Maps, mostly Hamiltonian, toward the KAM Theorem

1. Poincaré Sections of Hamiltonian Trajectories

Our studies of Hamiltonian systems, in more than one degree of freedom, is very much implemented by the use of Poincaré sections. For a system with N degrees of freedom, this corresponds to keeping track of individual intersections of the trajectory with a chosen $(2N-2)$-dimensional hyperplane, of course keeping track of the direction so that one only, for instance, "counts" the intersections coming one direction. These intersections clearly do not constitute a continuous curve on the hyperplane in question, but rather a discrete sequence of points, labelled by some index that increases as the time increases. This discrete sequence can be conceived of as the sequence generated by a mapping, from the previous point to the next one. In addition to the fact that the use of a Poincaré section often makes simpler the task of visualizing the actual (higher-dimensional) trajectories, the use of a mapping, rather than the integration of differential equations, also constitutes a much simpler method of generating the desired points.

A mapping for the Poincaré section of a Hamiltonian system is often referred to as the *Twist Map*. The simplest case is an **integrable** Hamiltonian problem, which is therefore describable by a collection of N actions, J_i and N angles, ϕ^j. In an integrable case, of course, having resolved the problem, any particular trajectory would have a constant value for the action. To describe our section, we pick out some single one of these pairs to be held constant and look at the intersections with the hyperplane defined by those constant values. Of course the actions have the same value each time, for each consecutive intersection, while the angles change by some amount that depends on all the actions. We denote these changes in the following way, using a function $\alpha = \alpha(J_i)$, which determines the change in the angle as a fraction of a full "turn." We denote the first such intersection by $n = 0$ and use the index n as a subscript on the action and the angle, suppressing the index for the degree of freedom, to simplify the notation:

$$\begin{aligned} J_{n+1} &= J_n \ , \\ \phi_{n+1} &= \phi_n + 2\pi\,\alpha(J_{n+1}) \ . \end{aligned} \tag{1.1}$$

Since it is clear that this mapping fills in (at least some) points on a circle, it is reasonable to refer to this as a "twist map." Clearly when α takes on rational values, there are "fixed points" on the circle, which are repeated indefinitely, while if α is irrational eventually "all" points on the circle will be approached at least arbitrarily closely. We will want to consider these two options separately, because they have very different behavior when perturbations are added.

On the other hand, the more interesting systems will in fact not be integrable; instead, we could suppose that we are studying a "near-integrable" system, by which we mean one which differs from an integrable one by a function which is multiplied by a "small" quantity, which we denote by ϵ as usual:

$$H(\phi, J) = H_0(J) + \epsilon\, H_1(\phi, J) \ . \tag{1.2}$$

We then anticipate a more general such mapping, referred to as the *perturbed twist mapping*:

$$\begin{aligned} J_{n+1} &= J_n + \epsilon\, f(\phi_n, J_{n+1}) \ , \\ \phi_{n+1} &= \phi_n + 2\pi\alpha(J_{n+1}) + \epsilon\, g(\phi_n, J_{n+1}) \ , \end{aligned} \tag{1.3}$$

where we require that f and g are periodic functions of ϕ_n, but must still determine what constraints they must satisfy, if any, in order that this mapping correspond to the Poincaré sections of a Hamiltonian motion. It turns out to be more convenient to say that the arbitrary functions depend on J_{n+1}, rather than J_n. As we will see shortly this makes the description in terms of a (Type 2) generating function much simpler, since such a generating function should depend on the original angles and the final actions. Using it as a canonical transformation from the n-th intersection with the Poincaré section to the $(n+1)$-st intersection, we want it to be a function of ϕ_n and J_{n+1}. This causes some small amount of "grief" in the first line, where one does not have an explicit equation for the new action, J_{n+1}, but, rather, must solve for it. However, it is a single equation, and, numerically, causes little grief to have to do this. As well, in many common cases, the function f is actually independent of the action, as we will discuss below.

At this point we ask what are the restrictions on this map that it correspond to the projection from an area-preserving, continuous set of trajectories. The requirement for area-preservation may be re-stated in terms of the canonical form of the Poisson brackets, which, again, may be re-stated as having the possibility of being stated in terms of a canonical transformation of type. We suppose, then, that F^2 is a transformation of Type 2, which should then be a function of the original angle and the final action, which we write in such a way that it is near the identity, with two additional functions that are to be determined so that this will indeed be the desired generating function:

$$F^2 = F^2(\phi_n, J_{n+1}) \equiv \phi_n J_{n+1} + 2\pi\mathcal{A}(J_{n+1}) + \epsilon\mathcal{G}(\phi_n, J_{n+1}) \ . \tag{1.4}$$

Using the standard rules, we then have

$$\begin{aligned} J_n &= \frac{\partial F^2}{\partial \phi_n} = J_{n+1} + \epsilon\frac{\partial \mathcal{G}}{\partial \phi_n} \ , \\ \phi_{n+1} &= \frac{\partial F^2}{\partial J_{n+1}} = \phi_n + 2\pi\frac{\partial \mathcal{A}}{\partial J_{n+1}} + \epsilon\frac{\partial \mathcal{G}}{\partial J_{n+1}} \ . \end{aligned} \tag{1.5a}$$

Comparing these to the actual mapping, we see that we need to require

$$\begin{aligned} &\frac{\partial \mathcal{A}}{\partial J_{n+1}} = \alpha \ , \\ &\frac{\partial \mathcal{G}}{\partial \phi_n} = -f \ , \quad \frac{\partial \mathcal{G}}{\partial J_{n+1}} = +g \ . \end{aligned} \tag{1.5b}$$

However, we see that in fact such a canonical transformation will exist only when there is such a function $\mathcal{G}$, whose existence is determined by the obvious integrability condition of mixed partial derivatives:

$$\begin{aligned} &+\frac{\partial g}{\phi_n} = \frac{\partial^2 \mathcal{G}}{\partial J_{n+1}\phi_n} = \frac{\partial^2 \mathcal{G}}{\partial \phi_n J_{n+1}} = -\frac{\partial f}{\partial J_{n+1}} \\ &\quad \Longrightarrow \quad \frac{\partial f}{\partial J_{n+1}} + \frac{\partial g}{\phi_n} = 0 \ . \end{aligned} \tag{1.5c}$$

Therefore the two functions f and g must actually be determined from one original function. We now discuss a method to do this, to order ϵ, for a perturbed Hamiltonian system, such as the one noted above in Eq. (1.2).

