx}{\omega}^{\hat{\theta}} \wedge \underset{\approx}{\omega}^{\hat{\varphi}} \,, && \Longrightarrow \quad \underset{\sim}{\Gamma}_{\hat{r}\hat{\varphi}} = -\frac{J}{r} \underset{\approx}{\omega}^{\hat{\varphi}} \,,\ \underset{\sim}{\Gamma}_{\hat{\theta}\hat{\varphi}} = -\frac{\cot\theta}{r} \underset{\approx}{\omega}^{\hat{\varphi}} \,,\ \underset{\sim}{\Gamma}_{\hat{\varphi}\hat{t}} \propto \underset{\approx}{\omega}^{\hat{t}} \,; \\
d\underset{\approx}{\omega}^{\hat{t}} &= J' \underset{\approx}{\omega}^{\hat{r}} \wedge \underset{\approx}{\omega}^{\hat{t}} = \frac{\Phi'}{J} \underset{\approx}{\omega}^{\hat{r}} \wedge \underset{\approx}{\omega}^{\hat{t}} \,, && \Longrightarrow \quad \underset{\sim}{\Gamma}_{\hat{r}\hat{t}} = \frac{\Phi'}{J} \underset{\approx}{\omega}^{\hat{t}} \,,\ \underset{\sim}{\Gamma}_{\hat{\theta}\hat{t}} = 0 \,,\ \underset{\sim}{\Gamma}_{\hat{\varphi}\hat{t}} = 0 \,,
\end{aligned}$$

where a useful summary may now be presented, remembering that the various 1-forms are skew-symmetric in the indices shown:

$$\underset{\sim}{\Gamma}_{\hat{r}\hat{\theta}} = -\frac{J}{r}\underset{\sim}{\omega}^{\hat{\theta}} \;,\; \underset{\sim}{\Gamma}_{\hat{r}\hat{\varphi}} = -\frac{J}{r}\underset{\sim}{\omega}^{\hat{\varphi}} \;,\; \underset{\sim}{\Gamma}_{\hat{\theta}\hat{\varphi}} = -\frac{\cot\theta}{r}\underset{\sim}{\omega}^{\hat{\varphi}} \;,\; \underset{\sim}{\Gamma}_{\hat{r}\hat{t}} = \frac{\Phi'}{J}\,\underset{\sim}{\omega}^{\hat{t}} \;,\; \underset{\sim}{\Gamma}_{\hat{\theta}\hat{t}} = 0 = \underset{\sim}{\Gamma}_{\hat{\varphi}\hat{t}} \;.$$

c. In terms of the 4-velocity $\widetilde{u}$, the four geodesic equations for a timelike observer moving under the action of no outside forces are then the following:

$$\begin{aligned}
\frac{du^{\hat{r}}}{d\tau} &= -u^\alpha \Gamma^{\hat{r}}{}_{\alpha\beta}u^\beta = \frac{J}{r}[(u^{\hat{\theta}})^2 + (u^{\hat{\varphi}})^2] - \frac{\Phi'}{J}(u^{\hat{t}})^2 \;, \\
\frac{du^{\hat{\theta}}}{d\tau} &= -u^\alpha \Gamma^{\hat{\theta}}{}_{\alpha\beta}u^\beta = -\frac{J}{r}u^{\hat{r}}u^{\hat{\theta}} + \frac{\cot\theta}{r}(u^{\hat{\varphi}})^2 \;, \\
\frac{du^{\hat{\varphi}}}{d\tau} &= -u^\alpha \Gamma^{\hat{\varphi}}{}_{\alpha\beta}u^\beta = -\frac{J}{r}u^{\hat{r}}u^{\hat{\varphi}} - \frac{\cot\theta}{r}u^{\hat{\theta}}u^{\hat{\varphi}} \;, \\
\frac{du^{\hat{t}}}{d\tau} &= -u^\alpha \Gamma^{\hat{t}}{}_{\alpha\beta}u^\beta = +u^\alpha \Gamma_{\hat{t}\alpha\beta}u^\beta = -\frac{\Phi'}{J}u^{\hat{r}}u^{\hat{t}} \;.
\end{aligned}$$

