

Physics 570

Homework #9

Due Thursday, 5 April, 2007

Solutions

1. First show that both of the mappings

$$\vec{\mathcal{J}} \implies -i\frac{1}{2}\vec{\sigma}, \quad \vec{\mathcal{K}} \implies \pm\frac{1}{2}\vec{\sigma},$$

constitute representations of the generators of the Lorentz group, where the elements of $\vec{\sigma}$ are the usual σ_x , σ_y , and σ_z . Recall that the commutation relations of those six generators are the following

$$[\mathcal{J}_i, \mathcal{J}_j] = \sum_{k=1}^3 \epsilon_{ijk} \mathcal{J}_k, \quad [\mathcal{J}_i, \mathcal{K}_j] = \sum_{k=1}^3 \epsilon_{ijk} \mathcal{K}_k, \quad [\mathcal{K}_i, \mathcal{K}_j] = -\sum_{k=1}^3 \epsilon_{ijk} \mathcal{J}_k.$$

As well recall that a representation is a mapping from the generators to a new set of algebraic objects that preserves the commutation relations.

The representation where the components of \mathcal{K} have a plus sign is called $D(0, \frac{1}{2})$, while the one with the minus sign is called $D(\frac{1}{2}, 0)$. These are the two, distinct, irreducible representations of the Lorentz group by 2×2 matrices. Using these matrices, then construct the generators of the representation $D(0, \frac{1}{2}) \otimes D(\frac{1}{2}, 0)$, and show that there is a (complex-valued) change of basis in the 4-dimensional vector space underlying these 4-dimensional matrices such that these new generators are indeed the ones that we first derived in the ordinary Minkowski spacetime.

Generic group elements, sufficiently near the identity, are often said to be generated by some scalar parameter multiplied by an element of the Lie algebra associated with that group: Let J be an element of the Lie algebra which generates a one-parameter group of elements $g(\lambda)$, a subgroup of the complete group, as shown below. In addition let $D^{(1)}$ and $D^{(2)}$ be representations of the algebra and its group; then we may make the following statements about these representations and their direct products:

$$g = e^{\lambda J} \rightarrow D^{(n)}(g) = e^{\lambda D^{(n)}(J)} \implies D^{(1)}(g) \otimes D^{(2)}(g) = e^{\lambda [D^{(1)}(J) \otimes I_2 + I_1 \otimes D^{(2)}(J)]},$$

where I_n is the identity matrix on the representation (vector) space on which $D^{(n)}(g)$ acts, and, in all of the above, n takes on either of the values 1 or 2.

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With the information given in the problem, for the representations $D(0, \frac{1}{2})$ and $D(\frac{1}{2}, 0)$, and the statement that $D(\frac{1}{2}, \frac{1}{2}) = D(0, \frac{1}{2}) \otimes D(\frac{1}{2}, 0)$, it is straightforward to write down these (new)

representations of the generators for the Lorentz group:

$$\vec{\mathcal{J}} \implies -\frac{i}{2}\{\vec{\sigma} \otimes I_2 + I_2 \otimes \vec{\sigma}\}, \quad \vec{\mathcal{K}} \implies \frac{1}{2}\{\vec{\sigma} \otimes I_2 - I_2 \otimes \vec{\sigma}\}$$

$$\begin{aligned} \mathcal{J}^x &\implies -\frac{i}{2} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \quad \mathcal{J}^y \implies +\frac{1}{2} \begin{pmatrix} 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \quad \mathcal{J}^z \implies i \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\ \mathcal{K}^x &\implies \frac{1}{2} \begin{pmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{pmatrix}, \quad \mathcal{K}^y \implies +\frac{i}{2} \begin{pmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{pmatrix}, \quad \mathcal{K}^z \implies \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

It is perhaps relevant to note that you need to have been told the convention for taking the direct product of two matrices—which is indeed in one of the early handouts. Nonetheless, I remind you of it here: it is the analogue of scalar multiplication (on the left), shown on the first line and the direct product on the second:

$$\begin{aligned} \alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix}, \\ \begin{pmatrix} r & s \\ t & u \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} ra & sa & rb & sb \\ ta & ua & tb & ub \\ rc & sc & rd & sd \\ tc & uc & td & ud \end{pmatrix}. \end{aligned}$$