It is interesting to consider in more detail how to begin with a Hamiltonian system and create the mapping of the type above; i.e., how do we determine the functions f and g? We suppose that the map in question involves changes in the ϕ^1, J_1-plane, and begin with the changes in time as given by Hamilton's equations, and content ourselves with dealing with a system with only two degrees of freedom:

$$\frac{dJ_1}{dt} = -\frac{\partial H}{\partial \phi^1} = -\epsilon\frac{\partial H_1}{\partial \phi^1} \ . \tag{1.6}$$

Integrating this equation over one period of the motion in the other angle, which we take to have period T_2, we have the change in this action over that motion:

$$J_{1,n+1} - J_{1,n} = \Delta J_1 = -\epsilon \int_0^{T_2} dt \frac{\partial H}{\partial \phi^1}(J_{1,n+1}, J_2, \phi_n^1 + \omega^1 t, \phi^2 + \omega^2 t) \ , \tag{1.7a}$$

where the restriction to two degrees of freedom, and the existence of a Hamiltonian, ensures that J_2, ω^1, and ω^2 are all functions of $J_{1,n+1}$. To accomplish this integral to order ϵ, we may simply use their values from the unperturbed Hamiltonian, which then gives us the value of the desired function f, which will depend on $J_{1,n+1}$ and ϕ_n^1. To determine g, we use the area-preserving condition determined above:

$$g = -\int d\phi^1 \frac{\partial f}{\partial J_1} \ . \tag{1.7b}$$

An important simpler case of the full perturbed twist map is the **radial twist map**, which is the special case where the function f is independent of J_{n+1} altogether, so that—consistent with the requirement above for area preservation—the function g is identically zero:

radial twist mapping:

$$\begin{aligned} J_{n+1} &= J_n + \epsilon\, f(\phi_n) \ , \\ \phi_{n+1} &= \phi_n + 2\pi\alpha(J_{n+1}) \ , \end{aligned} \tag{1.8}$$

For this special case, the function f is independent of the action, making successive applications of this map very straight-forward. Let us now consider some point which is a period 1 fixed point, say J_0, i.e., we have $J_n = J_0 = J_{n+1}$. We also specialize to the case when $\alpha(J_0)$ is an integer, and consider nearby values for the action, and linearize near there:

$$J_n = J_0 + \Delta J_n \quad \Longrightarrow \quad \alpha(J_n) = 2\pi\alpha(J_0) + 2\pi\alpha'(J_0)\Delta J_n + O^2(\Delta J_n) \ . \tag{1.9a}$$

Since our angle variable is indeed really an angle, and $\alpha(J_0)$ is an integer, we ignore the first term in the series for α, and re-normalize the action to study simply these deviations near the fixed point by multiplying the action equation by the constant $2\pi\alpha'(J_0)$, to define $I_n \equiv 2\pi\alpha'(J_0)\Delta J_n$, thereby re-phrasing, locally, the mapping to the form

generalized standard mapping:

$$\begin{aligned} I_{n+1} &= I_n + K\, F(\phi_n) \, , \\ \phi_{n+1} &= \phi_n + I_{n+1} \, , \quad \mathrm{mod} 2\pi \ , \end{aligned} \tag{1.9b}$$

where F is the original function f normalized so that it has maximum value +1, i.e., divided by its maximum value over the relevant range, 0 to 2π, or $F \equiv f(\phi_n)/f_{\max}$ and the new, **small** constant K has absorbed both these factors, including the ϵ: $K \equiv 2\pi\epsilon\alpha'(J_0)f_{\max}$. It is worthwhile noting that the generalized standard map is area-conserving, i.e., it has Jacobian determinant +1, which we see by explicit calculation:

$$\begin{aligned} &\phi_{n+1} = \phi_n + I_n + K\, F(\phi_n) \, , \quad I_{n+1} = I_n + K\, F(\phi_n) \, , \\ &\Longrightarrow \quad J\left(\frac{\phi_{n+1}, J_{n+1}}{\phi_n, J_n}\right) = \begin{vmatrix} 1 + KF' & 1 \\ KF' & 1 \end{vmatrix} = 1 \ . \end{aligned} \tag{1.9c}$$

In the event that we now consider the special case when the function F is chosen as the (trigonometric) sine function, then this becomes what is called the standard map:

standard map:

$$\begin{aligned} I_{n+1} &= I_n + K\sin\phi_n\ , \\ \phi_{n+1} &= \phi_n + I_{n+1}\ , \quad \bmod 2\pi\ . \end{aligned} \tag{1.9d}$$

It is also interesting to suppose that we have been given some area-preserving map and we desire to convert it into the corresponding Hamiltonian system, with of course the integer playing the (approximate) role of the time. Such a thing is not really unique; however, we may insert Dirac delta's into a Hamiltonian in such a way that it creates exactly that map. We begin by defining a periodic Dirac delta:

$$\delta_1(n) \equiv \sum_{m=-\infty}^{+\infty} \delta(n-m) = 1 + 2\sum_{q=1}^{+\infty} \cos 2\pi q n\ . \tag{1.10}$$

Considering only now the radial twist map—where we don't have to worry about resolving the action equation for the new action—we may rewrite the difference $J_{n+1} - J_n$ in the form

$$\frac{dJ}{dn} = \frac{J_{n+1} - J_n}{n+1\ -n} = J_{n+1} - J_n\ , \tag{1.11a}$$

which of course really only makes sense when n is an integer. We resolve that problem by multiplying the remainder of our equation—Eq. (1.8), the one for the radial map—by the quantity $\delta_1(n)$, above, which gives us the apparently continuous form of the equation:

$$\frac{dJ}{dn} = \epsilon f(\phi)\delta_1(n)\ , \qquad \frac{d\phi}{dn} - 2\pi\alpha(J)\ . \tag{1.11b}$$

