

3. Hamilton-Jacobi Equation and Action-angle variables

Perhaps it is good to note that **for quite some time now we will only be considering systems with Hamiltonians that are independent of time!**

Remember that cyclic coordinates are those q^a which do not appear in the statement of the Hamiltonian. Since they are not there, the conjugate momenta are constant. Obviously a new set of coordinates, and new Hamiltonian, such that all of them were cyclic would be much simpler to work with:

all the momenta would be constant, and

therefore, the Hamiltonian would be constant, and

therefore, the time-derivatives of the coordinates would be constant, since they are just derivatives of the Hamiltonian with respect to the momenta, and therefore also constant, so that

the individual motions would each be linear in time.

If in fact the Hamiltonian is a constant, then, since it is never defined except to within a constant, we could always take that constant to be zero. This consideration leads us to the Hamilton-Jacobi (or HJ) equation, which we will acquire via some choice of canonical transformation. Choices of both Type 1 and Type 2 are common in the literature. I believe that there is some value in choosing the desired transformation to be generated by a function of Type 1. The rationale for this, as followed by José, is that we are going to use this as a way to find the canonical transformation that leads to action-angle variables, and that transformation is most plausible as one of Type 2. Choosing the first one to be of Type 1 makes it somewhat easier to keep the two of them separate, philosophically, since, mathematically, they are very closely intertwined.

Therefore we now look for **time-dependent** canonical transformations of Type 1 such that $K = 0$. Following long-standing tradition, instead of referring to the generator of this particular canonical transformation as $F^1(q, Q, t)$, we will instead label it by $S = S(q, Q, t)$. The transformation equations then have the form

$$0 = K = H[q, p(q, Q, t), t] + S_{,t}(q, Q, t) ; \quad p_a = S_{,q^a} , P_r = S_{,Q^r} . \quad (3.1)$$

As already noted, since the new Hamiltonian is zero, the equations of motion in the new coordinates simply tell us that both the $Q^r(q, p)$ and the $P_s(q, p)$ are constant. However, we of course have so far no idea of how to find such a transformation. Nonetheless, if we simply take the equation for K and insert into it the form for the original momenta, p_a , in terms of the yet-to-be-determined generating function, S , we acquire the (celebrated) Hamilton-Jacobi equation, where S is often referred to as *Hamilton's principal function*:

$$H(q^a, S_{,q^b}, t) + S_{,t} = 0 . \quad (3.2)$$

This gives us a **partial differential equation** for the unknown generating function S , which depends on the coordinates q^a and t . In principle S also depends on the new coordinates Q^r ; however, they do not appear explicitly in the pde. On the other hand, as we have just seen, were we to know what they were, they should all be constants; therefore, it is reasonable to suppose that they may be taken as the *constants of integration for this problem*. In principle a pde should have arbitrary functions in the so-called "general solution." On the other hand, we only need some n constants in the solution. In fact, pde's in p variables also have what are called *complete solutions*, which involve p independent constants. In our case, p is obviously equal to $n + 1$. On the other hand, since the

function S itself does not appear in the pde, it is obvious that we may always add some arbitrary constant to it, without changing any physics; therefore, we take that constant, say C , as one of our constants, and then (usually) throw it away, and suppose that the remaining ones determine the desired coordinates Q^r . It may well be that the simplest way to acquire them results in some other constants; nonetheless, we will acquire n such constants, and usually refer to them as the desired Q^r .

Still supposing that we have solved the HJ equation, so that we do indeed have such an S , we have argued that it contains the desired Q^r , which could in principle be resolved as functions of the q^a and p_b . What, however, about the new momenta? They also must be constant, although that they are is not immediately clear. We now show how and why the possession of a principal function implies this:

$$\frac{d}{dt}P_s = -\frac{d}{dt}\frac{\partial S}{\partial Q^s} = -\frac{\partial^2 S}{\partial Q^s \partial q^a}\dot{q}^a - \frac{\partial^2 S}{\partial Q^s \partial t}. \quad (3.3a)$$

However, Eq. (3.1) tells us that $\partial S/\partial t = -H$, so that the second term above is

$$-\frac{\partial^2 S}{\partial Q^s \partial t} = +\frac{\partial}{\partial Q^s}H\left[q, \frac{\partial S}{\partial q}(q, Q, t), t\right] = \frac{\partial H}{\partial p_b}\frac{\partial^2 S}{\partial Q^s \partial q^b}, \quad (3.3b)$$

where the partial derivative of H in the last term is with respect to p_b simply because that is the name of the second place in the arguments of H . We may now put these two terms together to find the desired quantity:

$$\frac{d}{dt}P_s = -\frac{\partial^2 S}{\partial Q^s \partial q^a}\left\{\dot{q}^a - \frac{\partial H}{\partial p_a}\right\}. \quad (3.3c)$$

