Physics 570

Homework #7

Due Thursday, 22 March, 2007

NOTE: Problems 2, 3, and 4 will require some sort of computer calculations.

1. Begin with the usual, flat Minkowski spacetime and transform to a new set of coordinates, $\{p, q, \theta, \varphi\}$, via the following equations:

$$t + r \equiv \tan p\,, \quad t - r \equiv \tan q\,.$$

Show that the metric of Minkowski spacetime may now be written in the (conformal) form:

$$\mathbf{g} = \phi^2 \left\{-4\,dp\,dq + \sin^2(p-q)[d\theta^2 + \sin^2\theta\,d\varphi^2]\right\}\,.$$

The to-be-determined function, ϕ^2 is a function of p and q which is never negative. What is it? In order to cover all of Minkowski spacetime, what are the ranges of the new coordinates p and q. Create a two-dimensional graph showing these ranges, but suppressing the angular coordinates. On this graph show a few lines of constant t, a few lines of constant r, and a few light rays starting at the origin.

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First we just do the algebra required. The equations given tell us that

$$\begin{aligned} 2r &= \tan p - \tan q \quad &\Longrightarrow \quad 2dr &= \sec^2 p\,dp - \sec^2 q\,dq\,,\\ 2t &= \tan p + \tan q \quad &\Longrightarrow \quad 2dt &= \sec^2 p\,dp + \sec^2 q\,dq\,. \end{aligned}$$

Inserting this into the metric, given in the form of spherical coordinates, we have

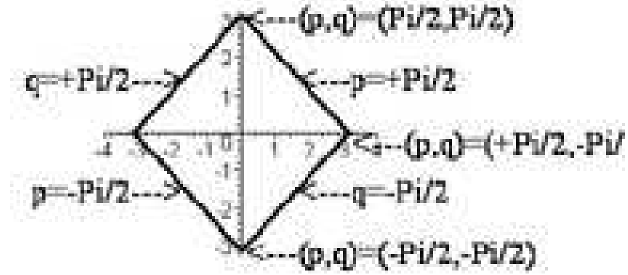
$$\begin{aligned} \mathbf{g} &= dr^2 + r^2\,d\Omega^2 - dt^2\\ &= \left(\frac{\tan p - \tan q}{2}\right)^2 d\Omega^2 + \tfrac{1}{4}\left\{[\sec^2 p\,dp - \sec^2 q\,dq]^2 - [\sec^2 p\,dp + \sec^2 q\,dq]^2\right\}\\ &= -\frac{dp\,dq}{\cos^2 p\cos^2 q} + \left(\frac{\tan p - \tan q}{2}\right)^2 d\Omega^2\\ &= \left(\frac{1}{2\cos p\cos q}\right)^2 \left\{-4dp\,dq + [\sin p\cos q - \sin q\cos p]^2 d\Omega^2\right\}\\ &= \left(\frac{1}{2\cos p\cos q}\right)^2 \left\{-4dp\,dq + \sin^2(p-q)\,d\Omega^2\right\}\,, \end{aligned}$$

which is indeed the form that was desired, with the desired conformal factor given explicitly as

$$\frac{1}{\phi} = 2\cos p\cos q\,.$$

Now, however, we need to concern ourselves somewhat more seriously, to determine the range of the new variables. As the entire system has spherical symmetry, we concentrate on the problem as we ignore the two spherical angles. This means that every point that we will study below really has two more dimensions to it, shaped like a sphere. As this is quite difficult to visualize, having 4 dimensions, the simplest approach to getting a better "feel" for it is to concentrate on the equatorial plane of that sphere: This may be accomplished fairly easily by taking the 2-dimensional (r, t), or (p, q) picture that we will create, in the plane of the paper, and then rotate it about a vertical axis, referring to the angle of rotation of φ in that equatorial plane.

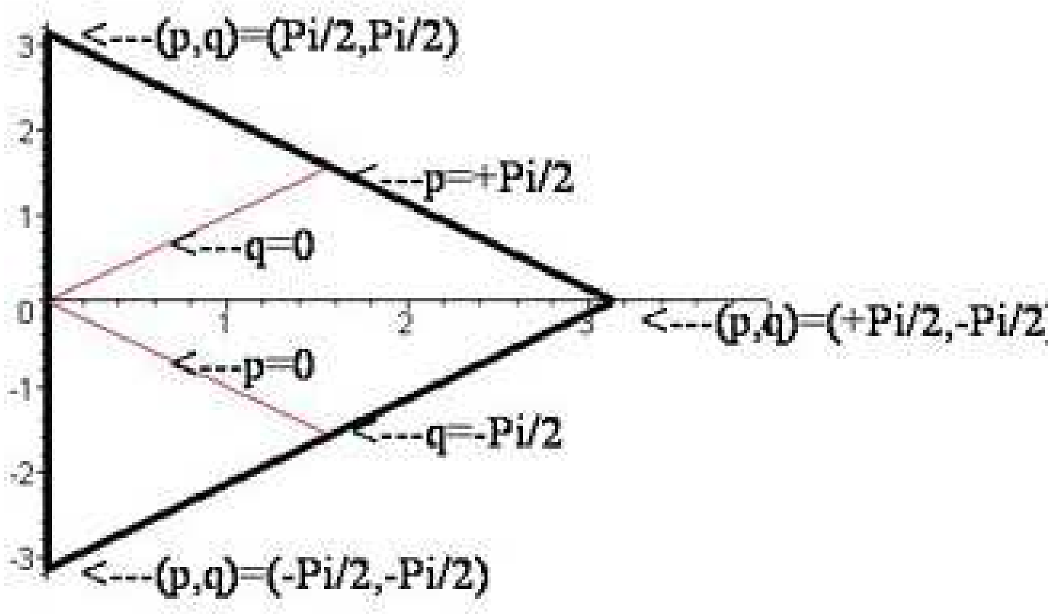
It is simplest to first ignore the fact that we must have $r \geq 0$. Instead, just suppose that both r and t would vary from $-\infty$ to $+\infty$, so that we are looking at a simple 2-dimensional plane, with r and t as horizontal and vertical axes. Then of course the intermediate coordinates, $t + r$ and $t - r$ also vary over that same range. However, the lines which show constant values for them are at angles relative to the original lines of constant r—lines parallel to the t-axis, and therefore vertical—and the original lines of constant t—lines parallel to the r-axis, and therefore horizontal. Instead, the lines of constant value for $t - r$ are at an angle of $+45°$ to the horizontal; the line where this constant value is zero goes through the origin, while if that constant value is larger (or smaller) than zero, then the lines are parallel to the one through the origin, but intersect the t-axis at values which are larger (or smaller) than zero. Similarly, the lines of constant value for $t + r$ are at an angle of $+135°$ to the horizontal; the one with $t + r = 0$ intersects the origin, while the ones with that constant value being larger (or smaller) than zero cut the t-axis at locations which are larger (or smaller) than zero. Each of these lines is of course of infinite extent, going from $-\infty$ to $+\infty$; however, that means that the arctangents, i.e., p and q, vary only from $-\pi/2$ to $+\pi/2$. There has been an infinite scale compression, so that now this entire 2-plane takes on the form of a rhombus, with 4 diagonal boundary lines. In the first quadrant, where both r and t are positive, the boundary is the line where $p = \pi/2$, and p varies along it, from $+\pi/2$ at the point where it intersects the t-axis to $-\pi/2$ at the point where it intersects the r-axis.