Unfortunately these 4×4 matrices for \mathcal{J} and \mathcal{K} do not look like the ones that we obtained in class earlier this semester, which were that

$$\begin{aligned} \mathcal{J}^x &\implies \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{J}^y \implies \begin{pmatrix} 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{J}^z \implies \begin{pmatrix} 0 & -1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ \mathcal{K}^x &\implies \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{K}^y \implies \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \mathcal{K}^z \implies \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \end{aligned}$$

where it is to be pointed out that these are the matrices with one contravariant index and one covariant index, as is appropriate for matrices representing transformations of vectors into (other) vectors. [NOTE to the grader: there are also conventions where the \mathcal{J}^i matrices above are the negative of those written; this is of course also fine if the student continues to the end with them.]

As usual, the problem is that the two sets of matrices are given with respect to a different set of basis vectors. We therefore must search for a matrix that describes a change of basis—call it W —such that for all of the matrices above we have the relationship

$$W M^{(\frac{1}{2}, \frac{1}{2})} W^{-1} = M^{(3+1)},$$

where by M we mean any of the 6 generators \mathcal{J}^i or \mathcal{K}^j and the notation $M^{(\frac{1}{2}, \frac{1}{2})}$ refers to those matrices as we calculated them via the direct products above, while the notation $M^{(3+1)}$ refers to the same matrices but with respect to the original $\{x, y, z, t\}$ -basis that we used when we were studying these as generators of the Lorentz group in ordinary Minkowski space in a Cartesian basis. As the matrix W must be the same for all of these 6 matrices, the calculation is moderately unique, and leads us to the following form for W , where its normalization is chosen so as to make both sets of basis vectors orthonormal:

$$W = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & -1 \\ i & 0 & 0 & i \\ 0 & -1 & -1 & 0 \\ 0 & +1 & -1 & 0 \end{pmatrix} .$$

This tells us, as expected, that the standard matrices for $D(1/2, 1/2)$ are simply given with respect to a different basis, built on $(x \pm iy)/\sqrt{2}$ and $(\pm z - t)/\sqrt{2}$, a choice of new basis which is not too surprising.

Although not required for a proper solution to the problem, I believe it might also be reasonable to give some greater detail concerning how one finds generators for direct product representations. We understand that the matrices representing the group elements of a product representation correspond to direct products of the individual representations of that group element. However, one must be somewhat more careful with respect to the generators. The following is the right approach for the generators, for any Lie group, G , with Lie algebra, \mathcal{G} . Let us suppose that we have been an element $g \in G$ such that there exists $Q \in \mathcal{G}$ such that $G = e^Q$. Further let us consider the situation when we have two distinct representations of G , which we label by $D_1(g)$ and $D_2(g)$. Therefore, we also can find, and label, the representations of the generator Q , simply by $Q^{(1)}$ and $Q^{(2)}$, so that

$$D_1(g) = e^{Q^{(1)}} , \quad D_2(g) = e^{Q^{(2)}} .$$

Then we find that the generators of the product representation are related to those of the individual representations in the following way:

$$\{D_1 \otimes D_2\}(g) \equiv [D_1(g)] \otimes [D_2(g)] \equiv e^{Q^{(1 \oplus 2)}} , \quad Q^{(1 \oplus 2)} \equiv Q^{(1)} \otimes I_2 + I_1 \otimes Q^{(2)} ,$$

where I_i indicates the matrix which is the identity matrix in representation D_i . To verify this we consider the following, setting, first $T \equiv Q^{(1)}$:

$$\begin{aligned} e^{T \otimes I_2} &= I_{1 \otimes 2} + T \otimes I_2 + \frac{1}{2!}(T \otimes I_2)(T \otimes I_2) + \frac{1}{3!}(T \otimes I_2)(T \otimes I_2)(T \otimes I_2) + \dots \\ &= \left\{ I_1 + T + \frac{1}{2!}T^2 + \frac{1}{3!}T^3 + \dots \right\} \otimes I_2 = D_1(g) \otimes I_2 . \end{aligned}$$

We have an entirely analogous construction for the exponential of $I_1 \otimes Q^{(2)}$, so that we may finally write

$$\begin{aligned} e^{Q^{(1)} \otimes I_2 + I_1 \otimes Q^{(2)}} &= e^{Q^{(1)} \otimes I_2} e^{I_1 \otimes Q^{(2)}} = \{D_1(g) \otimes I_2\} \{I_1 \otimes D_2(g)\} \\ &= D_1(g) \otimes D_2(g) \equiv \{D_1 \otimes D_2\}(g) , \end{aligned}$$

where we have used the “obvious” fact that $Q^{(1)} \otimes I_2$ commutes with $I_1 \otimes Q^{(2)}$.