Provided that the Hessian determinant of our principal function is not zero, so that the equation is really meaningful, this tells us that, yes, the P_s are constant if and only if the dq^a/dt half of Hamilton's equations are satisfied in the originally coordinates, as desired. Now we continue and consider the other half of Hamilton's equations, in the original sense, involving the original momenta which are to be defined in terms of derivatives of the generating function S as well:

$$\frac{d}{dt}p_a = \frac{d}{dt}\frac{\partial S}{\partial q^a} = \frac{\partial^2 S}{\partial q^a \partial q^b}\dot{q}^b + \frac{\partial^2 S}{\partial q^a \partial t} \quad (3.4a)$$

As before we may recover the original Hamiltonian in the second term:

$$\frac{\partial^2 S}{\partial q^a \partial t} = -\frac{\partial}{\partial q^a}H\left[q, \frac{\partial S}{\partial q}(q, Q, t), t\right] = -\frac{\partial H}{\partial q^a} - \frac{\partial H}{\partial p_b}\frac{\partial^2 S}{\partial q^a \partial q^b}, \quad (3.4b)$$

which gives us the desired equation:

$$\frac{d}{dt}p_a = -\frac{\partial H}{\partial q^a} + \frac{\partial^2 S}{\partial q^a \partial q^b}\left\{\dot{q}^b - \frac{\partial H}{\partial p_b}\right\}. \quad (3.4c)$$

From this we learn that the satisfaction of that first half of the original version of Hamilton's equations also, with our current constraints, tells us that the second half is satisfied as well, completing our proof that the finding of a solution of this pde is equivalent to solving the original Hamilton's equations.

It is also important to complete the following computation, which displays the physical significance of the function S . The obvious statement of the total derivative along the trajectory is the following:

$$\frac{d}{dt}S = \frac{\partial S}{\partial q^a}\dot{q}^a + \frac{\partial S}{\partial Q^r}\dot{Q}^r + \frac{\partial S}{\partial t}. \quad (3.5a)$$

However, we know that the Q^r are constant, and that $S_{,t} = -H$ [from Eqs. (3.1)], which gives us the equation

$$\frac{d}{dt}S = p_a \dot{q}^a - H = L, \quad S = \int dt L, \quad (3.5b)$$

so that we may think of S as that function which is minimized by the actual trajectory, compared to other, nearby possible trajectories.

It is appropriate to now present a relatively simple example of all this. Rather more serious examples will come somewhat later, after we have also presented the method to go from here to the action-angle variables. For the moment we simply consider a one-dimensional, simple harmonic oscillator:

$$H = \frac{p^2}{2m} + \frac{1}{2}kq^2 \quad \Longrightarrow \quad \frac{\partial S}{\partial t} + \frac{1}{2}kq^2 + \frac{1}{2m} \left(\frac{\partial S}{\partial q} \right)^2 = 0. \quad (3.6a)$$

Since we only need a solution which has one arbitrary constant, i.e., our complete solution, we look for a so-called (additively) *separable solution*: We suppose there exists W and T such that

$$S = W(q, Q) + T(t, Q) \quad \Longrightarrow \quad -T' = \frac{1}{2} \left\{ kq^2 + \frac{1}{m} \left(\frac{\partial W}{\partial q} \right)^2 \right\}. \quad (3.6b)$$

Since the left- and right-hand sides depend on different functions, it must be that each side is a constant. For the moment, we will call it α , which is some function or other of the needed Q . This separation then tells us that

$$\begin{aligned} S = W(q, Q) - \alpha t, \quad \left(\frac{\partial W}{\partial q} \right)^2 &= 2m\alpha - mkq^2, \\ \Longrightarrow \quad W &= \sqrt{mk} \int dq \sqrt{2\alpha/k - q^2}. \end{aligned} \quad (3.6c)$$