However, now let us consider the boundary line, where $r = 0$, a bit more. It must have $p = q$. Therefore, it cuts off all the above lines at the point where they intersect the (original) t-axis, leaving us with a "triangular " region, rather than the rhomboidal one. As that axis is the usual t-axis, the coordinate t must vary along it from $-\infty$ to $+\infty$. However, with $p = q$, we acquire $t = -\infty$ by setting $p = q = -\pi/2$, at the bottom corner. Similarly, we acquire $t = +\infty$ along this left boundary by setting $p = q = +\pi/2$. **It is usual to refer to the top point of this left boundary, where $p = q = +\pi/2$ (or $r = 0$ and $t = +\infty$), as future timelike infinity, and denote it by $\mathbf{i}^+$. Analogously, the point at the bottom of the left boundary, corresponding to either $r = 0$ and $t = -\infty$ or $p = q = -\pi/2$, is denoted $\mathbf{i}^-$, and referred to as past timelike infinity**.
Similarly, the r-axis, i.e., the line $t = 0$, begins on this left boundary, at $r = 0$, and therefore $p = q = 0$, and varies until $r = +\infty$, with values $p = +\pi/2$ and $q = -\pi/2$, so that the re-scaling has taken this formerly infinite line to a length of $\pi/2$. **It is usual to refer to this last point as spatial infinity, and to denote it by the symbol $\mathbf{i}^0$.**

Our vertical cutoff makes the various lines of constant p or q have varying lengths, since we must always have $p \geq q$. For instance the line where $q = -\pi/2$ puts no constraint on allowed values of p, so that p may take all possible values along that line, intersecting the left boundary at the bottom, i.e, at i^- and the central horizontal, i.e., the line $t = 0$ at spatial infinity, at i^0; therefore, it has length π. As we are ascribing different sorts of names to different sorts of infinities, that particular line has a special name also, **the entire line $q = -\pi/2$ is called $\mathcal{J}^-$, and is referred to as past null infinity.** [**The symbol is pronounced "scri," which stands for "script I."**] **All geodesics that describe light rays begin on this line.** However, the "line" which has $q = +\pi/2$ only allows one value for p, of $+\pi/2$; therefore, in fact this "line" is just a single point, namely future timelike infinity, i^+, and of course the lines of constant q become shorter

and shorter as they approach this single point.
Contrariwise the value $p = +\pi/2$ puts no constraint on the allowed values of q, so that this corresponds to a boundary line across the top of our graph, which passes from i^+ down to i^0 at an angle of $-45°$, with q taking on values from $q = +\pi/2$ at i^+ down to $q = -\pi/2$ at i^0. **This upper boundary line is referred to as future null infinity, and with the symbol $\mathfrak{I}^+$. All geodesics that can describe light rays end on this line.** [See diagram.]