2. A stationary observer uses rockets to maintain himself at fixed values of $r > 6m$, φ , and $\theta = \pi/2$, in the spacetime exterior to a rotating black hole, i.e., in the Kerr metric in the (usual) Boyer-Lindquist coordinates. He also arranges mirrors that cause a light ray he sends out to follow a circular trajectory of constant r , in the equatorial plane; this photon trajectory is of course not a geodesic. Calculate the time required for this photon to go all the way around and back to where our observer receives it, determining the two different values one gets for this answer depending on which direction the photon goes around the circular trajectory. A second observer rotates about the black hole at constant values of $r > 6m$ and $\theta = \pi/2$, but moving with angular velocity

$$d\varphi/dt = \omega = \frac{2ma}{r(r^2 + a^2) + 2ma^2} .$$

She performs the same experiment and measures that the two photons, traveling in opposite directions around the system of mirrors, take exactly the same time for transit. Please also demonstrate this with your equations.

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We have a light ray propagating around a trajectory that stays at a constant value of r and at the fixed value of $\theta = \pi/2$, i.e., around a circle in the equatorial plane. Therefore, using the coordinate basis, its trajectory must have the property that

$$0 = ds^2 = g_{\varphi\varphi} (d\varphi)^2 + 2g_{\varphi t} dt d\varphi + g_{tt} (dt)^2 = (dt)^2 \left\{ g_{\varphi\varphi} \left(\frac{d\varphi}{dt} \right)^2 + 2g_{\varphi t} \left(\frac{d\varphi}{dt} \right) + g_{tt} \right\}$$

$$\implies \frac{d\varphi}{dt} = \frac{-g_{\varphi t} \pm \sqrt{(g_{\varphi t})^2 - g_{\varphi\varphi}g_{tt}}}{g_{\varphi\varphi}} = +\omega \pm \sqrt{\omega^2 - g_{tt}/g_{\varphi\varphi}} \equiv \omega \pm X(r) ,$$

where I have used

$$\omega = -\frac{g_{\varphi t}}{g_{\varphi\varphi}} .$$

[I note that a student could just begin here, copying this equation from Schutz’s book, equations (11.71) and (11.81).]

While it is not quite essential to do so I note that for the Kerr metric we have the following equalities—in the equatorial plane:

$$\omega = -\frac{g_{\varphi t}}{g_{tt}} = \frac{2ma}{r(r^2 + a^2) + 2ma^2} , \quad -\frac{g_{tt}}{g_{\varphi\varphi}} = \frac{r - 2m}{r(r^2 + a^2) + 2ma^2} ,$$

$$X(r) = \sqrt{\omega^2 - g_{\varphi t}/g_{tt}} = \frac{r\sqrt{r^2 + a^2} - 2mr}{r(r^2 + a^2) + 2ma^2} ,$$

where the denominator in all cases is just A/r evaluated in the equatorial plane. We see that both of them depend only on r , and are both positive for all $r > 2m$, and of course are also constant everywhere on this circular orbit! However, since $-g_{tt}/g_{\varphi\varphi}$ is positive the square root, $X(r)$, is always greater than ω . This allows us to conclude that the solution for $d\varphi/dt$ with the plus sign is positive, and in fact greater than $2\omega > 0$, while the one with the minus sign is negative. Since $d\varphi/dt$ should be positive or negative, depending on which direction it is going around, it is clear that the one solution is for a ray going around in the “co-rotating direction,” and the other one for a ray going around in the opposite direction. The light-ray travel time for the two routes is then very straightforward, since this ratio is a constant. We simply have to integrate the equation for $d\varphi/dt$ around the loop:

i.) For travel around the loop in the positive sense, we have

$$\frac{d\varphi}{dt} = \omega + X \quad \implies \quad T \equiv \int dt = \int_0^{2\pi} \frac{d\varphi}{\omega + X} = \frac{2\pi}{\omega + X} .$$

ii.) For travel around the loop in the opposite sense, we have

$$\frac{d\varphi}{dt} = \omega - X \quad \implies \quad T \equiv \int dt = \int_0^{-2\pi} \frac{d\varphi}{\omega - X} = \frac{2\pi}{X - \omega} .$$