We could determine the integral explicitly; however, that particular integral is not all that interesting since it is just the generating function when what we really want is the transformation itself. Additionally we note that of course there is a constant of integration associated with this (indefinite) integral; nonetheless, we already have the one constant we need, so that we will simply ignore that constant. Note that we have already said that there would be some constant that one could add to S but which would be irrelevant so that we would ignore it; this constant is clearly that particular one. However, to complete the transformation, in this 1-dimensional problem, we do need to determine P , the conjugate momentum to Q . We do this using Eqs. (3.1), but, for now simply look for the conjugate momentum to α , referring to it as β :

$$\begin{aligned} \beta = \frac{\partial S}{\partial \alpha} &= \frac{\partial W}{\partial \alpha} - t = -t + \sqrt{\frac{m}{k}} \int \frac{dq}{\sqrt{2\alpha/k - q^2}} = \sqrt{\frac{m}{k}} \sin^{-1} \left(\sqrt{\frac{k}{2\alpha}} q \right) - t \\ \Longrightarrow \quad q &= \sqrt{\frac{2\alpha}{k}} \sin \left(\sqrt{\frac{k}{m}} (t + \beta) \right) \\ \Longrightarrow \quad p = \frac{\partial S}{\partial q} &= \sqrt{mk} \sqrt{2\alpha/k - q^2} = \sqrt{2\alpha m} \cos \left(\sqrt{k/m} (t + \beta) \right) \\ \Longrightarrow \quad H &= \frac{1}{2}kq^2 + \frac{1}{2m}p^2 = \alpha. \end{aligned} \quad (3.6d)$$

This tells us the (expected) physical meaning of the constants α and β ; i.e., α is really the original Hamiltonian, in terms of the new variables, $\alpha = H[q(\alpha, \beta), p(\alpha, \beta)]$, while β clearly takes the form

of some initial value for t , i.e., we may think of it as t_0 . As we know that energy and time are “conjugate” to one another, this makes sense, and tells us that it is reasonable to just take them directly as the new canonical variables, i.e., we choose $Q = \alpha$ and $P = \beta$. Therefore, the Eqs. (3.6d) do indeed give us the desired canonical transformation: they give us $\alpha = \alpha(q, p)$ and $\beta = \beta(q, p, t)$. We may also invert them, as has also been written out explicitly there, to obtain $q = q(\alpha, \beta)$ and $p = p(\alpha, \beta)$. Of course the new Hamiltonian is given by $K = H + S_{,t} = \alpha - \alpha = 0$, as expected. We have everything we need. Although it is not exactly needed explicitly, as we see, we could also have an explicit form for S if we were to simply evaluate the integral form given for $W = W(q, \alpha)$ given in Eqs. (3.6c).

It should be clear that whenever the Hamiltonian is independent of time the separation “trick” above will in fact work, so as to allow one to simplify the equation somewhat:

$$\begin{aligned}
 H_{,t} &= 0 \\
 &\iff \\
 \exists \quad W \text{ and } Q^1, &\text{ constant, such that } S = W(q, Q) - Q^1 t, \\
 \text{and } W \text{ is determined as a solution of the pde (in } n \text{ independent variables}
 \end{aligned} \tag{3.7}$$

$$H\left(q^a, \frac{\partial W}{\partial q_b}\right) = Q^1.$$

Under these circumstances, W is referred to as *Hamilton’s characteristic function*, and the equation that it satisfies is called the *time-independent Hamilton-Jacobi equation*. As well we see that

$$dW = \frac{\partial W}{\partial q^a} dq^a = p_a dq^a \implies W = \int p_a dq^a = \int \boldsymbol{\varrho}; \tag{3.8}$$

integrals of this form are commonly referred to as an *action*.

The original form of our equation, Eq. (3.6a), had quite a formidable appearance. The equation was nonetheless resolved because of our *ansatz* for additive separability. This was of course just an *ansatz*, i.e., a guess toward a solution. However if the guess does in fact generate a complete solution, that is quite alright; any complete solution is equivalent to any other. It is this possibility of separability that is by far the most usual approach to finding a solution of this equation. In more variables, of course, we could have just partial separability. Suppose that we can write out an additive *ansatz* for $W = W(q^a, Q^r)$ involving some m functions, $W_A = W_A(q^a, Q)$, each of which depends only on its one, particular q^a [although perhaps on all of the (constant) Q^r ’s], for $a = 1, \dots, m$. (We use a capital A to mean the same value as a , but to indicate that there is no intention of generating a sum because of the twice-repeated occurrence of that index.) In the (general) case, when $m < n$, then we will also need some sort of “remainder” function, which will depend on the remainder of the coordinates $\{q^a\}_{m+1}^n$, which then allows us a separability *ansatz* for the action:

$$W = \sum_{A=1}^m W_A(q^a, Q) + \hat{W}_m. \tag{3.9}$$

When such a solution exists, then we have been able to separate out at least some of the coordinates, and we say that the HJ equation in this case is *partially separable*. In the (desirable) event that $m = n$, so that all the coordinates have been separated out, and of course the function $\hat{W}_m = \hat{W}_n = 0$, then