A "Hamiltonian" which describes such equations is also easily created:

$$H(\phi, J, n) = 2\pi \int dJ\,\alpha(J) - \epsilon\delta_1(n)\int d\phi\, f(\phi)\ . \tag{1.11c}$$

Note that the Hamiltonian in question is *non-autonomous.*

There are physical systems in which this is not a particularly unreasonable description of what is occurring. The "kicked rotator," as described in José is a particularly simple one, that is interesting for this purpose nonetheless. We suppose a rigid body of moment of inertia I relative to some single fixed point about which it is free to rotate, with no outside influence of gravity, etc. On the other hand, with a period T it is impulsively kicked, by a force of fixed magnitude and direction, so that it receives an impulsive torque of amount $\tau = \epsilon\sin\phi$. We may then write down a Hamiltonian of the form

$$H = H_0 + \epsilon H_1 = J^2/(2I) + \epsilon\cos\phi \sum_{n=0} \delta(t - nT) \implies \begin{cases} \dot\phi &= J/I\ , \\ \dot J &= \epsilon\sin\phi \sum_{n=0} \delta(t-nT)\ . \end{cases} \tag{1.12}$$

As can be easily seen, J is constant between kicks, but of course has a discontinuous change in its value when it is kicked. On the other hand, ϕ changes continuously, but with a discontinuous change in its otherwise constant frequency. We can easily make this into a map by defining J_{n+1} as the value of $J(t)$ at a time t greater than nT, and just before $(n+1)T$, and the same for ϕ, we can write

$$\begin{aligned} J_{n+1} - J_n = \lim_{\sigma \to 0} \int_{nT-\sigma}^{nT+\sigma} dt \dot{J} = \epsilon \sin \phi_n \; ; \\ \phi_{n+1} = \phi_n + (J_{n+1}/I)T \quad \mod 2\pi \; . \end{aligned} \tag{1.13}$$

Of course the Hamiltonian here is neither conserved, nor time-independent, nor the energy; nonetheless, it gives the correct equations of motion. In principle, such a system cannot go on forever, since the energy would then approach infinity. As well, it cannot be a truly impulsive kick, since those don't really happen that way; nonetheless, it is a reasonable approximation to a physical problem, and provides a reasonable map. It is also relevant that it has exactly the form of the *standard map*, described above in Eqs. (1.9d). Therefore it is area-preserving. To study the behavior of this map, we choose an arbitrary initial point, i.e., (ϕ_0, J_0), and then generate new, related points by using the map, as n varies from 0 to some large number. It is obviously arranged so that both ϕ_n and J_n only take on values that vary over a range of 2π. José thinks it reasonable to make plots of these where both of them actually take on values from $-\pi$ to $+\pi$, covering that range. The fixed points are at $(0,0)$ and $(\pi, 0) = (-\pi, 0)$, and, when ϵ is positive, the fixed point at the origin is hyperbolic, while the one at $\phi_n = \pi$ is elliptic. Therefore, this method of presentation puts a hyperbolic point in the middle of the graph and the elliptic point divided in its presentation at the right- and left-boundaries. Beginning one's studies by considering the unperturbed case, with $\epsilon = 0$, the values of J do not change, while ϕ varies across the graph. This will generate straight lines in a flat presentation, or circles if we think of the diagram properly, as drawn on a cylinder, or, even better, a torus. As the value of ϵ increases away from zero, so that the system is now perturbed, the straight lines that pass through, or near, the fixed points will begin to change. In particular the particular straight line that corresponds to the initial value $J = 0$, and therefore passes directly through the fixed points, will be destroyed. It and those lines quite nearby will be replaced in such a way that the elliptic fixed points will acquire small ellipses around them, while the hyperbolic fixed points will take on the character of a separatrix. This is of course expected because we have seen that the standard map has, quite generally, the character of the behavior of a reasonably arbitrary map that is being looked at near one of its period-one fixed points. We will plan to come back and study these in more detail.

At the moment, let me regress to the more general, perturbed twist map, and divide its behavior into the two very different sorts of behavior it has available to it, as already mentioned, depending on whether the function $\alpha = \alpha(J_{n+1})$ is rational or not. We recall that canonical perturbation theory often constituted a reasonable approximation to the true behavior of a system, provided that (a) it was actually integrable, and (b) the perturbation expansion parameter, ϵ, was sufficiently small. The word "often" in this statement comes from the fact that this was only true when the initial conditions did not cause the denominators in the perturbation expansion to become too small, or even zero. We also recall that these denominators become small when the system is near some sort of resonance in the various individual frequencies that it has at its disposal. From a mathematical point of view these resonances occur when the ratio of the individual frequencies is a rational number, which allows the system to be truly periodic, as opposed to simply being periodic in the individual projections.

In the same way we now consider the general, perturbed twist map, given in Eq. (1.3), subject to the area-preserving constraint specified in Eqs. (1.5c); alternatively, we may simply suppose that we are looking at one of the versions of the (perturbed), radial twist map specified in Eq. (1.8), or

(1.9b) or even (1.9d), all of which are already arranged so as to be explicitly area-preserving. We begin with the unperturbed version, Eq. (1.1). In that case the quantity J is fixed, so that we may plot the points generated by the map as a circle, of constant radius J and angles ϕ_n around that circle. There is then surely some (initial) value of J, say J_r, such that the function $\alpha(J_r)$ is a rational number, say, j/k:

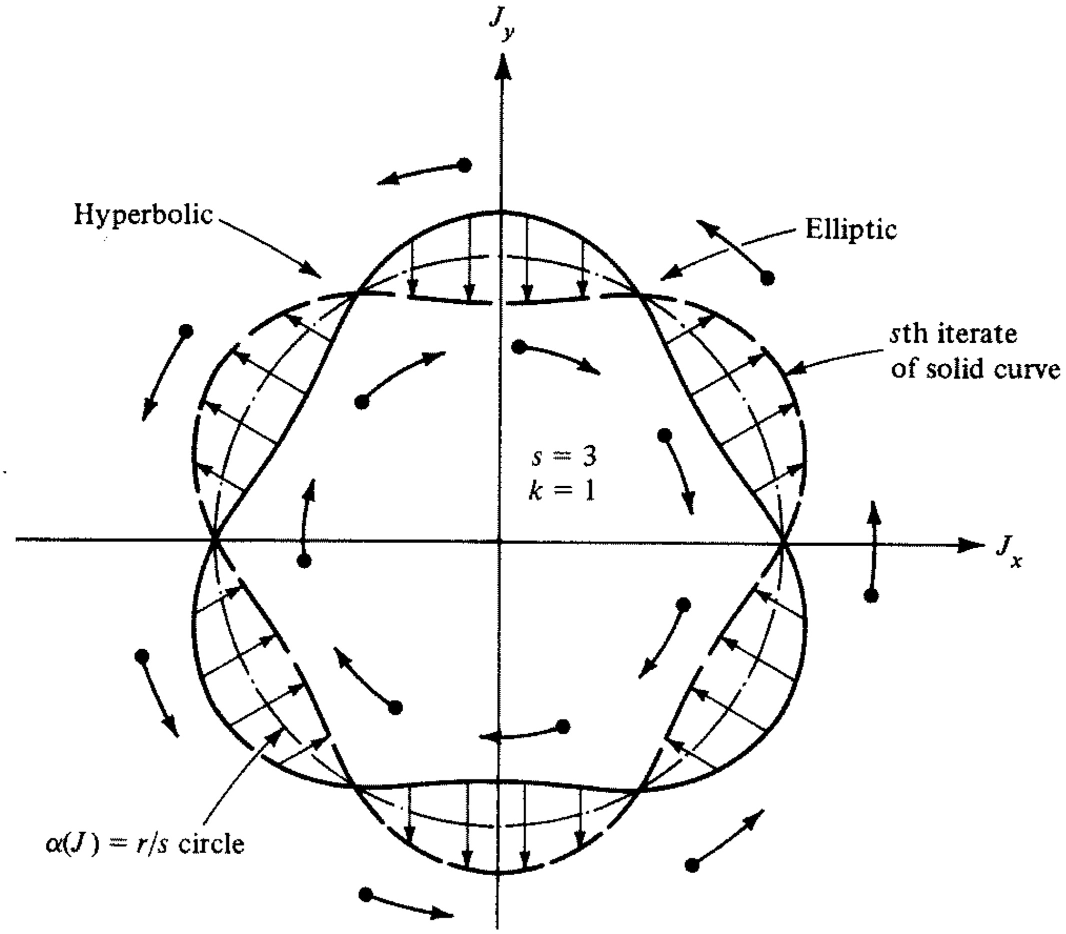
$$\alpha(J_r) = j/k \ . \tag{1.14}$$

Referring to the map itself as Z_ϵ, then that map takes some particular initial value, ϕ_0, and moves it around the circle, in either a clockwise or counterclockwise direction. Acting with the map again will push this new point further yet, in the same direction, so that the k-th power of Z_0 surely returns it back to its original value; i.e., acting on that initial point, the range of the map is some k individual points on that circle, to which the map returns over and over again, as one acts with it yet more and more times. As we vary that initial point, for instance over the piece of the circle that lies between the original point and its first-mapped point, the set of the ranges of all those individual maps will plot out the entire circle, k points at a time. This is the same as saying that the k-th power of the map, Z_0^k, simply sees this entire circle as a set of fixed points, leaving it unchanged under its action. We will refer to this circle as **an invariant curve** for the original map, Z_0, in the sense that the curve is simply mapped into itself, and will now label this particular circle as C. Of course each point on this curve is invariant under the action of Z_0^k, the k-th power of the original map.

Still studying the unperturbed map, we now consider values of J which are only slightly different from J_r. For points on that circle, the k-th power of the map will rotate the points of the circle either clockwise or counterclockwise. We choose to normalize things so that circles with radius determined by $J_+ > J_r$ are mapped counterclockwise by Z_0^k, while those with $J_- < J_r$ are mapped clockwise by Z_0^k.

We may now progress to the case where we have some small non-zero value for the perturbation parameter, ϵ, and consider the behavior of points in phasespace under the mapping Z_ϵ^k. As the mapping is a continuous function of ϵ, the circles described above must now be mapped into other closed curves. In particular, for sufficiently small ϵ, there must exist a closed curve, which we label as C_ϵ, which lies between the (unperturbed) circles associated with J_+ and J_- above, which is such that Z_ϵ^k leaves invariant all the values of ϕ, changing only the values for J, so that the circle is mapped in only radial directions. Since the map is area-preserving the area interior to this curve C_ϵ must be the same as the area inside the original circle, C, which means that they must surely intersect. (If they did not intersect, then C_ϵ would either lie entirely inside or entirely outside C and could not have the same area.) As they are both closed curves, they must intersect an even number of times. At each of these intersections the value of J on C is the same as the value on C_ϵ, and of course the values of ϕ are also the same. Therefore **each of the intersection points must therefore be a fixed point for Z_ϵ^k**, and it is only these points on C_ϵ that are fixed points for Z_ϵ^k. More generally, as we consider the action on our entire curve of Z_ϵ^k we know that it leaves fixed all the values of ϕ, so that an arbitrary point on the curve will either move radially outward, to larger values of J, or radially inward, to smaller values of J, leaving fixed our special fixed points. Obviously between any two such fixed points, the motion will all be in the same direction, either inward or outward, so that we can visualize this motion as sort of a wave—under the action of Z_ϵ^k—with the $2sk$ points held fixed, and oscillation inward or outward between them.

Given such an intersection the action of Z_ϵ^j, for $1 \le j \le k-1$, on that point will again generate a fixed point for the k-th power, so that there are a total of k such fixed points when any one is given. Of course, as already noted they come in pairs. In fact, for each such pair, one is an elliptic fixed point while the other is a hyperbolic one, as can be seen by the behavior as one changes α slightly, with some points moving toward the one for $\epsilon = 0$ and the alternate ones away. Therefore we actually have $2k$ such points. However, this may not be all of them; there might yet be another set entirely, for a total of s independent such points, each of which will generate $2k$ other, associated ones, which gives us a grand total of $2sk$ fixed points, where $s \geq 1$ is a number not given to us at the moment. It is the content of *Poincaré-Birkhoff Theorem* that all this is true for some non-zero neighborhood of values of ϵ for an arbitrary, area-preserving map that began from an integrable one.



