Physics 570

Homework #6

Due Thursday, 8 March, 2007

1. Using the Second Bianchi Identities, verify that the Einstein tensor is divergenceless; i.e., show that

$$\nabla_{\underset{\sim}{e}_\mu} G^{\mu\nu} = 0 \; .$$

2. The Schwarzschild (vacuum) metric has 4 Killing vectors, namely ∂_t and those 3 that pertain to a metric with spherical symmetry, namely

$$\begin{aligned} \widetilde{K}_1 &= \partial_\varphi = r \sin\theta \, \widetilde{e}_{\hat{\varphi}} \\ \widetilde{K}_2 &= \cos\varphi \, \partial_\theta - \cot\theta \, \sin\varphi \, \partial_\varphi = r[\cos\varphi \, \widetilde{e}_{\hat{\theta}} - \cos\theta \, \sin\varphi \, \widetilde{e}_{\hat{\varphi}}] \; , \\ \widetilde{K}_3 &= -\sin\varphi \, \partial_\theta - \cot\theta \, \cos\varphi \, \partial_\varphi = -r[\sin\varphi \, \widetilde{e}_{\hat{\theta}} + \cos\theta \, \cos\varphi \, \widetilde{e}_{\hat{\varphi}}] \; . \end{aligned}$$

a. Please show that each of these satisfies Killing's equations. You may decide whether it is easier to show that these do indeed satisfy Killing's equations in one of several modes: i.e., in coordinate basis or orthonormal basis, and with ordinary derivatives or covariant derivatives.

b. Please also determine their commutator algebra; i.e., determine the 3 independent commutators they possess, and show that they, themselves, span the set of these 3 commutators.

3. Let us try to choose new coordinates that make the Schwarzschild metric look somewhat more like the usual sort of isotropic coordinates by considering the following pair of coordinate transformations.

a. First define a new coordinate p such that

$$r = p(1 + m/2p)^2 \; .$$

Next, perform the coordinate transformation on the metric to rewrite it in terms of the coordinates (p, θ, φ, t). Then define isotropic coordinates

$$x \equiv p \sin\theta \cos\varphi \; , \quad y \equiv p \sin\theta \sin\varphi \; , \quad z \equiv p \cos\theta \; ,$$

and then transform the metric a second time, showing that in these coordinates it has the form

$$\mathbf{g} = (1 + m/2p)^4 (dx^2 + dy^2 + dz^2) - [(1 - m/2p)/(1 + m/2p)]^2 dt^2 \; , \qquad p^2 \equiv x^2 + y^2 + z^2 \; .$$

b. In the usual Schwarzschild coordinates the angular coordinates behave as expected, while so do the radial and timelike coordinates, i.e., r varies from 0 to $+\infty$, while t varies from $-\infty$ to $+\infty$. What are the corresponding ranges for the coordinates $\{p, \theta, \varphi, t\}$ and how do they map to and from the more usual Schwarzschild coordinate ranges?

4. Using the timelike geodesic equations in the usual Schwarzschild coordinates, determine the minimum radii that a stable elliptical orbit may have. Give a good definition of stable in this context.