we say the HJ equation is (*completely*) *separable*. The guarantee that this has occurred, of course, is when this form is inserted into our HJ equation, and we are able to generate from it a complete solution of the equation. It is usually the case that such insertion also requires overall multiplication by some specific function, perhaps just a constant, in order to be able to bring the equation into the form of a function of only one variable on one side of the equation and no dependence on that one variable on the other side of the equation. When that happens we may safely infer the existence of a *separation constant*, and in the case of full separability these various constants will generate the desired constant quantities Q^r . We also note that it is actually quite unusual to be able to resolve the Hamilton-Jacobi equation when it is not possible to separate it. In cases with more than one degree of freedom, such separability often occurs only in certain classes of coordinate systems, which one could then say are “adapted” to that physical problem, rather than in all. An interesting, non-trivial example of this is described in José, Worked Example 6.3, where he considers the HJ equation for a particle attracted to two fixed gravitational centers, and considers its motion in a single plane. This problem is separable in confocal elliptical coordinates.

We now **begin to concentrate on the case of complete separability**. For such a problem, we will use sometimes the symbol Q^1 for that separation constant that comes from separating off the time dependence; however, since in fact it is usually the energy, we may also often use the symbol E for it, using the symbol H when we mean, instead, the function on phase space. The For a problem where the HJ equation is completely separable we may write

$$p_a = \frac{\partial W}{\partial q^a} = \frac{\partial W_A}{\partial q^a} = p_a(q^A, Q), \quad P_a = t \delta_a^1 - \frac{\partial}{\partial Q^a} \sum_b W_b, \\ \implies W_a = \int p_a(q^A, Q) dq^A. \quad (3.10)$$

When Q^1 is just the energy, as already noted in the one case, the constant quantity P_1 should have simply the interpretation as some sort of initial time, and is therefore often not of very much importance. **However**, our separability ansatz clearly tells us the quite important fact that $p_a = p_a(q_A, Q)$ only, i.e., without any dependence on the other q^b 's. Therefore, for a given set of values for the Q^r 's, the relation between the values of p_a versus q^a , along the trajectory, is simply a 2-dimensional graph, totally **independent** of how the other generalized coordinates are changing. These 2-dimensional graphs, in the variables (q^a, p_a) , are referred to as *orbits*, and clearly are the projection of the entire orbit down to this particular 2-plane. For any given value of a , we will refer to this graph as the curve C_a . As their behavior is independent of the rest of the motion, it is quite useful to use them to set up detailed study of the entire trajectory, so that often one may find out some important properties of the system, even when it is in fact impossible to obtain the complete, explicit solution of the problem of the motion of the system!

Complete separability clearly gives us an orbit for each physical degree of freedom, although we should be clear that the word orbit is here is being used rather differently than is often the case. They are not the entire trajectory, nor are they the graphs of, say, r versus θ often used in a problem with two physical degrees of freedom and also referred to as orbits.

We now go on to our next particular **specification of the problems we are considering**: **We now restrict ourselves to those physical cases where, in addition, all these orbits are restricted to staying in bounded regions of the plane**. An individual orbit is then a problem that at least locally looks like the motion of a (nonlinear) pendulum, we may expect that there are only two possibilities for the shape of the orbit.

- 1.) We may have *libration*, where the values of q^a oscillate back and forth, with the momentum changing sign back and forth, so that it is multi-valued as a function of q^a . We say that

the orbit has *turning points*, where the momentum vanishes. In this case the curves C_a are closed, and may be thought of as deformed circles.

- 2.) Or, we may have *rotation*, where the momentum always has a single sign, and the (apparent) values of q^a simply continue to increase. One could therefore say that those values are multi-valued, since there will be many different values of q^a that correspond to the same physical configuration. However, as the actual values of q^a must remain bounded, it is more reasonable to view the motion on some sort of a cylinder, where the values of q^a are restricted to some compact region. This sort of motion will then only occur for a system where the coordinates are periodic, such as a pendulum.

In general, of course, there will be some special motion, some sort of *separatrix*, that divides the plane into areas where one or the other of these types may occur. For a sufficiently complicated system there could be several separatrices and therefore a division into several regions, but always of one or the other of the types above, when we insist that motions remain in bounded regions of the plane. It is worth pointing out that the system may change between the two options above depending on the particular values of the constants Q^r . As well, while the system may oscillate back and forth, it need not be nearly as simply-behaved as a true 1-dimensional system, since what it does depends on what the projections of the trajectory onto its other 2-planes as well.