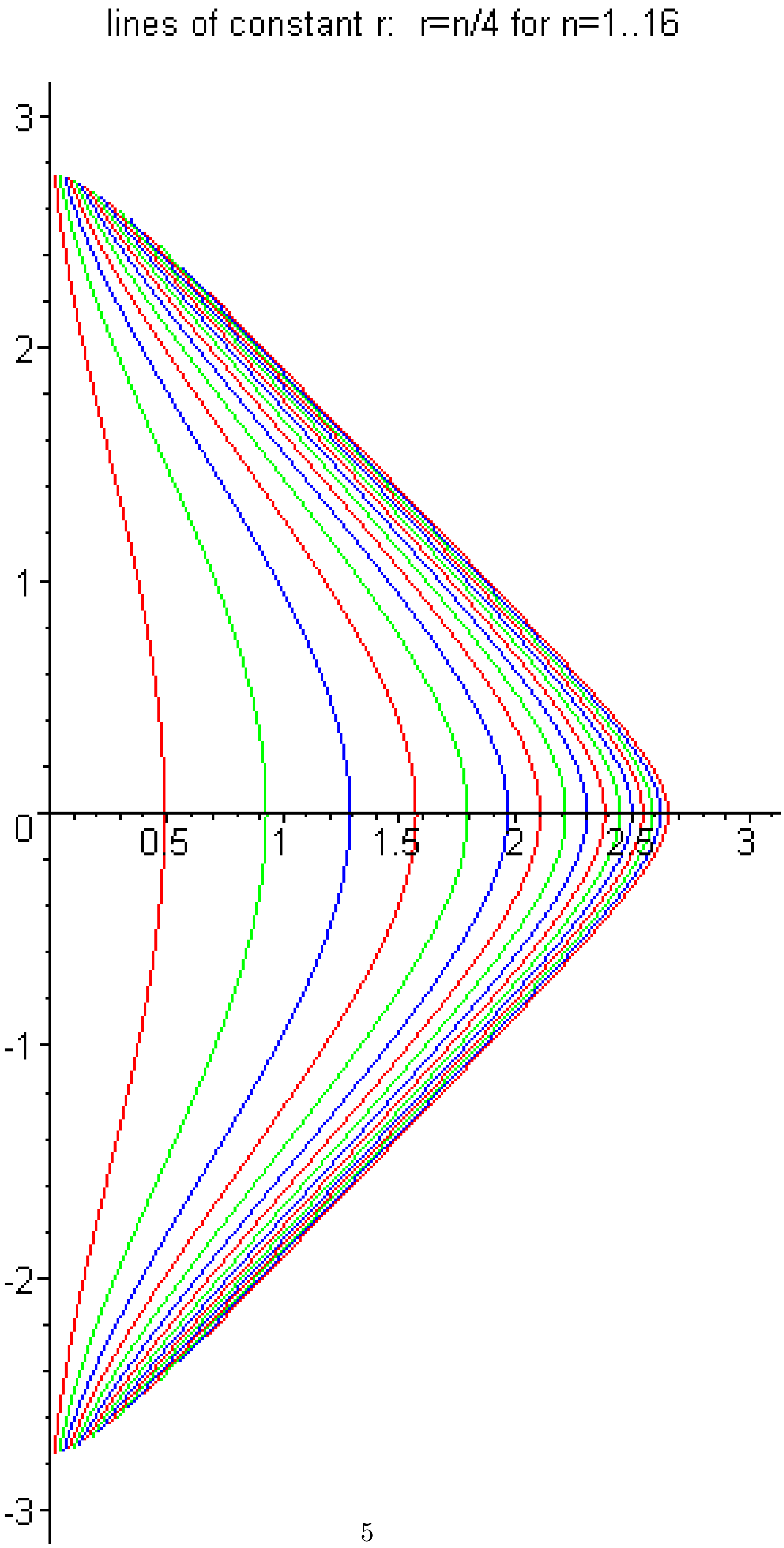


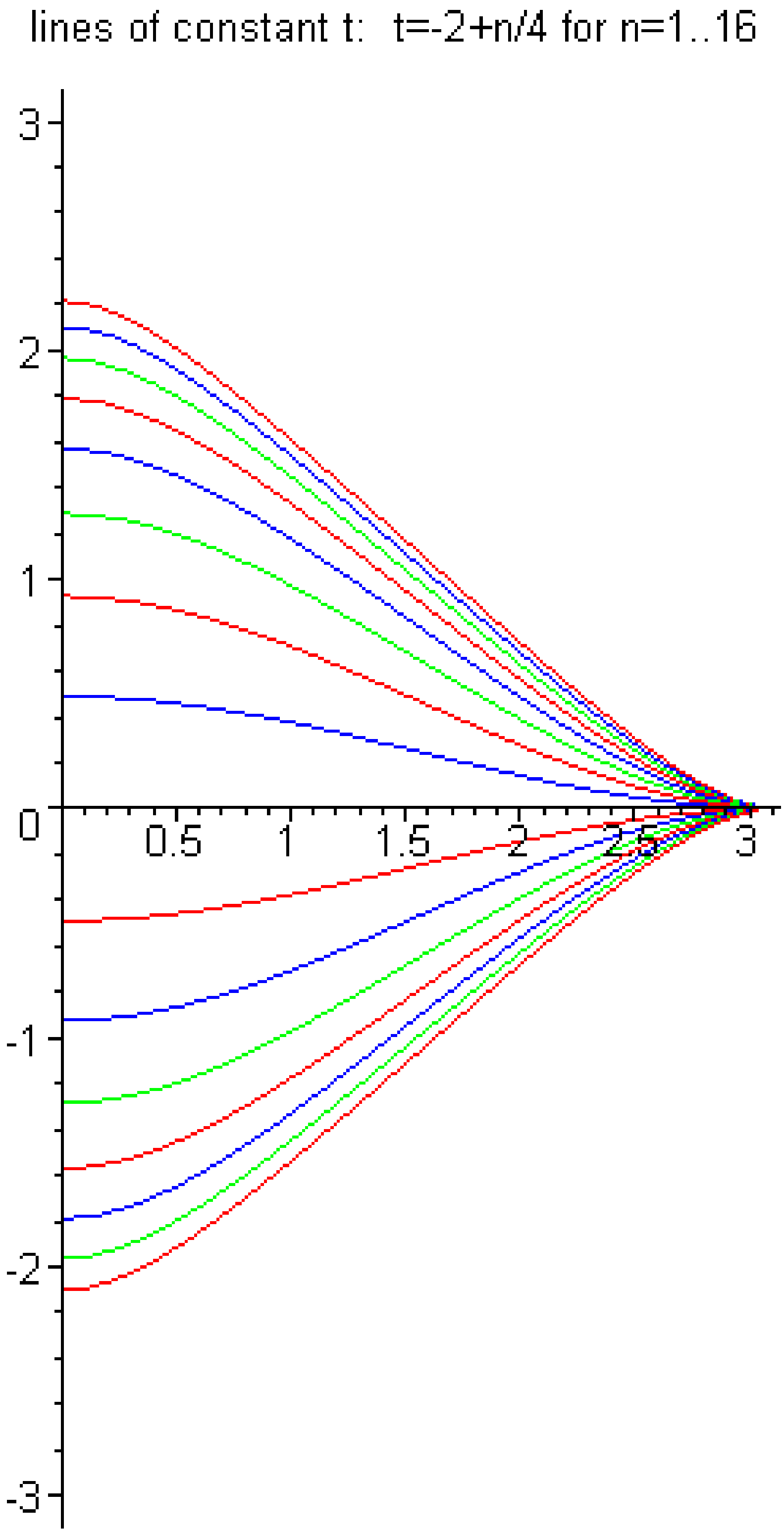
Remember that each point of the above is actually a sphere's worth of points; therefore, to acquire a better visualization of the entire thing, do please rotate the triangle just above about the vertical $r = 0$ axis, some 360°. This generates a figure rather like a pair of cones, with their bases attached.

For lines of constant r see the following graphs; note that, being timelike lines, they all begin at i^- and end at i^+, as some parameter describing the curves varies from $-\infty$ to $+\infty$. The ordinary value of t would be a reasonable parameter to do this.

Likewise for lines of constant t see the graphs on the page after that. These all begin at the vertical border $r = 0$ and end at i^0. An appropriate parameter that varies along them from $-\infty$ to $+\infty$ might be $\log r$.

