Physics 570

Homework #5

Due Thursday, 22 February, 2007

1. Using the fact that the covariant derivative of a vector is again a vector, determine the transformation of an affine connection from one basis set to another.
More precisely, let us define our transformation via the transformations on the reciprocal bases of 1-forms and tangent vectors as follows:

$$\underset{\sim}{\omega}'^{\alpha} \equiv X^{\alpha}{}_{\mu}\, \underset{\sim}{\omega}^{\mu}\ , \qquad \widetilde{e}'_{\beta} \equiv Y^{\nu}{}_{\beta}\, \widetilde{e}_{\nu}\ , \qquad Y^{\nu}{}_{\beta}\, X^{\beta}{}_{\mu} = \delta^{\nu}_{\mu}\ .$$

Then show that the affine connections relative to the two choices of basis sets are related as follows:

$$\underset{\sim}{\Gamma}'^{\alpha}{}_{\beta} = X^{\alpha}{}_{\mu}\, Y^{\nu}{}_{\beta}\, \underset{\sim}{\Gamma}^{\mu}{}_{\nu} + X^{\alpha}{}_{\mu}\, dY^{\mu}{}_{\beta}\ .$$

Now consider as an example the 2-dimensional, flat plane, and choose as the unprimed basis sets the usual Cartesian, coordinate basis vectors, i.e., $\widetilde{e}_a = \partial_{x^a}$, and as the primed basis set the orthonormal, non-holonomic basis vectors in polar coordinates $\widetilde{e}'_{\hat{r}} = \partial_r$ and $\widetilde{e}'_{\hat{\theta}} = \frac{1}{r}\partial_{\theta}$. Determine the 2×2 matrices X and Y and determine the Levi-Civita connection for the primed basis, using the known fact that the Levi-Civita connection in Cartesian coordinates vanishes in flat space.

2. Consider the metric on a 3-sphere, i.e., a 3-dimensional sphere in 4-space. Use the generalization of the usual angular coordinates that is appropriate for this "higher-dimensional" sphere, namely $\{\psi, \theta, \varphi\} \equiv \{\lambda^i\}_1^3$, and the usual metric is then given by the following:

$$\mathbf{g} = ds^2 = (d\psi)^2 + \sin^2\psi(d\theta^2 + \sin^2\theta\, d\varphi^2) \equiv (\underset{\sim}{\omega}^1)^2 + (\underset{\sim}{\omega}^2)^2 + (\underset{\sim}{\omega}^3)^3\ ,$$

where the set $\{\underset{\sim}{\omega}^i\}_1^3$ is a non-holonomic, orthonormal frame for the problem, at least in some "good" neighborhood of some generic point.

a. Please consider Cartan's First Structure equations, for a metric-compatible, torsion-free connection:

$$d\underset{\sim}{\omega}^i = \underset{\sim}{\omega}^j \wedge \underset{\sim}{\Gamma}^i{}_j\ ,$$

and use these to determine the 3 independent 1-forms that determine the connection, i.e.,

$$\underset{\sim}{\Gamma}_{12}\ , \quad \underset{\sim}{\Gamma}_{23}\ , \quad \underset{\sim}{\Gamma}_{31}\ .$$

Use the usual "guess" method to obtain them.

b. Determine the 3 independent 2-forms that determine the curvature, as given by the Second Structure equations:

$$\underset{\sim}{\Omega}^i{}_j \equiv d\underset{\sim}{\Gamma}^i{}_j + \underset{\sim}{\Gamma}^i{}_k \wedge \underset{\sim}{\Gamma}^k{}_j\ .$$

Be sure and remember that these forms are skew-symmetric on their indices when both are at the same level, i.e.,

$$\underset{\sim}{\Gamma}_{jk} = -\underset{\sim}{\Gamma}_{kj} \ , \qquad \text{and} \qquad \underset{\sim}{\Omega}_{jk} = -\underset{\sim}{\Omega}_{kj} \ .$$

As these are 2-forms, it is also valuable to write out their components explicitly, which we write as

$$\underset{\sim}{\Omega}_{jk} = \frac{1}{2} R_{jklm} \, \underset{\sim}{\omega}^l \wedge \underset{\sim}{\omega}^m \ .$$

As there are only three independent values for the pairs (jk) and/or (lm), the quantities R_{jklm} may be displayed as a 3×3 matrix. Please give such a presentation.
Also please confirm Eq. (3.191) from Carroll's text as applied to this example, noting that $R \equiv g^{jl} g^{km} R_{jklm}$, i.e., it's the full-trace of the curvature tensor.
As a motivation for the problem, it is worth noting that this is the metric for "constant-time" slices of the standard expanding-universe in the case where the universe is closed, i.e., has at each moment the symmetry appropriate to a 3-sphere. [It is also a local metric for the manifold for the group SU(3).]

3. Continuing in the vein where one does interesting problems that do not require too much algebra, consider the metric referred to as the Poincare hyperboloid, a 2-dimensional surface which we take to have coordinates ξ and η, and a metric given by

$$\mathbf{g} = \left(\frac{a}{\eta}\right)^2 (d\xi^2 + d\eta^2) \equiv (\underset{\sim}{\omega}^\xi)^2 + (\underset{\sim}{\omega}^\eta)^2 \ ,$$

where of course the last equality defines a non-holonomic, orthonormal basis for this manifold. Please again determine the metric-compatible, torsion-free connection and its associated curvature. As this is only a 2-dimensional manifold, the curvature should be determined by one, non-zero quantity, $R_{\xi\eta\xi\eta}$. Show that this quantity is $-1/a^2$.

4. PLEASE NOTE THAT I HAVE CHANGED THIS PROBLEM. I do not think we have gotten yet quite far enough with a discussion of Lie derivatives and Killing vectors to ask you to do this now. Therefore, what comes next is rather different—changed this Friday afternoon. Please return to the Brinkman metric, from the last problem set. Using the non-holonomic basis set from that problem, please determine the affine connection 1-forms and the curvature 2-forms.