We now begin a study of the bounded motions. The projection into any given 2-plane will have bounded motions that look something like circles, which we denote topologically like S^1 ; i.e., the (closed) curve in that 2-plane is homotopic to a circle. If, on the other hand, there were two degrees of freedom the trajectory would be simultaneously on both of those closed circuits, which would allow us to describe it as lying on a torus, $T^2 = S^1 \times S^1$. In yet the most general case, it would lie on an n -torus, T^n . Since the trajectory lies on a single such T^n , these n -tori are often referred to as *invariant tori* for the problem in question. Therefore the labels which specify the various tori must be invariants of the motion; i.e., they would be “constants of the motion.” The set of such n -tori foliates the $2n$ -dimensional cotangent space, T^*Q . The initial conditions, for instance, can provide constants to label which of the sheets is the one on which the trajectory will occur; i.e., which T^n . (We think of the 1-dimensional torus as the same as the circle; i.e., $T^1 \equiv S^1$.) The text by José refers to such a physical system as ***completely integrable***. On the other hand, he does also point out that the definition is slightly broader than that, and that it is also referred to as **Liouville integrability**. The full version is the following notion. In order to satisfy the (generalized, or perhaps original) definition of Liouville integrability, a given Hamiltonian dynamical system should have n independent constants of the motion, $\{I_a\}_1^n$, that are *in involution*, and have bounded motions in the sense to be described just below. We consider the subspace of T^*Q that corresponds to any particular set of constant values for all these constants of the motion. When that subspace is both n -dimensional and **compact**, then the system is completely integrable in the sense of Liouville. We note that the property of being in involution simply means that all the Poisson brackets between them must vanish: $\{I_a, I_b\} = 0$. (Note that the Hamiltonian is certainly allowed to be one of these constants, when it is indeed constant, i.e., time independent.)

For a completely integrable system there is a **different canonical transformation** that takes us from (q, p) to yet new coordinates (ϕ^r, J_s) . The new momenta $\{J_a\}$ will be used to specify the individual tori onto which the motion may be projected, so that they will be constant along any given trajectory. In particular, for a multi-dimensional problem they will also allow us to pick out the projections based on the idea that the HJ equation is separable in those coordinates. We may then use the (new) coordinates ϕ^a as describing the motion along given trajectory, which requires them to be periodic, so that it is reasonable to treat them as if they had the status of “angles,” and

normalize them so that they increase by 2π each time the circuit is traversed. Since they have the dimensions of angles, the dimensions of the J_b will be those of angular momentum, or of action. It is traditional to begin this process by supposing that the transformation is of Type 2. We will call it $\widetilde{W} = \widetilde{W}(q, J)$, so that the usual CT equations will tell us that

$$p_a = \frac{\partial \widetilde{W}}{\partial q^a}, \quad \phi^r = \frac{\partial \widetilde{W}}{\partial J_r}; \quad K = K(\phi, J) = H[q(\phi, J), p(\phi, J)], \quad (3.11)$$

where the last equality comes from the fact that these transformations are not explicitly dependent on the time.

To create these new coordinates, we first return to the closed, projected curves, C_a , described by the variables (q^a, p_a) , **in the picture appropriate to the completely separable Hamilton-Jacobi equation, considering only closed orbits**, and consider in more detail the normalization requirement around the curve C_a in a particular 2-plane:

$$2\pi = \oint_{C_A} d\phi^a = \oint_{C_A} d\left(\frac{\partial \widetilde{W}}{\partial J_a}\right) = \oint_{C_A} \frac{\partial^2 \widetilde{W}}{\partial q^b \partial J_a} dq^b = \frac{\partial}{\partial J_a} \oint_{C_A} \frac{\partial \widetilde{W}}{\partial q^b} dq^b = \frac{\partial}{\partial J_a} \oint_{C_A} p_b dq^b \quad (3.12a)$$

This suggests the consideration of the closed integral around the curve as a coordinate that would characterize that curve. We recall from Eq. (1.28) that this integral is one of the Poincaré invariants, along a trajectory, so that it would be a consistent way to label the individual projections of the trajectory with a quantity that is invariant under the motion of the entire trajectory. Therefore we define the desired coordinates J_a as follows:

$$\implies J_a \equiv \frac{1}{2\pi} \oint_{C_A} p_a dq^a = \frac{1}{2\pi} \int_{S_A} dp_a \wedge dq^a = \frac{1}{2\pi} \int_{S_A} dp_b \wedge dq^b. \quad (3.12b)$$

where the last equality is true since, for a surface S_a , on a given torus, the sum under the integral will vanish except for the one term which corresponds to $b = a$, and in that case it will give the area in question. From the point of resolving the motion, we also note that the equation that defines the various $W_a(q^A, Q)$, namely Eqs. (3.10), involves determining exactly the same integral as is used to define J_a above, **except for the fact** that the one for J_a is a definite integral between endpoints where the integrand vanishes. Therefore, the calculation of the J_a is expected to both be simpler and to give much simpler results; in addition, of course, they do not depend upon the q^a 's. This is to be emphasized because it is often possible to proceed this way without ever having explicitly determined the involved integral(s) that determine the various W_a , but nonetheless to acquire considerable information about the motion and its general behavior.