We now consider an observer who simply orbits the earth at the equator, i.e., moves so that r and $\theta = \pi/2$ are constant.
In principle, we may only insist on these values at the beginning of the trajectory, i.e., at $\tau = 0$, and we must then study the equations of motion to ensure that these conditions can be maintained as the motion progresses. However, of course since we want them to be constant, we may also insist that at this initial time that their first derivatives vanish, i.e., we take as initial conditions for our equations of motion that $u^{\hat{r}}(\tau = 0) = 0$ and $u^{\hat{\theta}}(\tau = 0) = 0$. Then we must evaluate the above equations at $\tau = 0$, and check to see if these conditions of constancy can be maintained. [As an aside, just from our usual knowledge of "great circles" on spheres, we know that the equator is a great circle that has a constant value of θ, while other such circles do not; therefore, it is not a completely trivial matter that one should check this out.] We may now write out the equations evaluated at this same initial time:

$$\begin{aligned}
\left.\frac{du^{\hat{r}}}{d\tau}\right|_{\tau=0} &= \frac{J}{r}(u^{\hat{\varphi}})^2(0) - \frac{\Phi'}{J}(u^{\hat{t}})^2(0) \;, \\
\left.\frac{du^{\hat{\theta}}}{d\tau}\right|_{\tau=0} &= 0 \;, \\
\left.\frac{du^{\hat{\varphi}}}{d\tau}\right|_{\tau=0} &= 0 \;, \\
\left.\frac{du^{\hat{t}}}{d\tau}\right|_{\tau=0} &= 0 \;.
\end{aligned}$$

These last three equations tell us immediately that these particular initial conditions arrange for the first derivatives of $u^{\hat{\theta}}$, $u^{\hat{\varphi}}$, and $u^{\hat{t}}$ to all also vanish at the beginning. Therefore, we could differentiate the first-order ode's again, and show that the second derivatives must all vanish at the beginning. Continuing such a cycle indefinitely, all their derivatives must vanish at the beginning; therefore, the functions themselves must be constant, as all their derivatives vanish—via an expansion using Taylor's theorem. Therefore, it is indeed possible to maintain θ at its fixed value, since an initial value of $u^{\hat\theta}$ of zero allows it never to change. It also tells us that $u^{\hat\varphi}$ and $u^{\hat t}$ may then stay constant, allowing us to determine the angular frequency.

However, we also want $u^{\hat r}$ to remain zero. Following the same line of reasoning as above, this requires that its derivative should vanish at $\tau = 0$. That constraint gives us a relation between the constant values of $u^{\hat\varphi}$ and $u^{\hat t}$:

$$0 = \frac{d}{d\tau} u^{\hat r}\Big|_{\tau=0} \quad \Longrightarrow \quad \left(\frac{u^{\hat\varphi}}{u^{\hat t}}\right)^2 = \frac{r\Phi'}{J^2} \text{ ,or } \frac{u^{\hat\varphi}}{u^{\hat t}} = \pm\frac{\sqrt{r\Phi'}}{J} \ ,$$

where the $\pm$ simply relates to which direction she is travelling around the equator. However, now we need to know how this ratio is related to the angular frequency:

$$\pm\frac{\sqrt{r\Phi'}}{J} = \frac{u^{\hat\varphi}}{u^{\hat t}} = \frac{r\sin\theta d\varphi}{J\,dt}\Big|_{\theta=\pi/2} = \frac{r}{J}\frac{d\varphi}{dt} = \frac{r}{J}\omega \quad \Longrightarrow \quad \omega = \sqrt{\frac{\Phi'}{r}} = \sqrt{M/r^3} \ .$$

This is in fact the same result as if we were just following Newton's laws of motion. NOTE that we are not surprised since we are looking at the lowest-order, Newtonian approximation. HOWEVER, we are quite surprised since we have put in no gravitational forces to cause this, but have simply found the geodesic in this curved-space metric.

