The Pendulum as a Useful Example: Action-Angle Variables and Perturbation Theory

1. The Simple Pendulum: Setup and Motion in Terms of Action Angle Variables

We consider a simple pendulum, of mass m and length ℓ, swinging under gravity from a single, fixed point at one end, assuming all the mass at the other end. Measuring an angle, θ, upward from its rest position hanging straight down, to whatever location it happens to have at some time t, we may write the following, where we normalize the potential energy so that it is zero when the pendulum is at right angles to that rest position, i.e., when $\theta = \pi/2$. However, since we intend to use this as a demonstration example for one degree of freedom, we will refer to the angle θ as q:

$$\begin{aligned} & V = -mg\ell\cos q\,, \quad E = \frac{1}{2}m\ell^2\dot{q}^2 + V\,, \\ \Longrightarrow \quad p = m\ell^2\,\dot{q} \quad & \Longrightarrow \quad H = H(q,p) = \frac{p^2}{2m\ell^2} - mg\ell\cos q = \tfrac{1}{2}G\,p^2 - F\cos q\,, \\ & \Longrightarrow \quad \begin{cases} \dfrac{d}{dt}q = \dfrac{\partial H}{\partial p} = \dfrac{p}{m\ell^2} \equiv G\,p\,, \\ \dfrac{d}{dt}p = -\dfrac{\partial H}{\partial q} = -mg\ell\sin q \equiv -F\sin q\,, \end{cases} \end{aligned} \tag{1.1}$$

where we have given shorter names to the arrangements of constants that show up in the problem.

In principle we can resolve this problem by direct integration, involving elliptic functions. One approach gives the equations for the motion in the form:

$$\left.\begin{aligned} \sin(q/2) &= k\ \mathrm{sn}(u;k)\,, \\ \tfrac{1}{2}\,p &= \sqrt{F/G}\,k\ \mathrm{cn}(u;k)\,, \end{aligned}\right\} \quad u \equiv \omega_0\,t + \zeta\,, \quad E \le F\,,$$

$$\left.\begin{aligned} \sin(q/2) &= \mathrm{sn}(u;1/k)\,, \\ \tfrac{1}{2}\,p &= \sqrt{F/G}\,k\ \mathrm{dn}(u;1/k)\,, \end{aligned}\right\} \quad u \equiv \omega_0\,k\,t + \zeta\,, \quad E \ge F\,, \tag{1.2}$$

$$\omega_0 \equiv \sqrt{FG}\,, \quad k \equiv \sqrt{\tfrac{1}{2}(1 + E/F)} = \sqrt{\tfrac{1}{2}(1 + E/mg\ell)}\,,$$

where ω_0 is the "natural" frequency in the problem, i.e., the one it would have if the system responded linearly, and ζ is simply a constant of integration to be determined by initial conditions, and $\mathrm{sn}(u;k)$, etc. are the (usual) Jacobi elliptic functions.

On the other hand, it is conceivable that your knowledge of the elliptic functions is not everything I might wish it were, and we want to use this as an example to better see how to do such things when the integrals are even more difficult to perform. Therefore, we now proceed to finding the appropriate action angle variables.

1. The first step is to solve the equations so as to obtain $p = p(q,Q)$. In our case the only Q we have is the energy, E, which is the same as the Hamiltonian, so we have

$$p = \pm\sqrt{\frac{2}{G}(E + F\cos q)}\,. \tag{1.3a}$$

2. We may then find the time-independent HJ generating function, which depends on q and Q:

$$W = \int dq\, p \;=\; \int_0^q dq \sqrt{\frac{2}{G}(E + F\,\cos q)}\;. \tag{1.3b}$$

3. Along the same lines we find the action, by performing the same integral, but by doing it over either an entire cycle, if the motion is closed, or over a period it the motion is simply periodic in the coordinate. For our problem, the determining factor is the energy, which is either greater than or less than F:

$$J = \frac{1}{2\pi}\oint dq\, p = \frac{1}{2\pi}\oint dq \sqrt{\frac{2}{G}(E + F\,\cos q)} = \frac{2}{\pi}\int_0^{q_{\rm max}} dq \sqrt{\frac{2}{G}(E + F\,\cos q)}\;, \tag{1.3c}$$

where $q_{\rm max}$ will be either π for a rotational motion, i.e., an unbounded but periodic one, or $\cos^{-1}(-E/F)$ for a libratory one, i.e., a bounded motion.

4. In order to determine the angular coordinate, we need $\widetilde{W} = \widetilde{W}(q, J) = W[q, Q(J)]$, which we so far have only in the form on an integral. Nonetheless, we may proceed from that as follows, since Q depends only upon J and not q:

$$\begin{aligned}\phi = \frac{\partial \widetilde{W}}{\partial J} = \frac{\partial}{\partial J} W[q, Q(J)] = \frac{\partial Q}{\partial J}\frac{\partial W}{\partial Q} = \left(\frac{\partial J}{\partial E}\right)^{-1} \frac{\partial}{\partial E}\int_0^q dq \sqrt{\frac{2}{G}(E + F\,\cos q)}\\ = \frac{1}{G\,(\partial J/\partial E)}\int_0^q dq\, \frac{1}{\sqrt{\frac{2}{G}(E + F\,\cos q)}}\end{aligned} \tag{1.3d}$$

where we have switched, part-way through, from Q to E because that it is the only relevant Q for this problem with only one degree of freedom, and also we have inverted the derivative because it is already J that we know, at least via the quadrature above in the form of a definite integral, as a function of the constant E. These two equations give us $J = J(Q)$ and $\phi = \phi(q, Q)$; we may of course resolve the first equation to obtain $Q = Q(J)$ and insert it into the form for ϕ if desired.