As for light rays, as already noted, the lines of constant p are incoming light rays, while the lines of constant q are outgoing light rays; there should not be a need to draw any more of them, since they have already been sufficiently well described, and are just diagonal, straight lines (at $\pm 45°$) that pass from $\mathfrak{I}^-$ to the border at $r = 0$, or from that border to $\mathfrak{I}^+$.


lines of constant r: r=n/4 for n=1..16
3
2
1
0
-1
-2
-3
0.5
1
1.5
2
2.5
3


lines of constant t: t=-2+n/4 for n=1..16
3
2
1
0
-1
-2
-3
0.5
1
1.5
2
2.5
3

2. A (test) particle with non-zero mass falls radially toward the horizon of a Schwarzschild black hole of mass m. The geodesic it follows has $A = 0.95$.

a. At what value of r is at rest? At that value of r, what is its physical distance from $r = 2m$?

b. How much proper time is required for it to fall from $r = 3m$ to $r = 2m$? How much coordinate time is required?

c. How much proper time is required for it to fall from the horizon at $r = 2m$ to the singularity at the center at $r = 0$? In order to compute this quantity show that a solution to the differential equation determining $r = r(\tau)$ may be given via the following equations:

$$r = \tfrac{1}{2}R(1 + \cos\eta) \ , \quad \tau = \tfrac{1}{2}R\sqrt{\frac{R}{2m}}(\eta + \sin\eta) \ ,$$

for some particular constant R. What is the meaning of R, and what is its value in terms of the "energy constant," A?

d. What is its ordinary 3-velocity as a function of its radial coordinate r?

e. As it passes the radius $r = 2.01m$, it emits a radio signal back toward observers who have safely decided to wait at the earth. What is the redshift of the signal when it is received at the earth? Do not forget to include the Doppler shift because of its earth-measured speed at the time of emission.

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The relevant geodesic equations are

$$\frac{d\phi}{d\tau} = 0 = \frac{d\theta}{d\tau} \ , \qquad \frac{dt}{d\tau} = \frac{A}{H} \ , \quad \left(\frac{dr}{d\tau}\right)^2 = A^2 - H = A^2 - 1 + \frac{2m}{r} \ .$$

c. The last equation, for $r = r(\tau)$ can be done, but it gives reasonably nasty-looking results; therefore, we now follow the suggestion in part (c) and show that the solution is best realized in terms of a parameter η:

$$\text{First set} \quad r = \tfrac{1}{2}R(1 + \cos\eta) \ , \quad \tau = \tfrac{1}{2}R\sqrt{\frac{R}{2m}}\,(\eta + \sin\eta) \ ,$$

$$\Longrightarrow \quad \frac{dr}{d\eta} = -\tfrac{1}{2}R\sin\eta \ , \quad \frac{d\tau}{d\eta} = \tfrac{1}{2}R\sqrt{\frac{R}{2m}}\,(1 + \cos\eta) = \sqrt{\frac{R}{2m}}\,r \ ;$$

$$\Longrightarrow \quad \frac{dr}{d\tau} = -m\sqrt{\frac{R}{2m}}\,\frac{\sin\eta}{r} \ .$$

Next we must re-write the $\sin\eta$ in terms of r:

$$\sin\eta = \sqrt{1 - \cos^2\eta} = \frac{2r}{R}\sqrt{\frac{R}{r} - 1} \ .$$

Inserting this back into the equation above we have

$$\frac{dr}{d\tau} = -\sqrt{\frac{2m}{r} - \frac{2m}{R}} = \pm\sqrt{A^2 - 1 + \frac{2m}{r}} \ .$$

a. We see that, yes, this does form a solution to the geodesic equations—at least for the infall question where $\frac{dr}{d\tau}$ is negative. [For an outgoing, radial geodesic we would either change $1 + \cos\eta$ to $1 - \cos\eta$, or arrange for η to be in the third and fourth quadrants.] We then immediately notice that R is the maximum value that r is allowed to have, or, differently phrased, it is the desired rest point for this motion, and that it is related to the energy as follows:

$$R = \frac{2m}{1 - A^2} \underset{A=0.95}{\longrightarrow} 20.513\, m \; .$$

To determine the proper distance between $r = 2m$ and R, we must integrate the spacelike separation:

$$L \equiv \int_{2m}^{R} \frac{dr}{\sqrt{1 - 2m/r}} = 2m \int_1^X dx \sqrt{\frac{x}{x-1}} \; ,$$

where $x \equiv r/2m$ and $X \equiv R/2m$. The best way to perform the integral is actually to use a (different) useful substitution:

$$x \equiv \cosh^2\sigma \implies \begin{cases} x - 1 = \sinh^2\sigma \; , \\ dx = 2\cosh\sigma\, \sinh\sigma\, d\sigma \; , \end{cases}$$

$$\implies \int_1^X dx \sqrt{\frac{x}{x-1}} = 2\int_0^S d\sigma\, \cosh^2\sigma = \{\sinh\sigma\, \cosh\sigma + \sigma\}\Big|_0^S = \sqrt{X(X-1)} + \cosh^{-1}\sqrt{X} \; ;$$

$$\implies L = \sqrt{R(R-2m)} + 2m\cosh^{-1}\sqrt{R/2m} \; ;$$

$$\implies L = R - m + 2m\log(2\sqrt{R/2m}) + O(m/R) \; ,$$

or, for $A = 0.95$, we have $L \underset{A=0.95}{\longrightarrow} 23.151\, m$.

We do notice that this is substantially larger than the quantity $R - 2m = 18.513\, m$, which we would have obtained were this all happening in flat spacetime.

b. To determine the proper time, we have already shown a simple form for the dependence of τ on η, and also $r = r(\eta)$. Since we want the proper time from $r = 3m$ to $r = 2m$, we first notice that, of course, $r = 2m$ is a rather special place. We define $\eta = \eta_{2m}$ as the value which it has when $r = 2m$. We can easily find a more-or-less explicit expression for this:

$$2m = r_{2m} = R\, \cos^2(\eta_{2m}/2) = \frac{2m}{1 - A^2} \cos^2(\eta_{2m}/2)$$

$$\implies \cos^2(\eta_{2m}/2) = 1 - A^2 \; , \quad \sin^2(\eta_{2m}/2) = A^2 \; , \quad \sin(\eta_{2m}) = 2A\sqrt{1 - A^2} \; ,$$

$$\implies \tau_{2m}/m = \left(\frac{R}{2m}\right)^{3/2} [\eta_{2m} + \sin(\eta_{2m})] = 2(1 - A^2)^{-3/2} \left[A\sqrt{1 - A^2} + \sin^{-1}(A)\right]$$

$$\underset{A=0.95}{\longrightarrow} 101.817 \; .$$

Now, we have to do roughly the same thing for $r = 3m$; however, since it is not so special we will simply find η_{3m} and insert it into the equation for τ, keeping track of the fact that