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3. Take $\underset{\sim}{\alpha}$ as an arbitrary 2-form; therefore, in terms of a basis of 1-forms, $\{\underset{\sim}{\omega}^\mu\}_1^4$, it may be written as

$$\underset{\sim}{\alpha} = \tfrac{1}{2}\alpha_{\mu\nu}\underset{\sim}{\omega}^\mu \wedge \underset{\sim}{\omega}^\nu \ .$$

Let the reciprocal basis of tangent vectors be labeled as $\{\widetilde{e}_\lambda\}_1^4$. Show in sufficient detail the calculation that shows that the components of $\underset{\sim}{\alpha}$ are determined by

$$\alpha_{\mu\nu} = \underset{\sim}{\alpha}(\widetilde{e}_\mu, \widetilde{e}_\nu) \ .$$

. . . .

$$\begin{aligned}\underset{\sim}{\alpha}(\widetilde{e}_\mu, \widetilde{e}_\nu) &= \tfrac{1}{2}\alpha_{\kappa\lambda}\left\{\underset{\sim}{\omega}^\kappa \wedge \underset{\sim}{\omega}^\lambda\right\}(\widetilde{e}_\mu, \widetilde{e}_\nu) = \tfrac{1}{2}\alpha_{\kappa\lambda}\left\{\underset{\sim}{\omega}^\kappa \otimes \underset{\sim}{\omega}^\lambda - \underset{\sim}{\omega}^\lambda \otimes \underset{\sim}{\omega}^\kappa\right\}(\widetilde{e}_\mu, \widetilde{e}_\nu) \\ &= \tfrac{1}{2}\alpha_{\kappa\lambda}\left\{\underset{\sim}{\omega}^\kappa(\widetilde{e}_\mu)\underset{\sim}{\omega}^\lambda(\widetilde{e}_\nu) - \underset{\sim}{\omega}^\lambda(\widetilde{e}_\mu)\underset{\sim}{\omega}^\kappa(\widetilde{e}_\nu)\right\} = \tfrac{1}{2}\alpha_{\kappa\lambda}\left\{\delta^\kappa_\mu\delta^\lambda_\nu - \delta^\lambda_\mu\delta^\kappa_\nu\right\} = \tfrac{1}{2}\left\{\alpha_{\mu\nu} - \alpha_{\nu\mu}\right\} \\ &= \alpha_{\mu\nu} \ ,\end{aligned}$$

which is what was desired, and of course the very last equality is because the coefficients of a 2-form are skew-symmetric.

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4. The Brinkman metric is a solution of the Einstein vacuum field equations that describes plane gravitational waves. In terms of coordinates $\{a, b, u, v\}$ and one arbitrary function of one variable, $h = h(u)$, it may be written as follows:

$$\mathbf{g} \equiv ds^2 = 2\,da\,db + a^2\,h(u)\,du^2 - du\,dv\ .$$

We will also use the following basis for 1-forms:

$$\underset{\approx}{\omega}^a \equiv da\ , \quad \underset{\approx}{\omega}^b \equiv db\ , \quad \underset{\approx}{\omega}^u \equiv du\ , \quad \underset{\approx}{\omega}^v \equiv \tfrac{1}{2}(a^2\,h\,du - dv)\ , \quad \mathbf{g} \equiv g_{\alpha\beta}\,\underset{\approx}{\omega}^\alpha \otimes \underset{\approx}{\omega}^\beta\ .$$

a. First write out the 4×4 matrix that presents the quantities $g_{\alpha\beta}$, just defined above, i.e., the components of the given metric, relative to the given basis of 1-forms.

b. Then write down definitions for the basis of tangent vectors, $\{\widetilde{e}_\beta\}_1^4$, that is reciprocal to this basis for 1-forms.