Moving onward now to the behavior of the trajectories in the near neighborhood of these fixed points, our study of perturbations near elliptic fixed points convinces us that there will be small neighborhoods of elliptic orbits near those elliptic fixed points. In general, at least to some level of accuracy, for some sufficiently small value of ϵ, there will be a "last" such, retained elliptic orbit, which we refer to as a KAM surface. We will come back to that later.

The neighborhood of the hyperbolic points is however much more complicated. We first recall the situation for the case of the simple pendulum, where there are only two singularities; one is elliptic and one is hyperbolic. At that hyperbolic fixed point, some four curves are joined there, corresponding to the trajectories coming into the point, from either side, and those leaving that point, toward either side. Of course the incoming trajectories take infinitely much time to arrive. In the more general case, we anticipate that at any hyperbolic singularity, four curves join, corresponding to the two outgoing trajectories. We will refer to those trajectories as the stable and unstable manifolds corresponding to that singularity, where a definition could be that

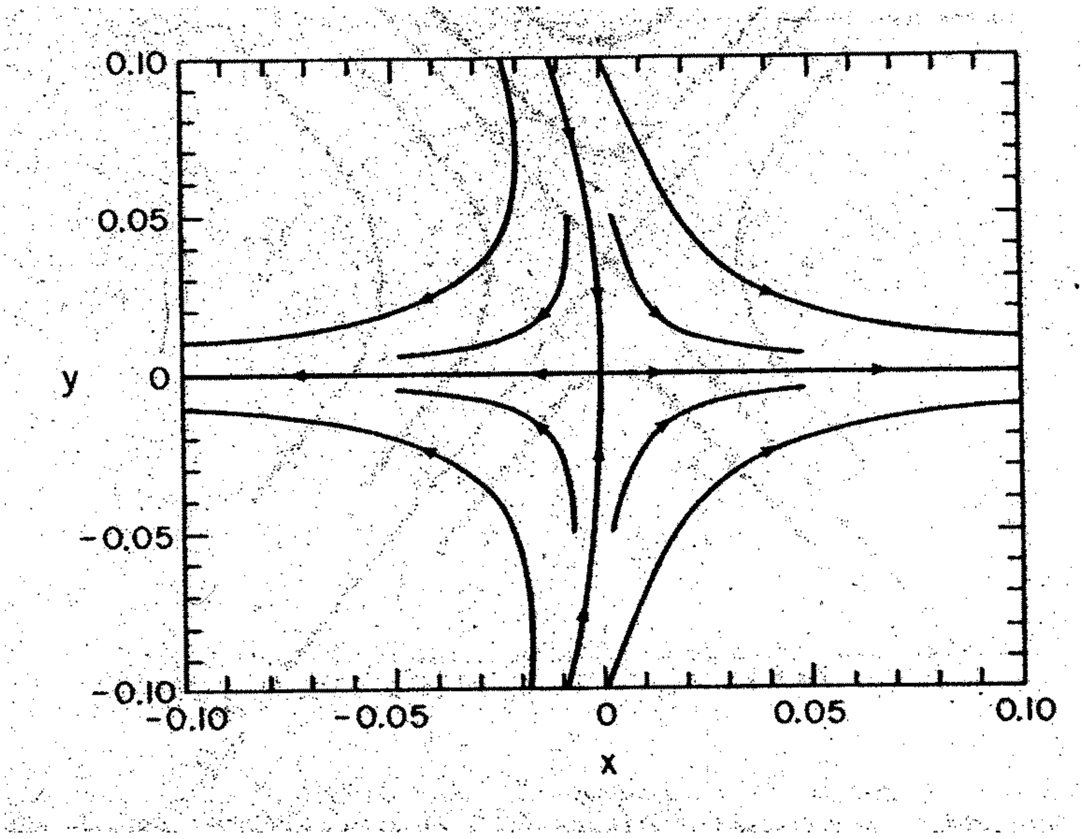
a.) a point lies on the stable manifold if the limit as n goes to infinity of the action of Z_ϵ^n on that point is the hyperbolic fixed point, i.e., if the point will eventually get there; while

b.) a point lies on the unstable manifold if the limit as n goes to infinity of the action of Z_e^{-n} on that point is the hyperbolic fixed point, i.e., if the point originated there, perhaps in the infinite past.

It is worth noting that the map must have an inverse since it comes from a Hamiltonian system, which can always be considered as running backwards in time, if desired. In this more generic case there can be more than one of each of the types of fixed points. In the very near neighborhood of a fixed point the linearization procedure provides good data, so that we expect the stable and unstable manifolds to enter into the join, at the fixed point, along lines that are approximately straight. The two parts of each will come in along the same angle, but in opposite directions; on the other hand, they will have some non-zero direction relative to one another. The graph below is from Dragt and Finn, "Insolubility of Trapped Particle Motion in a Magnetic Dipole Field," J. Geophys. Res., **81**, 2327-2340 (1976), which shows this situation for a simple, nonlinear, area-preserving map which they chose for ease of presentation, and named the Cremona map:

$$L: \quad \begin{cases} x_{n+1} = \lambda[x_n + (x_n - y_n)^2] \, , \\ y_{n+1} = \lambda^{-1}[y_n + (x_n - y_n)^2] \, . \end{cases} \tag{1.15}$$

This map has a fixed point at the origin, which will be hyperbolic when $\lambda > 1$.



The figure is drawn for the case $\lambda = 3$ as is clearly rather near the origin. The 4 curves that join at the origin are the stable and unstable manifolds for that fixed point. The other curves are invariant curves for nearby points, showing the hyperbolic nature of this fixed point.

The general situation is then as follows. If we continue to apply the map to points on either of these manifolds, then the curves will be extended further and further away from the fixed point, and therefore further and further away from the linear regime. This process does not of course modify the curves, but simply gives us more numerical data about them, since they are invariant curves under any power of our map. As this extension is performed, there are generically two possibilities:

1.) They may go off toward a different hyperbolic fixed point, to join there with, possibly, some other such trajectories, or

2.) they may eventually head back toward this same hyperbolic fixed point.