$R/2m = 10.2564$:

$$3m = r = R\cos^2(\eta_{3m}/2) \quad\Longrightarrow\quad \eta_{3m} = 2.357\ ,$$
$$\tau_{3m} = m\frac{R}{2m}^{3/2}[\eta_{3m} + \sin(\eta_{3m})] = 100.625m$$
$$\Longrightarrow \quad \Delta\tau = 1.192m\ .$$

The question concerning Δt is of course a trick question: We know that $t \to +\infty$ as $r \to 2m$; therefore, from any point on the trajectory

$$\Delta t = t_{2m} - t_{3m} = +\infty\ .$$

Nonetheless, while not required for the problem, I will put here an expression for the form of t as a function of either η, or r:

$$\frac{dt}{d\eta} = \frac{dt}{d\tau}\frac{d\tau}{d\eta} = \frac{A}{\sqrt{1-A^2}}\frac{r}{H(r)} = \frac{A}{\sqrt{1-A^2}}R\frac{\cos^4(\eta/2)}{A^2-\sin^2(\eta/2)}$$
$$= 2m\frac{A}{(1-A^2)^{3/2}}\left\{1 - A^2 + \cos^2(\eta/2) + \frac{(1-A^2)^2}{A^2-\sin^2(\eta/2)}\right\}\ .$$

This can be integrated, normalized so that $t(0) = 0$, i.e., the clock is set to zero at the rest point of the infall, to give

$$\frac{t}{2m} = \frac{t(\eta)}{2m} = \sqrt{\frac{R}{2m}-1}\left[\eta + \left(\frac{R}{2m}\right)\frac{\eta+\sin\eta}{2}\right] + 2\tanh^{-1}\left[\frac{\tan(\eta/2)}{\sqrt{\frac{R}{2m}-1}}\right]\ .$$

If we recall, from calculations above, that

$$\tan(\eta_{2m}/2) = \frac{\sin(\eta_{2m}/2)}{\cos(\eta_{2m}/2)} = \frac{A}{\sqrt{1-A^2}} = \sqrt{\frac{R}{2m}-1}\ ,$$

then we see that it is this last term, the inverse hyperbolic tangent that arranges for t to become infinite because $\tanh^{-1}(1) = +\infty$.

d. The 3-velocity is of course radial, and inward, and with magnitude given by

$$|v_r| = \frac{|u^{\hat{r}}|}{u^{\hat{t}}} = \frac{\frac{1}{\sqrt{H}}\left|\frac{dr}{d\tau}\right|}{\sqrt{H}\frac{dt}{d\tau}} = \frac{\sqrt{A^2-H}}{A} = \frac{1}{A}\sqrt{\frac{2m}{r}-\frac{2m}{R}}\ .$$

It is clear that it begins at rest, when $r = R$, and increases as r decreases. As $r \to 2m$, it approaches 1. This can perhaps be seen most easily by recalling that $H(r = 2m) = 0$. Of course the observer using these coordinates never sees this person actually arrive at $r = 2m$—until $t = +\infty$—so that its speed only gets closer and closer to 1, without ever actually reaching that value.

e. In particular, at the currently interesting value of $r = 2.01m$, we have $v = 0.99724$, while the value of $H(r = 2.01m) = 0.004975$.

As noted in the problem, the Doppler shift of light emitted here, and sent back to infinity, i.e., to the Earth, is due to two distinct causes: the rising up out of the gravitational potential well, and the fact that the source is moving. To obtain a complete formula for this shift—not actually necessary to complete this problem correctly, since I suppose it is alright if you just copy it from somewhere—let's first concern ourselves with the effect caused by the motion of the source. We can determine that by just using a Lorentz boost from the rest frame of the falling particle back to the rest frame of our observer, at infinity and therefore presumably also at rest, acting on the energy of the light-ray, remembering that the magnitude of its momentum is the same as its energy:

$$E = \gamma(E' - v\,E') = E'\sqrt{\frac{1-v}{1+v}}\ ,$$

where the v in question is just the one already calculated above, and is positive, since the signs have already been chosen to take account of the fact that the source is infalling.

However, let us recall that there is no effect of the gravitational field—or of location on the (curved) manifold—in this expression, so it is best to interpret it as a relationship between energies measured at the same place, but by two observers in relative motion at that place. Therefore, I will treat the quantity E above as "the energy at infinity," or measured at infinity. Then, next we have to think about the fact that the energy depends on the location of the person measuring it—in a gravitational field. The simplest way to think about that is to say that for an observer with 4-velocity $\widetilde{u}$ and a particle with 4-velocity $\widetilde{w}$, that observer measures the energy per unit mass as the negative of the scalar product of the two 4-vectors, or, in the case of a light ray, simply its energy, when the affine parameter along the ray is properly normalized:

$$E = -\widetilde{u}\cdot\widetilde{w} = -\widetilde{e}_{\hat{t}}\cdot\widetilde{w} = w^{\hat{t}} = A/\sqrt{H}\ ,$$

where we have noted that the observer's measurement of his own 4-velocity is $\widetilde{u} = \widetilde{e}_{\hat{t}}$, and in the last equality we have used our conserved quantity, A, that comes from the existence of the Killing vector ∂_t. The physical statement as to the nature of A is clear from this statement; i.e., A is the energy when measured at infinity (where $H = 1$). Therefore it is this quantity A that should be inserted for the symbol E above, in the Lorentz transformation relating things measured in the same (nearby) region but in different relative states of motion. We may therefore now insert all of this into our equation and create a final (desired) result:

$$\sqrt{H(r_{\rm recv})}\;E_{\rm recv} = A = \sqrt{\frac{1-v}{1+v}}\;A = \sqrt{\frac{1-v}{1+v}}\;\sqrt{H(r_{\rm emit})}\;E_{\rm emit,moving}$$
$$\Longrightarrow \qquad \omega_{\rm recv} = \sqrt{\left(\frac{1-v}{1+v}\right)\frac{H(r_{\rm emit})}{H(r_{\rm recv})}}\;\omega_{\rm emit}\ .$$

where in the second line we have noted that the energy of a photon is just proportional to its frequency, ω. We then simply insert the values of H at the two points, and the relative velocity of the source and receiver, $v = |v_r|$ calculated above, and we have the ratio of the two frequencies:

$$\frac{\omega_{\rm recv}}{\omega_{\rm emit}} = 0.00262 \quad \Longrightarrow \quad Z \equiv \frac{\lambda_{\rm recv} - \lambda_{\rm emit}}{\lambda_{\rm emit}} = \frac{\lambda_{\rm recv}}{\lambda_{\rm emit}} - 1 = \frac{\omega_{\rm emit}}{\omega_{\rm recv}} - 1 = 380.4 \ ,$$

where we have used the standard definition for the Doppler shift—a shift in wavelengths—and denoted it by Z.