c. An arbitrary tangent vector may be described by giving explicitly its components with respect to that reciprocal basis:

$$\widetilde{w} = w^\alpha\,\widetilde{e}_\alpha \equiv \rho\,\widetilde{e}_a + \sigma\,\widetilde{e}_b + \psi\,\widetilde{e}_u + \phi\,\widetilde{e}_v\ .$$

Please determine its associated 1-form

$$\underset{\approx}{w} \equiv w_\beta\,\underset{\approx}{\omega}^\beta \equiv (g_{\beta\alpha}w^\alpha)\,\underset{\approx}{\omega}^\beta\ .$$

....

Comparison of the given metric with the basis 1-forms tells us that the metric matrix is

$$g_{\alpha\beta} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\ ,$$

where we have used the same order as in the listing of the basis 1-forms.
The reciprocal basis has to satisfy the usual equations; therefore, we have

$$\widetilde{e}_a = \partial_a\ , \quad \widetilde{e}_b = \partial_b\ , \quad \widetilde{e}_u = \partial_u + a^2\,h(u)\partial_v\ , \quad \widetilde{e}_v = -2\,\partial_v\ ,$$

where the first two are obvious, while the last two must surely be spanned on ∂_u and ∂_v, so we simply write them both as combinations of those two, and solve the two equations in two unknowns.

For the desired 1-form, we use the metric given above:

$$w_a = g_{a\beta}w^\beta = g_{ab}w^b = \sigma\ , \quad w_b = g_{ba}w^a = \rho\ , \quad w_u = g_{uv}w^v = \phi\ , \quad w_v = g_{vu}w^u = \psi\ ,$$
$$\Longrightarrow \quad \underset{\approx}{w} = \sigma\,\underset{\approx}{\omega}^a + \rho\,\underset{\approx}{\omega}^b + \phi\,\underset{\approx}{\omega}^u + \psi\,\underset{\approx}{\omega}^v$$

$$= \sigma\,da + \rho\,db + \phi\,du + \tfrac{1}{2}\psi[a^2\,h(u)\,du - dv] = \sigma\,da + \rho\,db + [\phi + \tfrac{1}{2}a^2h(u)\psi]\,du - \tfrac{1}{2}\psi\,dv\ ,$$

where the GRADER should NOTE that the last line is not required in order to have answer the question properly.

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5. Consider the following 4×4 matrix:

$$L \equiv \begin{pmatrix} -1 & 0 & a & -a \\ 0 & 1 & 0 & 0 \\ -a & 0 & -1+a^2/2 & -a^2/2 \\ a & 0 & -a^2/2 & 1+a^2/2 \end{pmatrix} .$$

a. Please show that L is in fact a special, orthochronous Lorentz transformation, for all real values of the constant parameter a. However, also give arguments that it is neither a pure rotation, nor a pure Lorentz boost.

b. Even though the matrix L is in the connection part of the Lorentz group that contains the identity, it is nevertheless a Lorentz transformation that is sufficiently far from the identity that it may NOT be written as a single exponential. I will not ask you to show such a non-existence statement; however, do show that it can be written as a product of a rotation $R(\theta;\hat{y})$ multiplied (on the right) by a pure boost. What is the direction of the velocity associated with this boost?

. . . .

We begin by showing that L is a Lorentz transformation, i.e., that $L^T H L = H$, or, with indices, that $L^\mu{}_\alpha L^\nu{}_\beta \eta_{\mu\nu} = \eta_{\alpha\beta}$:

$$\begin{pmatrix} -1 & 0 & -a & a \\ 0 & 1 & 0 & 0 \\ a & 0 & -1+a^2/2 & -a^2/2 \\ -a & 0 & -a^2/2 & 1+a^2/2 \end{pmatrix} \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 & a & -a \\ 0 & 1 & 0 & 0 \\ -a & 0 & -1+a^2/2 & -a^2/2 \\ a & 0 & -a^2/2 & 1+a^2/2 \end{pmatrix}$$
$$= \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} .$$