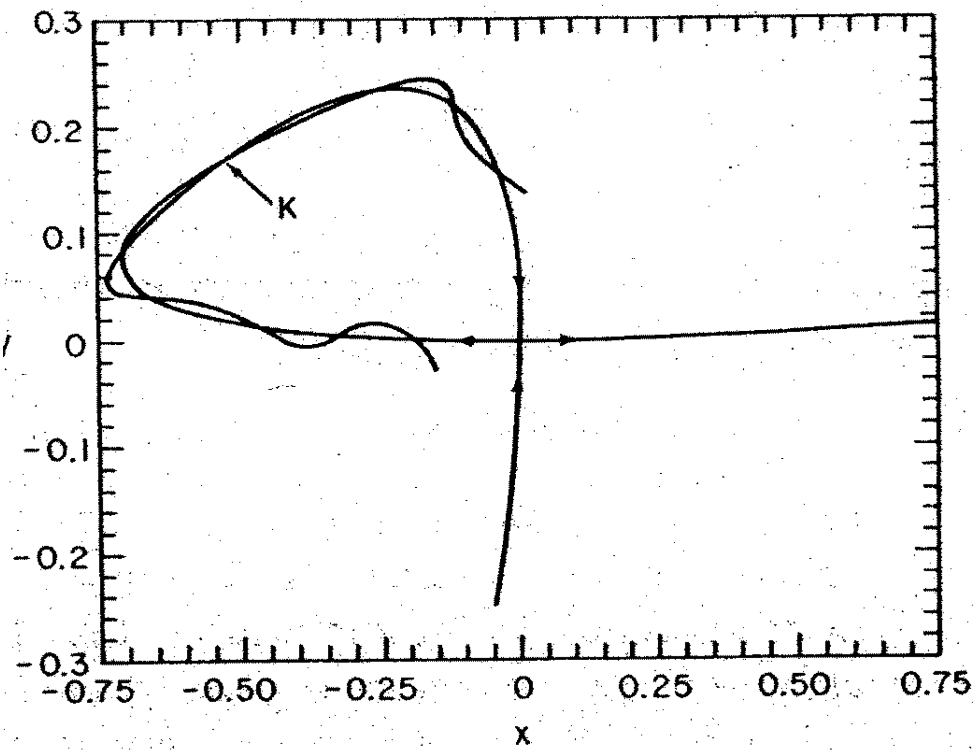
In either of these cases they are then very far from the linear regime so that, except in very special (completely integrable) cases such as the pendulum, they are very far from the linear regime, and may therefore have the possibility of doing quite exotic things. Indeed they might intersect one another. We stop for a moment and note that the invertibility of our maps prevents a manifold of either kind from intersecting with itself: Under such circumstances it would contain a loop. Under the action of the map the curve must be mapped into itself, but not that part of the curve that constitutes just the loop; therefore, the loop would be mapped into a portion of the curve with two endpoints. However, the inverse of that map would then be mapping those two distinct points into the single point where the intersection is, violating the continuity of the map. As well, two manifolds of the same kind but coming from, or going to, two different (hyperbolic) fixed points cannot intersect, since in the limit as n goes to either $+\infty$ or $-\infty$ the intersection point would have to approach both of the two distinct fixed points. **However,** in fact there is nothing to prevent unstable manifolds from intersecting with stable manifolds, although of course it does not happen in all examples, as we already know from the example of the pendulum. When such an intersection does occur, we, first of all, label them differently, depending on the two different ways in which it can occur:

1.) An intersection of a stable and an unstable manifold of a single (hyperbolic) fixed point is referred to as a *homoclinic point*; while

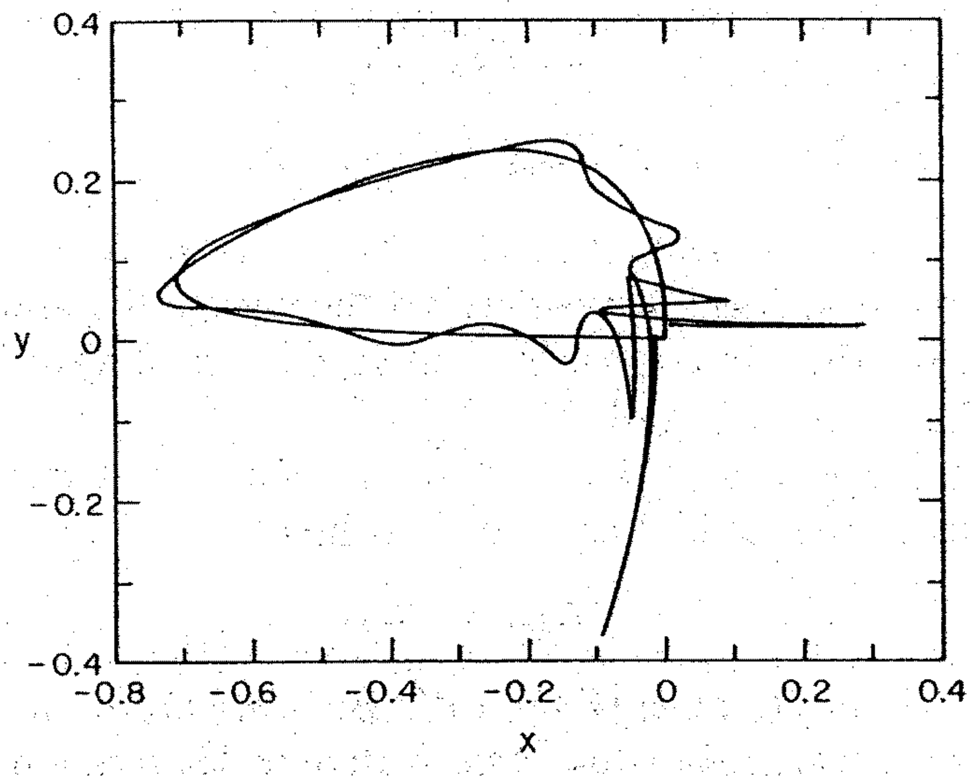
2.) an intersection of a stable and an unstable manifold pertaining to two distinct (hyperbolic) fixed points is referred to as a *heteroclinic point*.