Clearly we may consider the coordinate J_a as a function of the original coordinates (q, p) , which is what is desired in order to define the canonical transformation between them. On the other hand, since it is an integral around a particular trajectory, in order to evaluate this relationship, i.e., to determine $J_a(q, p)$, we must have first solved the problem so as to determine the motion. It is of course in fact quite the opposite thing that we want to do; we want to utilize the method being created so as to obtain simple methods to solve the problem. Therefore, we need a different approach. We begin toward that goal with the following notions. As discussed after Eqs. (3.10), when the motion is projected onto a given 2-plane, we have p_a as a function only of the particular q^a on that surface, as well as (possibly all) the constant, HJ coordinates, Q^r . Since our definition of

J_a requires the integration over a given trajectory, defined by that momentum and coordinate, the result is simply that we must have J_a depending only on the choices of the invariants $\{Q^r\}_1^n$, and, in particular, not on the conjugate momenta, $\{P_s\}_1^n$, in that choice of (canonical) coordinates. In more detail, since we presume the HJ equation has been resolved—not necessarily quite the same as having been solved, since, perhaps, it has only been reduced to the determination of some set of quadratures—we may begin with the solution of that problem, $W = W(q, Q)$ and calculate the following, where we are also beginning to define a way to find the generating function (of Type 2), $\widetilde{W} = \widetilde{W}(q, J)$, between the (q, p) and the (ϕ, J) :

$$\begin{aligned} J_a = J_a(q, p) &\equiv \frac{1}{2\pi} \oint_{C_A} p_a dq^A = \frac{1}{2\pi} \oint_{C_A} \left\{ \frac{\partial W_a}{\partial q^A} \right\} dq^A \\ &= \frac{1}{2\pi} \oint_{C_A} \left\{ \frac{\partial W}{\partial q^a} \right\} dq^A \equiv \mathcal{J}_a(Q) = \mathcal{J}_a[Q(q, p)] , \end{aligned} \quad (3.13)$$

where the last equality required us to insert our knowledge of $Q^r = Q^r(q, p)$ obtained, again, from the resolution of the HJ problem. Given those equations, i.e., $J_a = \mathcal{J}_a(Q)$, they may be inverted, as will be desired soon, to obtain the inverse transformation, i.e., $Q^r = Q^r(J)$, where the very important fact in this transformation, in both directions, is that they are functions only of each other, without the other parameters, either $\{q^a\}$ or $\{\phi^s\}$ becoming involved. All that then suggests the definition of the generating function, \widetilde{W} , for this second CT, which maps between the (q, p) and the (ϕ, J) :

$$\widetilde{W} = \widetilde{W}(q, J) \equiv \sum_{a=1}^n W_a[q^A, Q(J)] = W[q, Q(J)] , \quad (3.14)$$

where we have used the fact that our HJ system is completely separable, and we used the inverse relation mentioned after Eqs. (3.13), namely $Q^r = Q^r(J)$. Since the Q^r and the J_a transform between themselves, the partial derivatives with respect to the other coordinates, q^a , act the same in both sets of coordinates; i.e., we have the following equalities:

$$p_a = \frac{\partial W}{\partial q^a} = \frac{\partial \widetilde{W}}{\partial q^a} , \quad (3.15)$$

where in the first equality we were holding fixed the coordinates $\{Q^r\}$ while in the second we were holding fixed the coordinates J_a .

In principle at this point the work is done: We have created a generating function and transformation equations between the original coordinates and the action-angle variables. However it is worth noting that since the Q^r are all constant, all the J_a are constant as well. Since they are constant, it must be that the Hamiltonian, in these coordinates, is such that all the conjugate coordinates are cyclic; i.e., the Hamiltonian does NOT depend on any of the entire set of angle variables, $\{\phi^a\}_1^n$. Therefore we may immediately determine the frequencies of the various angle coordinates, and thereby determine their motion in a simpler way than to explicitly determine the \widetilde{W}_a 's. The calculation of these frequencies is often easier, and we may often determine many of their properties into only integrals of quadrature:

$$\frac{d}{dt} \phi^a = \frac{\partial K}{\partial J_a} \equiv \nu^a \implies \phi^a = \nu^a t + \phi_0^a . \quad (3.16)$$