Comparing these two calculations, we see that they are indeed surely different, and we may calculate

$$\Delta T = 2\pi \left\{ \frac{1}{X - \omega} - \frac{1}{X + \omega} \right\} = \frac{4\pi\omega}{-\frac{g_{tt}}{g_{\varphi\varphi}}} = 4\pi \frac{2ma}{r - 2m} .$$

However, when we have the light travel time measured by our ZAMO observer, we recall that he is travelling around the circular orbit with angular velocity ω . Therefore, his (local) measurement of the angular velocity of the light ray is just

$$\left. \frac{d\varphi}{dt} \right|_{\text{measured by ZAMO}} = \left. \frac{d\varphi}{dt} \right|_{\text{measured by observ. at rest}} - \left. \frac{d\varphi}{dt} \right|_{\text{vel. of ZAMO}} = \omega \pm X(r) - \omega = \pm X(r) .$$

We see that the two light rays, as measured by the ZAMO, moving in opposite directions do indeed have different angular velocities, BUT that they are just the same in absolute value and differing in sign. Therefore, it is straightforward to see that the actual time travel around will be the same for the two as he measures them.

3. On the latest handout, concerning the Kerr metric, you are given the form of $\mathbf{g} = g_{\mu\nu} dx^\mu dx^\nu$, and an associated orthonormal tetrad basis for 1-forms. Use this information to determine the inverse metric and the associated (dual) orthonormal basis for tangent vectors.

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The handout on the Kerr metric allows us to already know what are the components, $g_{\mu\nu}$, in the coordinate basis. It also gives us an orthonormal basis for 1-forms and the associated orthonormal basis for tangent vectors. Using that basis for tangent vectors we may divine the components, $g^{\mu\nu}$ of the inverse metric:

$$\begin{aligned} ds^2 = \eta^{\alpha\beta} \tilde{e}_\alpha \otimes \tilde{e}_\beta &= \frac{1}{\Sigma} (\Delta \partial_r^2 + \partial_\theta^2) + \frac{\Sigma}{\Delta \sin^2 \theta} \partial_\varphi^2 - \frac{A}{\Sigma \Delta} (\partial_t + \omega \partial_\varphi)^2 \\ &= \frac{1}{\Sigma} (\Delta \partial_r^2 + \partial_\theta^2) - \frac{A}{\Sigma \Delta} \partial_t^2 - 2 \frac{\omega A}{\Sigma \Delta} \partial_t \partial_\varphi + \left(\frac{\Sigma}{A \sin^2 \theta} - \frac{A \omega^2}{\Sigma \Delta} \right) \partial_\varphi^2 \\ &= \frac{1}{\Sigma} (\Delta \partial_r^2 + \partial_\theta^2) + \frac{1}{\Delta \sin^2 \theta} \left(1 - \frac{2mr}{\Sigma} \right) \partial_\varphi^2 - \frac{2mar}{\Sigma \Delta} \partial_t \partial_\varphi - \frac{A}{\Sigma \Delta} \partial_t^2 \end{aligned}$$

If one had wanted to take a different route, by simply inverting the original matrix that contains the elements $g_{\mu\nu}$, then we first notice that the matrix has the form of the direct sum of a 2×2 diagonal matrix—for the dr and $d\theta$ terms—and a second 2×2 matrix—for the $d\varphi$ and dt terms, which we will call D . Therefore its inverse has again this same direct sum form. The inverse of the diagonal matrix is of course rather trivially determined; however, the inverse of the matrix D , being 2×2 , gives us the following:

$$\left. \begin{aligned} g^{tt} &= \frac{g_{\varphi\varphi}}{DD}, \\ g^{\varphi\varphi} &= \frac{g_{tt}}{DD}, \\ g^{\varphi t} &= -\frac{g_{\varphi t}}{DD}, \end{aligned} \right\}; \quad DD \equiv \text{determinant}(D) = \Delta \sin^2 \theta,$$

which is of course consistent with the calculation above by different means.