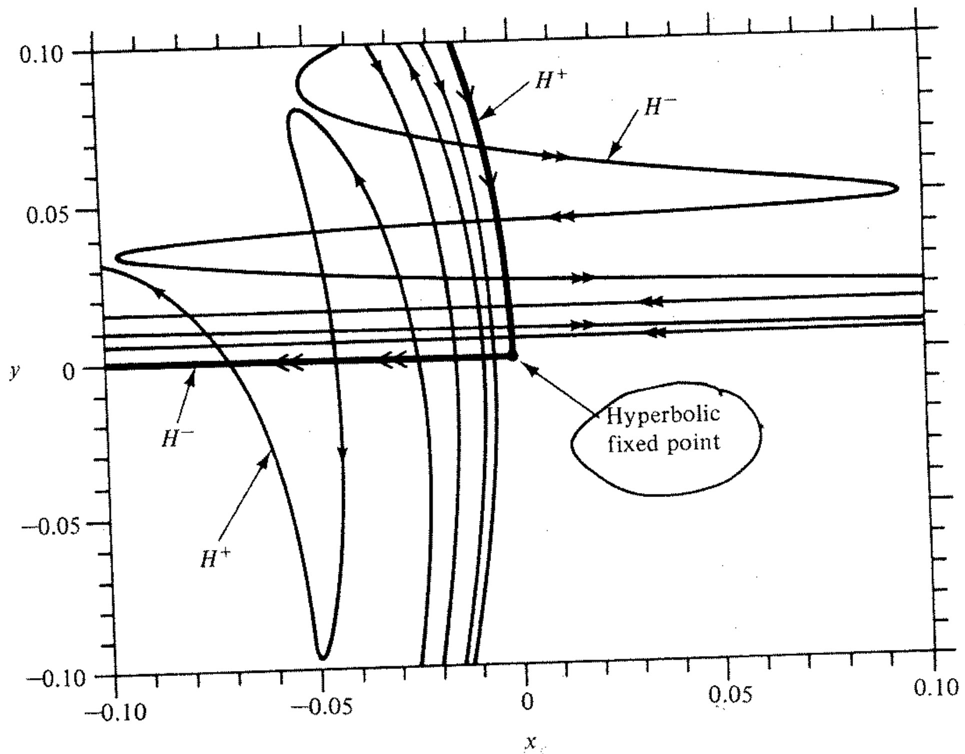
In the event that there is such an intersection then there must be more: Let K be such an intersection point of the two curves. As both of these curves are invariant curves, then surely the mapped point $L(K)$ also lies on each of the curves, somewhat further away from its limit point in the one case, and somewhat closer to its limit point in the other case. Therefore, there is another intersection point. The same argument says there are actually infinitely many more such intersection points. However, now pick two (adjacent) such intersection points, and consider the area between the two curves between those two points. Now pick another pair of (adjacent) intersection points; the area between them must be the same as the earlier area chosen, since the earlier area is mapped into this new area by an area-preserving map.

As we map the intersection point into the next one, by using additional copies of L, say, we are pushing the point closer and closer to the fixed point along the stable manifold, and further and further away along the unstable one. Therefore in the one case we will be nearing the regime where the linear approximation is valid and the manifold must be approximately straight. Nonetheless, there must be an infinite number of intersections of the other curve with this approximately straight one, and before the two get to the fixed point. This can obviously only happen if the other one, the non-straight one, makes more and more serious oscillations as it approaches the fixed point. On the other hand, since the area between any two adjacent intersections must be the same, but they are becoming closer and closer together, it must be so that the amplitudes of these oscillations, away from the other curve, say, must become ever larger. The following figure is an exact, numerical calculation taken from Dragt's paper, of the behavior of these manifolds coming out of the fixed point at the origin for the Cremona map. Notice that the area, in the horizontal and vertical directions, is much larger than that displayed in the earlier figure:



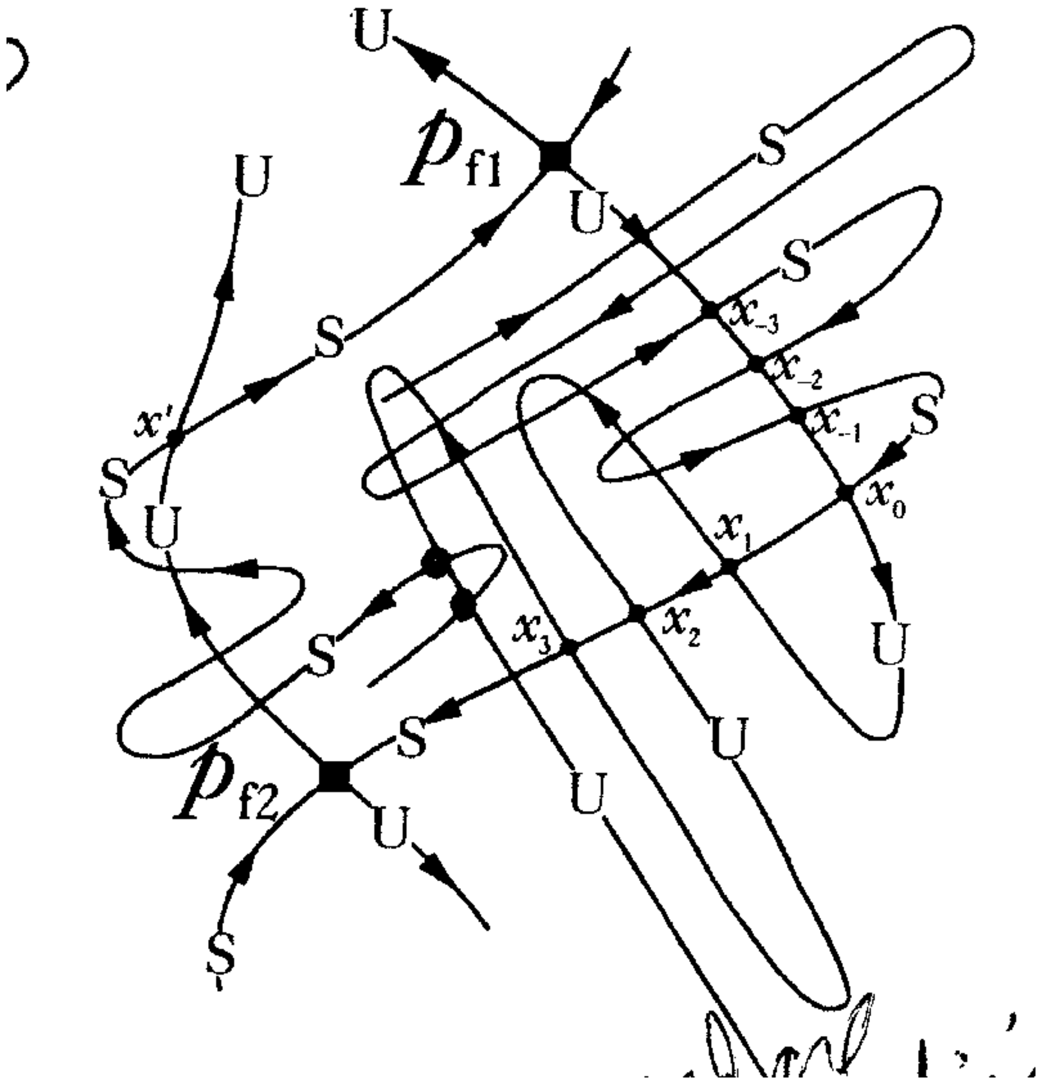
The graph is somewhat difficult to follow; nonetheless, try hard to follow the upward- and/or leftward-moving branch of the stable and unstable manifolds, respectively, that are shown there, now out to values of ± 0.75 instead of the earlier values of ± 0.10 used in the approximately linear map in the earlier figure. We see that each of these particular branches gets some distance away and then turns back, to "circle around" and return to their own fixed point. About the middle of that turning, they begin to become oscillatory, intersecting often with the other one, while it is still in a fairly steady behavior. The subsequent two graphs are similar, with approximately the same area of display as before, but the mappings have been pushed considerably further, so that one can see more and more intersections, with larger and larger oscillations.



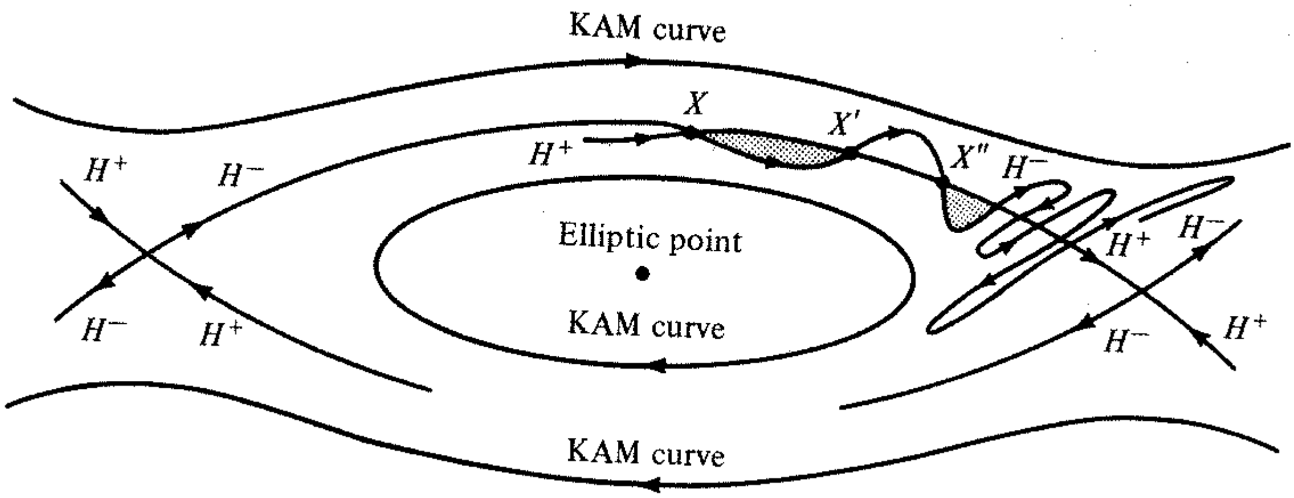


It is probably worth remembering that these are **not** actual trajectories, but, rather curves which contain the intersections with the Poincaré section for all possible trajectories. Any particular trajectory, with a fixed initial condition, hits only the points ascribed for it by the action of repeated copies of the map. These are infinitely many points, along any one of these curves, all approaching closer and closer to the fixed point. However, this is sensible because we know that a trajectory headed toward the fixed point requires an infinite time in order to actually arrive there. This tells us that, in the generic, non-integrable case, the intersections oscillate at ever wider amplitudes away from "the mean" as it does approach that fixed point. This surely begins to sound like something like an approach to chaos, so that very similar initial conditions will be quite far from each other as they approach closer and closer to this fixed point, i.e., as more and more time has passed.

All of these are homoclinic intersections. I do not have any real calculations, as those were, for heteroclinic intersections. Nonetheless, below is a figure from José, his Fig. 7.41d, showing "sketches" of what such things should look like, from two distinct hyperbolic fixed points, labeled pf1 and pf2.



In particular, it is quite possible that the wild oscillations of two curves involved in heteroclinic intersections may well cause them to oscillate so far that they will also join into homoclinic intersections closer to home; such a possibility is shown in this figure by two nearby, unlabeled dots, near x3 in the lower, left quadrant of the figure. Additionally the following figure is taken from Lichtenberg which shows the basic ideas involved there, which are not actually much different. In this diagram he also shows three typical KAM curves that might well exist, outside of the neighborhood where these wild oscillations are occurring.



(a)

Lastly, we should note that we have already stated that the stable and unstable manifolds may not cross themselves, nor their brethren; however, they also may not cross the (stable) elliptic orbits, which therefore form *barriers* to the oscillation of the manifolds for the hyperbolic points.