In these variables the motion is simply that the momenta, J_a , are constant, and the angular coordinates change linearly with time. It is of course worth commenting that the frequencies, ν^a , do

depend on all the various original coordinates, q^b , so that an inversion question can sometimes be quite troublesome, especially when there are several degrees of freedom in the system. Moreover, by finding a completely separable solution of the HJ equation we have a method to determine these variables. The only last thing that might perhaps be considered necessary to complete a proof is to show that the angular coordinates, created from this \widetilde{W} via Eqs. (3.11), actually do satisfy the normalization condition. However, we will actually do slightly slightly better than this. We calculate the change in ϕ^a as one of the original coordinates q^b goes around a complete circuit in its phase space, where b need not be the same as a :

$$\begin{aligned} \Delta_b \phi^a &= \oint_{C_b} d\phi^a = \oint_{C_b} d\left(\frac{\partial \widetilde{W}}{\partial J_a}\right) = \oint_{C_b} \left(\frac{\partial^2 \widetilde{W}_A}{\partial J_a \partial q^B}\right) dq^B \\ &= \frac{\partial}{\partial J_a} \oint_{C_b} \left(\frac{\partial W}{\partial q^B}\right) dq^B = \frac{\partial}{\partial J_a} \oint_{C_b} p_B dq^B = 2\pi \frac{\partial J_a}{\partial J_b} = 2\pi \delta_b^a. \end{aligned} \tag{3.17}$$

This tells us that ϕ^a changes by 2π in each a -cycle, while it is periodic as one traverses any other cycle. It should be noted that this calculation does **not** tell us that ϕ^a actually changes by 2π in an a -cycle during an actual motion. During that motion all the various q^c are changing, while in the calculation above, our progress around the a -cycle held all the other cycles fixed.

We need to make two sorts of general comments about how to use the set of orbits on each of the 2-planes to learn things concerning the actual trajectory, or motion in the entire phasespace. Firstly, given any two of the frequencies, $\nu^a = \nu^a(J) = \nu^a[J(q, p)]$, in order that the actual trajectory would be periodic it is necessary that their ratio must be a rational number, i.e., a ratio of integers such as a/b . Such (pairs of) frequencies are referred to as *commensurate*. In that case, at least when one orbit has repeated itself b times, then the other orbit will have repeated itself a times, so that, finally, both of them will come back to the same place at the same time. If this happens, for all pairs of frequencies, i.e., if all pairs are commensurate—or, if you prefer the language, if the entire set of frequencies is commensurate—then the system is actually periodic. In general this is unlikely; more precisely, since the frequencies are allowed to have any real, numerical value, the set that are commensurate has measure zero within the entire set of all possible frequencies.

A second comment has to do with whether or not one has chosen the particular set of J_a in an optimal way, or whether there is some set of functions of them that would have been a better choice. To try to see what we mean let us first consider an ordinary torus. There are two obvious sorts of circular paths around that torus: one which circles through the center hole and comes back to its origin as quickly as possible, and one that circles around the outer edge and comes back to its origin as quickly as possible without passing through the area “within” that center hole. (This is of course immediately clear when a figure is presented, such as, for instance, Fig. 6.10 on p. 322 of José.) We suppose, as shown in that figure, that each of these circles has some tangent vector that describes them. On the other hand, one could imagine proceeding in some direction, on the surface of the torus, with a tangent vector that was some linear combination of those two vectors. In that case, the curve following that tangent vector is most likely “condemned” to proceeding around the torus forever, passing both around the entire torus and through its central hole and infinite number of times. This occurs of course because when the curve completes one circuit, say, around the larger circle then it is probably only part way toward completing a circuit of the smaller circle, so that it continues again trying to come to a point where it competes both circuits at once. This last of course happens only when the coefficients multiplying the two tangent vectors, in this sum that we are following, happen to have a ratio which is a rational number: the commensurability problem again. However, when we

have determined two different, constant J_a 's, how do we know whether their flows are along those "special directions" that are tangent to the two "best" ways to circle the torus, or whether they are along two rather more general directions, non-commensurate sums of the two best ones, so that they will circle forever. The answer is fairly clear that we don't immediately know which sort we have chosen, for some given set of tori, for some particular dynamical problem. However, the proof of the Liouville theorem on Integrability establishes the existence of such "best directions," but, to the best of my current knowledge, does not establish algorithms to determine them. (They correspond, obviously, to some sort of "eigen-directions," but, of course, in some infinite-dimensional space.) We will not deal further with this problem, but be sure to note its existence, and endeavor, any time that it is needed, to use our physical intuition to ensure that we have chosen the directions appropriately. It is also worthwhile noting that when there are only two directions, and one of the constants is the energy, this is not really a serious difficulty; the problems really only occur when there are at least three real degrees of freedom.