4. If a particle of charge e and mass μ is moving in the neighborhood of a non-zero electromagnetic field, described by the usual Faraday tensor, F , its trajectory in spacetime would no longer be a geodesic, but rather $x^\mu = x^\mu(\tau)$ such that

$$u^\mu = \frac{dx^\mu}{d\tau}, \quad u^\nu u^\mu{}_{;\nu} = \frac{e}{\mu} F^\mu{}_\nu u^\nu.$$

Assuming such a particle is moving in the Reissner-Nordström manifold, exterior to any horizons, show that the quantity

$$A \equiv H \frac{dt}{d\tau} + e \frac{q}{\mu r} = \left(1 - \frac{2m}{r} + \frac{q^2}{r^2} \right) \frac{dt}{d\tau} + e \frac{q}{\mu r}$$

is conserved, and that it is surely what one would want to call the energy for the particle.

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We first copy down the \hat{t} -component of the geodesic equations from the usual handout, using $1/J = H = 1 - 2m/r + (q/r)^2$ for the Reissner-Nordström metric, inserting the right-hand side instead of zero:

$$\frac{d}{d\tau} u^{\hat{t}} + \frac{dH/dr}{2\sqrt{H}} u^{\hat{r}} u^{\hat{t}} = \frac{e}{\mu} F^{\hat{t}}{}_{\nu} u^{\nu} = \frac{e}{\mu} \left(\frac{q}{r^2} \right) u^{\hat{r}} = \frac{eq}{\mu} \frac{1}{r^2} \frac{1}{\sqrt{H}} \frac{dr}{d\tau},$$

where we have used the fact determined in an earlier homework problem that for this metric

$$F^{\mu}{}_{\nu} \implies \begin{pmatrix} 0 & 0 & 0 & E \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ E & 0 & 0 & 0 \end{pmatrix}, \quad E = \frac{q}{r^2}.$$

We now follow the line of reasoning used already twice recently, in the Schwarzschild metric computations, that the left-hand side of this equation may be re-written in the form

$$\frac{d}{d\tau} u^{\hat{t}} + \frac{dH/dr}{2\sqrt{H}} u^{\hat{r}} u^{\hat{t}} = \frac{1}{\sqrt{H}} \frac{d}{d\tau} \sqrt{H} u^{\hat{t}}.$$

Inserting this back into the earlier formulation, and multiplying both sides by \sqrt{H} we obtain

$$\frac{d}{d\tau} \sqrt{H} u^{\hat{t}} = \frac{eq}{\mu} \frac{1}{r^2} \frac{dr}{d\tau} = -\frac{eq}{\mu} \frac{d(1/r)}{d\tau}.$$

Lastly we recall the definition of $u^{\hat{t}} = \sqrt{H} dt/d\tau$, which gives us

$$\begin{aligned} \frac{d}{d\tau} \left\{ H \frac{dt}{d\tau} + \frac{eq}{\mu r} \right\} &= 0, \\ \implies A = H \frac{dt}{d\tau} + \frac{eq}{\mu r} &= \left(1 - \frac{2m}{r} + \left(\frac{q}{r} \right)^2 \right) \frac{dt}{d\tau} + \frac{eq}{\mu r}, \end{aligned}$$

where A is a name for this value which is constant along the trajectory.

[I do apologize that the problem had a couple of typographical difficulties in the sense that the mass of the test particle was called m which was also the mass of the central gravitating object, and also that that particular m was left out of the last term in the form for the conserved quantity. In this solution I have inserted the mass of the test particle in the correct places, and denoted it by μ .]

5. A physical observer is on a circular orbit around the center on a Reissner-Nordström manifold, with charge q and mass m . What is the magnetic field that this observer measures?

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From a previous problem we know the procedure to determine the 4-velocity for a geodesic on a circular orbit in a spherically-symmetric system. We insist that $du^{\hat{r}}/d\tau$ should vanish, that $u^{\hat{\theta}}$ should vanish, and that $(\tilde{u})^2 = -1$. In this case we therefore acquire the following, using our knowledge concerning the constants due to the Killing vectors, $\partial/\partial\varphi$ and $\partial/\partial t$:

$$\begin{aligned}\tilde{u} &= u^{\hat{\varphi}} \tilde{e}_{\hat{\varphi}} + u^{\hat{t}} \tilde{e}_{\hat{t}} = \frac{B}{r} \tilde{e}_{\hat{\varphi}} + \frac{A}{\sqrt{H}} \tilde{e}_{\hat{t}}, \\ 0 &= \frac{d}{d\tau} u^{\hat{r}} = \frac{\sqrt{H}}{r} (u^{\hat{\varphi}})^2 - \frac{dH/dr}{2\sqrt{H}} (u^{\hat{t}})^2 \\ &\implies \left(\frac{u^{\hat{\varphi}}}{u^{\hat{t}}} \right)^2 = \frac{1}{2} r \frac{dH/dr}{H} \equiv W^2, \\ &\implies W^2 = \frac{mr - q^2}{r^2 - 2mr + q^2}.\end{aligned}$$

