

Physics 570

Homework #8

Due Thursday, 29 March, 2007

1. An observer is moving on a circular orbit around the central mass in the Schwarzschild manifold; i.e., his location is on a geodesic with a constant value of r and $\theta = \pi/2$.
 - a. First write out the entire set of geodesic equations, but surely already including the constants of the motion, A and B . Since r is a constant, we must have $du^{\hat{r}}/d\tau = 0$. Therefore use that equation and the one that says that the geodesic has $(\tilde{u})^2 = -1$ to determine the values of A and B in terms of the constant radius r .
 - b. Now determine the locally-measured 3-velocity which this observer has, using the usual Schwarzschild coordinates, and then the Lorentz transformation that would go between the stationary, local observer's frame and that of the observer on the circular orbit. As well, describe a locally co-moving basis for our circularly-moving observer's tangent space, i.e., a frame so that she may make proper measurements of vectorial quantities. Denote her basis for tangent vectors by $\{\tilde{f}_\alpha\}_1^4$, require as usual that they constitute an orthonormal frame and that $\tilde{f}_4 = \tilde{u}$, her 4-velocity. Demonstrate some reasonable, simple choice for her basis in terms of the original, orthonormal Schwarzschild frame vectors.
2. The constant-time, equatorial plane of the Schwarzschild metric is a 2-dimensional, curved surface. Please determine an embedding of it in flat, 3-dimensional space. That is, take the metric in the 3-dimensional space as $dz^2 + dr^2 + r^2 d\varphi^2$ and determine $z = z(r, \varphi)$ so that the so-determined surface in that 3-space has the same metric as does that plane in the Schwarzschild manifold.
3. Let's begin preparations for considerations of rotating, compact sources of gravitational potential. A proper study of them will require that we utilize oblate spheroidal coordinates, since this sort of oblateness is what rotation causes for an originally spherically-symmetric fluid body. Of course that also means we have to study rotation a bit.
 - a. Therefore, we will first look at oblate spheroidal coordinates in flat space. We begin with the usual form of the Minkowski-space metric in cylindrical coordinates,

$$\mathbf{g} = d\rho^2 + \rho^2 d\varphi^2 + dz^2 - dt^2 ,$$

and transform to new (oblate, spheroidal) coordinates $\{\sigma, \theta, \phi\}$ via

$$\rho = \sqrt{\sigma^2 + a^2} \sin \theta , \quad z \equiv \sigma \cos \theta , \quad \phi = \varphi ,$$

where $a > 0$ is a positive constant. In these new coordinates, please show that the metric has the form

$$\mathbf{g} = (\sigma^2 + a^2 \cos^2 \theta) \left\{ \frac{d\sigma^2}{\sigma^2 + a^2} + d\theta^2 \right\} + (\sigma^2 + a^2) \sin^2 \theta d\phi^2 - dt^2 .$$

Then choose an orthonormal basis for this metric, in these coordinates. Although I am sure you could indeed calculate the connection 1-forms and the curvature 2-forms for this metric yourself, INSTEAD, please use the software package GRTensorII to determine the connection 1-forms in these coordinates, and also the curvature 2-forms. [These last quantities should of course all be zero!] This will be a very good time to begin practicing using this software, or other software for the same purpose, if you desire. This is because the upcoming discussions of the Kerr metric (rotating star) will probably be beyond your hand-written ability, or desire, to calculate the connections, so we need to begin practicing with something simpler. [Please append to your homework a printout of your session with GRTensorII, where this is done, or the other program if you choose to do it that way.]

- b. Give a description, in cylindrical coordinates, of the surfaces of constant σ , including some representative graphs. IF you don't want to make 3-dimensional graphs, because of the axial symmetry of cylindrical coordinates, it is sufficient to draw them in the plane which constitutes a particular value of ϕ and also that value plus π , so that one has a particular plane section. On such a section, the surface in question is reduced to simply a curve; a useful parameter along that curve is the other variable, θ .
 - c. Likewise, now look at points of constant θ , taking σ as the parameter along the planar sections of that surface, to give nice descriptions of what they "look like." Compare these sections to the similar sections one would have if we were dealing with ordinary spherical coordinates, instead of these "squashed" ones.
4. Let us re-consider the spherically-symmetric, static metric one last time, but consider different input to Einstein's equations, involving no matter, i.e., vacuum, but there is a non-zero cosmological constant, Λ . Using the values for the Einstein tensor, in terms of $J(r)$ and $H(r)$, re-solve the Einstein equations when $\Lambda \neq 0$, but we still maintain the desire that there is a central mass, of constant value M . Are there still horizons in this new manifold, in these coordinates? If so, where are they?