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3. The perihelion distance of Mercury may be taken as 46.00 million kilometers, while the aphelion distance is 69.82 million kilometers. Please use this data to determine accurate values for the integration constants, A, and B [or $b = (B/m)^2$]. Then perform a numerical integration of the equation of motion to obtain an accurate value for the perihelion shift, using the values of $H(r)$ and $J(r)$ that are given to us by Einstein's equations. You will need the mass of the sun in kilometers, which you should look up in one of your texts, or calculate yourself.

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Given the values above for the perihelion and aphelion distances for Mercury, and the fact that the mass of our Sun is $m = 1.477km$, we set $(dr/d\tau)^2 = 0$ and insert these two turn-around points into the equation, and solve for $A = 0.9999999872$ and $B = 9050.7km$. [I note that more exact values than these were necessary, in the resulting calculations, in order to get consistent results; one can see the complete values used in the Maple worksheet.] We then insert the two differential equations for $d\varphi/d\tau$ and $\frac{d^2r}{d\tau^2}$ into Maple and ask its differential-equation solver to study the problem. The "ini" statement sets the initial conditions for the differential equation so that $r = 46 \times 10^6$ km, the perihelion point, so that the initial rate of change, $dr/d\tau$ should be zero, and has normalized φ so that it begins there. I have also not put in quite the most exact values used for the numbers below, to save typing space; again see the worksheet:

> $Leq, r2eq;$

$$\frac{d\varphi}{d\tau} = \frac{9050.703152}{r(\tau)^2}\ ,\ \frac{d^2 r(\tau)}{d\tau^2} - \frac{0.8191522755\times 10^8}{r(\tau)^3}\left(1-\frac{2.954}{r(\tau)}\right) + 1.477\frac{0.9999999744-\left(\frac{dr(\tau)}{d\tau}\right)^2}{r(tau)^2\left(1-\frac{2.954}{r(\tau)}\right)}\ ,$$

> $iniMerc := r(0) = 46000000, D(r)(0) = 0, phi(0) = 0:$

> $EqMerc := dsolve(Leq, r2eq, iniMerc, r(tau), phi(tau), numeric, output = listprocedure);$

$$EqMerc := [\tau = (proc(\tau)...endproc), \varphi(\tau) = (proc(\tau)...endproc), r(\tau) = (proc(\tau)...endproc), dr(\tau)/d\tau = (proc(\tau)...endproc)]$$

We can now use this to create a plot of the orbit:

> polarplot([rhs(EqMerc(τ)[3]),rhs(EqMerc(τ)[2]),$\tau = 0..10^{(12)}$], scaling=constrained);

Please see the Maple worksheet included on the website, both in html form and in worksheet form, to see more details. The results that it gives me, having found it necessary to invoke some sort of "double precision" in the calculations—more precisely, insist that all calculations were done to a precision of 25 significant figures— are the following for one orbit:

$$\Delta\tau = 2.2783455\times 10^{12}\ \text{km}\ ,\quad \Delta\varphi = 6.28317989\ \text{radians}$$
$$\Longrightarrow\quad \Delta\varphi - 2\pi = -1.1173\ \text{sec of arc per orbit.}$$

Schutz claims that the value ought to be approximately one-tenth of this amount. I agree with this because the annual orbit of Mercury requires approximately 0.24 Earth years, so that one Earth century would multiply this number by about 400, and it ought, then, to be approximately 43 seconds of arc. Well, if you do this well in your calculation, that should be just fine!

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4. Consider the orbit of a (massive) test particle in the Schwarzschild metric with a value of $b \equiv (B/m)^2 = 14$, and the following two separate cases: Please show that if $A = 0.96$ then the orbit is an elliptic one, while in the case $A = 0.97$ the orbit does not have a perihelion but, rather, eventually falls below the horizon and into the singularity.

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This simply asks us to study the behavior of particles moving in the effective potential for this problem, inserting the given values of $b = (B/m)^2 = 14$ and the two values of A, where the oscillation problem is determined by

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V(r) = \tfrac{1}{2}(A^2-1)\ , \quad V(r) = -\frac{m}{r} + \frac{1}{2}\frac{B^2}{r^2} - \frac{mB^2}{r^3} = -\frac{1}{x} + \frac{b}{2x^2} - \frac{b}{x^3}\ , \quad x \equiv \frac{r}{m}\ ,$$

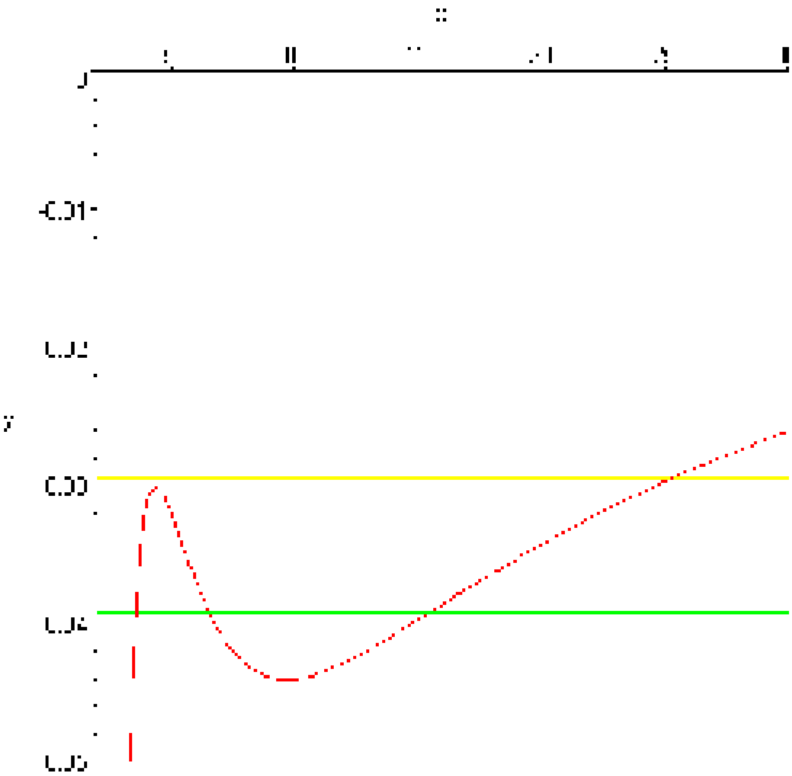
$$\underset{b=14}{\longrightarrow} \quad V(r) = -\frac{1}{x} + \frac{7}{x^2} - \frac{14}{x^3}\ , \quad \text{and} \quad \tfrac{1}{2}(A^2-1) = \begin{cases} -0.0440\ ,\ A = 0.096\ , \\ -0.0296\ ,\ A = 0.097\ . \end{cases}$$