It is now certainly useful to give some sort of summary of this process, where as above we consider only the case where the Hamiltonian is time independent and the motions are compact, so that the projections to any single 2-plane are closed, **and** where the HJ equation is completely separable, and we ignore the explicit indices in the summary below.

We first describe, and summarize, the two different, strongly-related canonical transformations:

- a.) We have CT¹: $(q, p) \longrightarrow (Q, P)$, with $K = 0$ and $F^1 \equiv S = -Q^1 t + W(q, Q)$, where W satisfies the HJ equation under a separable ansatz: $H(q, W_{,q}) = Q^1 = E$, and $W = \sum W_A(q^a, Q)$. The resultant equations are $p = W_{,q}$, $P = -W_{,Q}$ along with $\dot{Q} = 0$ and $\dot{P} = 0$.
- b.) We also have CT²: $(q, p) \longrightarrow (\phi, J)$, with $\tilde{K} = H$ and $F^2 \equiv \tilde{W}(q, J) = W[q, Q(J)]$. The resultant equations are $p = \tilde{W}_{,q}$, $\phi = \tilde{W}_{,J}$ along with $\dot{\phi} = \tilde{K}_{,J} \equiv \nu$ which are constant, and $\dot{J} = -\tilde{K}_{,\phi} = 0$.

We then give a (plausible) summary of the details of the method for deriving them:

- 1.) First find a completely separable solution of the HJ equation, determining the various W_a 's, and the associated momenta, $p_a = W_{,q^a}$.
- 2.) Use these momenta to determine the circuital integrals that give us J_a which are functions only of the Q^r 's, the other variables in W . Invert these equations to obtain the $Q^r = Q^r(J)$.
- 3.) Define the action-angle generating function $\tilde{W}(q, J) \equiv W[q, Q(J)]$. In addition, note that the Hamiltonian in these coordinates is just the original Hamiltonian, as a function of these variables, which in fact does not depend on the angular variables: $\tilde{K}(J) = H[q(\phi, J), p(\phi, J)]$.
- 4.) Use this generating function to define the angle variables, ϕ . Even if one cannot immediately resolve the integrals necessary to do this, an alternative approach is to find the recursive frequencies, $\dot{\phi} = \partial \tilde{K} / \partial J \equiv \nu$, which are constant.

In general, were one to acquire the complete details of the motion in action-angle variables, it would probably still be desired to transfer those details back into the original (q, p) variables. In general this is certainly possible, because we have insisted on the Hessian condition, i.e., that the equations are invertible; however, in fact this may be either very difficult or impossible to effect in practice because of the insolubility of various transcendental equations that could occur. Even more generally, there is the problem that we may not actually be able to complete all the calculations that

are needed to get to this point, even though, perhaps, many of them were done. Therefore, it is useful to think about ways in which our knowledge of the frequencies, ν^a , can be used to determine at least some useful information concerning the problem in the original coordinates even though we may not be able to learn everything. What we really want is to know about $q^a = q^a(t)$ and $p_b = p_b(t)$. On the other hand, one way to re-phrase what has been said just above is the following: if we consider, instead, the inverse canonical transformation, i.e., think of $\xi^\alpha = \xi^\alpha(\phi, J)$, then we know that

$$\xi^\alpha(\phi^a + 2\pi k^a, J_b) = \xi^\alpha(\phi^a, J_b) , \quad (3.18)$$

where the k^a are some set of n different integers. This tells us that, **for fixed values of the $\{J_b\}$** , we may write out the desired original coordinates as (multiple) Fourier series in the angle variables:

$$\xi^\alpha(\phi, J) = \sum_{[k_1, \dots, k_n]} A_{[k]}^\alpha e^{i(k_a \phi^a)} . \quad (3.19)$$

The problem with this approach, of course, is that we do not have a clear way to determine the coefficients A unless we have already solved the problem; nonetheless, we will find that it can be put to some value in determining some useful information concerning the system's behavior.

It is also worthwhile recalling that we have explicit information concerning the time behavior of the angle variables; therefore, we may insert that into this equation, whereby we acquire

$$\xi^\alpha(\phi, J) = \sum_{[k_1, \dots, k_n]} B_{[k]}^\alpha e^{it(k_a \nu^a)} . \quad (3.20)$$