We may then insert this relation into the normalization condition, which gives us

$$+1 = (u^{\hat{t}})^2 - (u^{\hat{\varphi}})^2 = (u^{\hat{t}})^2 [1 - W^2] \implies \begin{cases} u^{\hat{t}} = \frac{1}{\sqrt{1 - W^2}} = \sqrt{\frac{r^2 - 2mr + q^2}{r^2 - 3mr + 2q^2}}, \\ u^{\hat{\varphi}} = \frac{W}{\sqrt{1 - W^2}} = \sqrt{\frac{mr - q^2}{r^2 - 3mr + 2q^2}}. \end{cases}$$

We can of course also think of $u^{\hat{t}}$ as the usual γ for the 4-velocity, while $u^{\hat{\varphi}} = v\gamma$, where this v is of course in the $\tilde{e}_{\hat{\varphi}}$ -direction.

We now want the Lorentz transformation that moves this 4-velocity to the one that the observer herself would measure, i.e., $u^{\hat{\varphi}} = 0$ and $u^{\hat{t}} = 1$. That transformation is surely

$$\Lambda^{\alpha}_{\mu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & u^{\hat{t}} & -u^{\hat{\varphi}} \\ 0 & 0 & -u^{\hat{\varphi}} & u^{\hat{t}} \end{pmatrix}$$

where the matrix labels are $\tilde{e}_{\hat{r}}$, $\tilde{e}_{\hat{\theta}}$, $\tilde{e}_{\hat{\varphi}}$, and $\tilde{e}_{\hat{t}}$, in that order on both rows and columns. We now recall that the Faraday tensor is given by

$$F^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & -E \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ +E & 0 & 0 & 0 \end{pmatrix} \quad \text{i.e.,} \quad \vec{E} = E \tilde{e}_{\hat{r}} = \frac{q}{r^2} \tilde{e}_{\hat{r}}.$$

Therefore the Faraday measured in the observer's frame is

$$\begin{aligned}
F'^{\alpha\beta} &= \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu F^{\mu\nu} = \{\Lambda F \Lambda^T\}^{\alpha\beta} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & u^{\hat{t}} & -u^{\hat{\phi}} \\ 0 & 0 & -u^{\hat{\phi}} & u^{\hat{t}} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & -E \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ +E & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & u^{\hat{t}} & -u^{\hat{\phi}} \\ 0 & 0 & -u^{\hat{\phi}} & u^{\hat{t}} \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & u^{\hat{\phi}} E & -u^{\hat{t}} E \\ 0 & 0 & 0 & 0 \\ -u^{\hat{\phi}} E & 0 & 0 & 0 \\ u^{\hat{t}} E & 0 & 0 & 0 \end{pmatrix}.
\end{aligned}$$

This tells us that in the observer's frame, she sees

$$\vec{E}' = \gamma E \tilde{e}_{\hat{r}} = \sqrt{\frac{r^2 - 2mr + q^2}{r^2 - 3mr + 2q^2}} \left(\frac{q}{r^2}\right) \tilde{e}_{\hat{r}}, \quad \vec{B}' = -v\gamma E \tilde{e}_{\hat{\theta}} = -\sqrt{\frac{mr - q^2}{r^2 - 3mr + 2q^2}} \left(\frac{q}{r^2}\right) \tilde{e}_{\hat{\theta}},$$

where we have used the fact that for $i, j = 1, 2, 3$, the F^{ij} component is B_k , i, j , and k , being in cyclic order. We also note that the direction $-\tilde{e}_{\hat{\theta}}$ is straight **upward** perpendicular to the equatorial plane in which her orbit lies. [Note to grader: in principle I only asked them to determine her measurement of the magnetic field.]