In the graph of the effective potential, shown below, we see that it is an inverse cubic that asymptotes out to 0, as $x \to +\infty$ from below, stays always negative, but has a local minimum value of -0.04404 at $x = 9.645$ and then a local maximum of -0.03004 at $x = 4.354$, and then turns down and goes off to $-\infty$ as $x \to 0$.

One can see that the two values of A that are given generate values for $(A^2-1)/2$ that are below—plotted as the lower horizontal line, in green—and above—plotted as the upper horizontal line, in yellow—respectively, the height of that local maximum, as are needed for the required results.

i.) For $A = 0.96$, we have intersections of $(A^2-1)/2$ with the effective potential at two points to the right of the maximum, for a perihelion of $r = 6.1515m$ and an aphelion of $r = 15.4463m$.

ii.) For $A = 0.97$, we have only the one intersection, at $r = 25.1807m$. This means that a test particle put there, at rest, so that this is the correct value of A would be drawn in by the gravitational attraction, speeding up, inspiralling around all the while, but never turning back.



5. Please begin with the usual non-holonomic basis set for a metric which is spherically symmetric and static, but this time let us not assume the vacuum solution, but instead let us suppose that there is an electric field in the region exterior to the source, caused by an electric charge, q, at the location of that source. The solution to Einstein's equations for the matter tensor corresponding to such an electric field is given by the following metric:

$$\mathbf{g} = J\,dr^2 + r^2\,d\Omega^2 - H\,dt^2\ , \qquad J^{-1} = H = 1 - \frac{2m}{r} + \frac{q^2}{r^2}\ .$$

a. Please choose the standard orthonormal basis for such metrics, and then use general form of the connections and curvature, and Einstein tensor, given in the handout on such metrics to write out the independent components of the Einstein tensor, at least, explicitly.

b. Now we want to know what is the electric field caused by the central charge, q. To do this, it seems reasonable that, in an orthonormal basis, the field of a central charge would be of the form $\underset{\sim}{F} = E(r)\,\underset{\sim}{\omega}^{\hat{r}} \wedge \underset{\sim}{\omega}^{\hat{t}}$. Therefore assume that this is the case, and solve Maxwell's equations, on this manifold, to determine the explicit form for $E(r)$, in terms, presumably, of r and q. Do make sure that the so-constructed $E(r)$ satisfies ALL of Maxwell's equations, where I mean those equations to say that $d\underset{\sim}{F} = 0 = d * \underset{\sim}{F}$. Lastly, what is the value of the 1-form $\underset{\sim}{A}$ that is the 4-vector potential for this field?

. . . .

We begin with the handout on Spherically Symmetric, Static Spacetimes, with the given metric and the standard tetrad.

a. We simply copy the connections, curvatures, etc., from that handout, where they are given in terms of J and H, and then insert these given values. In particular, we need the 4 independent curvatures, where the statement that $JH = 1$ tells us that $J'/J + H'/H = 0$:

$$-\mathbf{A} = +\mathbf{C} = +\frac{H'}{2r} = \frac{m}{r^3} - \frac{q^2}{r^4}\ , \quad \mathbf{B} = \frac{1-H}{r^2} = 2\frac{m}{r^3} - \frac{q^2}{r^4}\ , \quad \mathbf{D} = \tfrac{1}{2}H'' = -2\frac{m}{r^3} + 3\frac{q^2}{r^4}\ .$$

We see that all of these quantities, for this problem, are expressible in terms of constant multiples of just 2 things. This will simplify the calculation of the Einstein tensor. The handout tells us the form of the Einstein tensor in terms of those 4 curvature quantities; we need now only insert their values from above, in terms of m/r^3 and q^2/r^4. We might perhaps note that we anticipate that all the m/r^3 terms will cancel, since we know the Einstein tensor vanishes for the Schwarzschild metric, which is determined only by m:

$$\begin{aligned}
\mathcal{G}_{\hat{r}\hat{r}} &= 2\mathbf{C} - \mathbf{B} = 2\frac{m}{r^3} - 2\frac{q^2}{r^4} - 2\frac{m}{r^3} + \frac{q^2}{r^4} = -\frac{q^2}{r^4}\ ,\\
\mathcal{G}_{\hat{\theta}\hat{\theta}} = \mathcal{G}_{\hat{\varphi}\hat{\varphi}} &= \mathbf{D} + \mathbf{C} - \mathbf{A} = 2\frac{m}{r^3} - 2\frac{q^2}{r^4} - 2\frac{m}{r^3} + 3\frac{q^2}{r^4} = +\frac{q^2}{r^4}\ ,\\
\mathcal{G}_{\hat{t}\hat{t}} &= 2\mathbf{A} + \mathbf{B} = -2\frac{m}{r^3} + 2\frac{q^2}{r^4} + 2\frac{m}{r^3} - \frac{q^2}{r^4} = +\frac{q^2}{r^4}\ ,
\end{aligned}$$

We see that in fact this Einstein tensor is very simple, vanishes when $q = 0$, and therefore it is reasonable that q can be interpreted as being a cause for a non-zero energy-momentum tensor, presumably/hopefully caused by an electric field, as desired.

b. As suggested we now take the Faraday, the electromagnetic field tensor, in the form

$$\underset{\approx}{F} \equiv E(r)\, \underset{\approx}{\omega}^{\hat r} \wedge \underset{\approx}{\omega}^{\hat t} \equiv \tfrac{1}{2} F_{\mu\nu}\, \underset{\approx}{\omega}^{\mu} \wedge \underset{\approx}{\omega}^{\nu} \implies \quad F_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & E \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -E & 0 & 0 & 0 \end{pmatrix} .$$