Next one calculates its determinant to be $+1$, and, lastly $L^4{}_4 > 1$ so that it is orthochronous, i.e., preserves the direction of time.
Lastly, we may say that it is

i.) not a rotation since a rotation should not mix the 4th components of vectors with the other 3, i.e., it should have zero entries in the 4 row and column, except for the 4,4-entry which should be 1, which is not true for this matrix, and

ii.) not a boost since all boosts are symmetrical matrices, i.e., equal to their own transposes, which again this matrix is not.

b. Next we should be able to determine the desired boost by writing $B = R(-\theta;\hat{y})L$:

$$\begin{pmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & a & -1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & -1+a^2/2 & -a^2/2 \\ a & 0 & -a^2/2 & 1+a^2/2 \end{pmatrix}$$
$$= \begin{pmatrix} -\cos\theta + a\sin\theta & 0 & a\cos\theta + (1-a^2/2)\sin\theta & -a\cos\theta + (a^2/2)\sin\theta \\ 0 & 1 & 0 & 0 \\ -\sin\theta - a\cos\theta & 0 & a\sin\theta - (1-a^2/2)\cos\theta & -a\sin\theta - (a^2/2)\cos\theta \\ a & 0 & -a^2/2 & 1+a^2/2 \end{pmatrix} .$$

Knowing that the 4th row and 4th column of a boost should be equal we quickly write down two equations to determine the sine and cosine of θ:

$$\begin{aligned} -a\,\cos\theta + \frac{a^2}{2}\,\sin\theta &= a \\ -a\,\sin\theta - \frac{a^2}{2}\,\cos\theta &= -\frac{a^2}{2} \\ \Longrightarrow \qquad \cos\theta = \frac{a^2-4}{a^2+4}\ ,\ \sin\theta &= \frac{4a}{a^2+4}\ . \end{aligned}$$

It is straightforward to show that, indeed, these are proper values for the cosine and sine, i.e., that $\cos^2\theta + \sin^2\theta = 1$. Note that for $a = 0$ the angle θ must equal π, while as a increases toward infinity, that angle decreases toward 0.
We must now insert these values into the remainder of the matrix, to acquire our desired boost. This gives

$$B(\vec{v}) = \begin{pmatrix} 1+\frac{2a^2}{a^2+4} & 0 & -\frac{a^3}{a^2+4} & a \\ 0 & 1 & 0 & 0 \\ -\frac{a^3}{a^2+4} & 0 & 1+\frac{a^4/2}{a^2+4} & -a^2/2 \\ a & 0 & -a^2/2 & 1+a^2/2 \end{pmatrix} .$$

Knowing that the 4th row and/or column is just the components of $\widetilde{u}$, we may immediately say that

$$\gamma v^x = a\ ,\quad \gamma v^y = 0\ ,\quad \gamma v^z = -a^2/2\ ,\quad \gamma = 1 + a^2/2\ ,$$
$$\Longrightarrow \qquad \vec{v} = \frac{1}{a^2+2}\left\{2a\,\hat{x} - a^2\,\hat{z}\right\}\ .$$

It is straightforward to insert these values into the standard form for boosts and to observe that the entire matrix is consistent with this choice for $\vec{v}$.
The conclusion is that

$$L = R(\theta;\hat{y})B(\vec{v})\ ; \qquad \cos\theta = \frac{a^2-4}{a^2+4}\ , \quad \vec{v} = \frac{2a\,\hat{x} - a^2\,\hat{z}}{a^2+2}\ .$$

Again we see that when $a = 0$ this is just the identity, i.e., the associated velocity is zero, while as a increases the velocity increases, making an angle in the x, z-plane with tangent of $-a/2$, and therefor beginning slowly in the $\hat{x}$-direction and rotating, as a increases, to point more and more in the negative $\hat{z}$-direction.

NOTE to GRADER: as usual, all this interpretation for varying values of a is NOT required.