We want to determine the so-far unknown function E so that this Faraday satisfies Maxwell's equations. On the other hand, it was already reasonable to suppose that it depends only on r, since we want it to describe the electric field of a charged, central body. We first determine the dual 2-form,

$$*\underset{\approx}{F} = E * \{\underset{\approx}{\omega}^{\hat r} \wedge \underset{\approx}{\omega}^{\hat t}\} = iE\, \underset{\approx}{\omega}^{\hat \theta} \wedge \underset{\approx}{\omega}^{\hat \varphi} .$$

Then Maxwell's equations simply require that the exterior derivative of each of these two tensors should vanish. Although we have explicit connections for this manifold, it is actually more straightforward to calculate directly the exterior derivatives of the 2-forms we have:

$$d\underset{\approx}{\omega}^{\hat r} = d\{\sqrt{J}\, dr\} = 0 , \quad d\underset{\approx}{\omega}^{\hat t} = d\{\sqrt{H}\, dt\} = \frac{H'}{2\sqrt{H}}\, dr \wedge dt = \frac{H'}{2\sqrt{H}}\, \underset{\approx}{\omega}^{\hat r} \wedge \underset{\approx}{\omega}^{\hat t} ,$$

$$\implies \quad d\{\underset{\approx}{\omega}^{\hat r} \wedge \underset{\approx}{\omega}^{\hat t}\} = 0 ,$$

$$d\underset{\approx}{\omega}^{\hat \theta} = d\{r\, d\theta\} = dr \wedge d\theta = \frac{1}{r\sqrt{J}} \underset{\approx}{\omega}^{\hat r} \wedge \underset{\approx}{\omega}^{\hat \theta} ,$$

$$\underset{\approx}{\omega}^{\hat \theta} \wedge d\underset{\approx}{\omega}^{\hat \varphi} = \underset{\approx}{\omega}^{\hat \theta} \wedge d\{r \sin\theta\, d\varphi\} = \frac{1}{r\sqrt{J}}\, \underset{\approx}{\omega}^{\hat \theta} \wedge \underset{\approx}{\omega}^{\hat r} \wedge \underset{\approx}{\omega}^{\hat \varphi} ,$$

$$\implies \quad d\{\underset{\approx}{\omega}^{\hat \theta} \wedge \underset{\approx}{\omega}^{\hat \varphi}\} = \frac{2}{r\sqrt{J}}\, \underset{\approx}{\omega}^{\hat r} \wedge \underset{\approx}{\omega}^{\hat \theta} \wedge \underset{\approx}{\omega}^{\hat \varphi} .$$

From this information we quickly find the desired equations:

$$\begin{aligned} d\underset{\approx}{F} &= d\{E\, \underset{\approx}{\omega}^{\hat r} \wedge \underset{\approx}{\omega}^{\hat t}\} = dE \wedge \underset{\approx}{\omega}^{\hat r} \wedge \underset{\approx}{\omega}^{\hat t} = 0 , \\ d * \underset{\approx}{F} &= d\{E\, \underset{\approx}{\omega}^{\hat \theta} \wedge \underset{\approx}{\omega}^{\hat \varphi}\} = dE \wedge \underset{\approx}{\omega}^{\hat \theta} \wedge \underset{\approx}{\omega}^{\hat \varphi} + E\, \frac{2}{r\sqrt{J}}\, \underset{\approx}{\omega}^{\hat r} \wedge \underset{\approx}{\omega}^{\hat \theta} \wedge \underset{\approx}{\omega}^{\hat \varphi} \\ &= \frac{1}{\sqrt{J}} \left\{ \frac{dE}{dr} + 2\frac{E}{r} \right\} \underset{\approx}{\omega}^{\hat r} \wedge \underset{\approx}{\omega}^{\hat \theta} \wedge \underset{\approx}{\omega}^{\hat \varphi} . \end{aligned}$$

In order for this last 3-form to vanish, we have a simple differential equation which must be satisfied by $E(r)$, i.e.,

$$0 = \frac{dE}{dr} + 2\frac{E}{r} \implies \quad E = K/r^2 ,$$

where K is some constant of integration. [This is hardly unexpected!] On the other hand, it is worth noting that this means that

$$\widetilde{E} = F^{\hat{t}\hat{r}}\widetilde{e}_{\hat{r}} = \frac{K}{r^2}\widetilde{e}_{\hat{r}} = \frac{K}{r^2}\sqrt{1-2m/r+q^2/r^2}\,\partial_r \ .$$

As well, at this point we go ahead and work out the associated vector potential, $\underset{\sim}{A}$, which is of course the "potential" for $\underset{\sim}{F}$, i.e., $\underset{\sim}{F} = d\,\underset{\sim}{A}$:

$$\underset{\sim}{F} = E\,\underset{\sim}{\omega}^{\hat{r}} \wedge \underset{\sim}{\omega}^{\hat{t}} = \frac{K}{r^2}\,dr \wedge dt = d\{-(K/r)\,dt\} \implies \underset{\sim}{A} = -\frac{K}{r}\,dt = -\frac{K}{r\sqrt{H}}\,\underset{\sim}{\omega}^{\hat{t}} \ .$$

From this we see that one of them, i.e., $\widetilde{E}$ and $\underset{\sim}{A}$, looks "simple" in the coordinate basis and the other in the non-holonomic basis!

To determine the value of this constant, we insist, now, that Einstein's equations are also satisfied. To do this we need the energy-momentum tensor for this Faraday tensor. We recall that

$$T_{\mu\nu} \equiv \frac{1}{4\pi}\left\{F_{\mu\alpha}F^{\alpha}{}_{\nu} + \tfrac{1}{4}\eta_{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right\} \ .$$

A very little bit of calculation, for such a simple Faraday tensor, gives us the desired values:

$$T_{\mu\nu} = \frac{1}{8\pi}\begin{pmatrix} -E^2 & 0 & 0 & 0 \\ 0 & E^2 & 0 & 0 \\ 0 & 0 & E^2 & 0 \\ 0 & 0 & 0 & E^2 \end{pmatrix} ,$$

where we notice, as it should be, that it is traceless. We now compare this to the form of $\mathcal{G}_{\mu\nu}$, and see that, indeed, it does have the same form. All that is needed is that we identify q^2/r^4 with $8\pi(1/8\pi)E^2 = (K^2/r^4)$;i.e., we must have $K = q$, the charge on the central body.