5. We may then determine the frequency of the motion in ϕ by calculating $\partial H/\partial J$. As H is just E, this is really just the inverse of the derivative we already needed to determine ϕ, namely $1/\omega = \partial J/\partial E$. We will characterize it by comparing it to the "natural" frequency in the problem, namely ω_0, as given in Eqs. (1.2), where in this context it is reasonable to look at ω_0 as *the frequency for linearized oscillation about the elliptic singular point, at* $(q, p) = (0, 0)$:

$$\frac{\omega_0}{\omega} = \sqrt{FG}\,\frac{dJ}{dE} = \frac{2}{\pi}\sqrt{\frac{F}{G}}\int_0^{q_{\rm max}} \frac{dq}{\sqrt{\frac{2}{G}(E + F\,\cos q)}}\;. \tag{1.3e}$$

Remembering now the basic structure of the phasespace for the pendulum, we recall that it has this *elliptic singular point* at $(0, 0)$ and a hyperbolic singular point at $(\pm\pi, 0)$ (where proper identification of the points in the phasespace to account for the angular nature of the coordinates just identifies these two coordinate presentations as the same point). There are elliptic trajectories in phase space surrounding the elliptic point, separatrix solutions that go through the hyperbolic point and which separate the elliptic trajectories from those which are hyperbolic (unbounded) trajectories lying on the other side. When we consider the limit toward the separatrix trajectories from the unbounded side, we have $q_{\rm max} = \pi$, then in the limit as $E \to F$ it is clear that the integral, for $\omega_0/\omega \propto T$, becomes infinite, where T is the period of the motion, as is desired for motion along a separatrix. Contrariwise, in the limit from the elliptic side, we simply have the limits in the opposite order since E is increasing up to F, so that $q_{\rm max}$ is becoming defined by $\cos(q_{\rm max}) = -1$, with the same result. We can obtain the limiting equations without too much effort by simply inserting $E = F$ into the original equations. Beginning with Eq. (1.3a) for $p = p(q,E)$, and denoting the variables along the separatrix trajectory by a subscript x, we have the following, where we have started the motion with $\phi(0) = 0$:

$$\begin{aligned} p_x = \ &\pm \frac{\omega_0}{G}\sqrt{2(1+\cos q)} = \pm 2\frac{\omega_0}{G}\cos(q/2) \\ \frac{d}{dt}q_x = Gp_x \implies \quad &\omega_0\, t = \log\{\tan[(q_x+\pi)/4]\} \quad \implies \quad q_x = -\pi + 4\tan^{-1}(e^{\omega_0 t}) \ , \end{aligned} \tag{1.4}$$

from which we can see the expected behavior of the motion along the separatrix, i.e., that it takes infinite time to move to the singular point at $\pi = \phi(+\infty)$, i.e., it is moving very slowly when near the singular point. It is reasonable to expect, and also true, that motion along curves very near the separatrix itself have similar behavior. In particular, when we have more degrees of freedom so that *a resonance* between the frequencies can occur, there will always be some (new) variable that is a very slowly varying function of the time. For instance, suppose that the two frequencies ω_1 and ω_2 is near a rational number, say r/s; then the new variable $r\phi^1 - s\phi^2$ will be such a slowly varying angle. If one then averages over the fast moving angles, the resulting problem will be 1-dimensional, and very much like the pendulum one now being studied.

Finally, to acquire more details we will have to actually determine at least the integral form for $J(E)$. Lichtenberg and Lieberman give the value of these integrals, in terms of elliptic integrals of the first and second kinds, which are basically inverse functions of the Jacobi elliptic functions:

$$\begin{aligned} J &= \frac{8}{\pi}\sqrt{\frac{F}{G}}\begin{cases} E(k) - k'^2\, K(k)\ , & k<1\ , \\ \tfrac{1}{2}k\, E(1/k)\ , & k>1\ , \end{cases} \\ \phi &= \frac{\pi}{2}\begin{cases} \dfrac{1}{K(k)}\, F(\eta,k)\ ,\ k<1\ , \\ \dfrac{2}{K(1/k)}\, F(q/2, 1/k)\ ,\ k>1\ , \end{cases} \end{aligned} \tag{1.5a}$$

where we have appropriate definitions:

$$\begin{gathered} u = F(w,k) = \mathrm{sn}^{-1}(\sin w) \quad , \qquad k\,\sin\eta \equiv \sin(q/2)\ , \\ K(k) \equiv F(\pi/2,k) = \int_0^{\pi/2}\frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}}\ , \quad E(k) \equiv E(\pi/2,k) = \int_0^{\pi/2} d\alpha\,\sqrt{1-k^2\sin^2\alpha}\ . \end{gathered} \tag{1.5b}$$

Using these same integrals, we may also determine the frequency more explicitly, from Eq. (1.3e) above:

$$\frac{\omega_0}{\omega} = \frac{2}{\pi}\begin{cases} K(k) \ , & k\le 1\ , \\ (1/2k)K(1/k)\ , & k\ge 1\ . \end{cases} \tag{1.5c}$$

It is also useful to have an understanding of the behavior of the first complete elliptic integral for arguments, k, that are not too large, and not too far from 1:

$$
\begin{aligned}
K(k) &= (\pi/2)[1 + k^2/4 + 3(k^2/8)^2 + \ \dots \ + \left[\frac{(2n-1)!!}{2^n n!}\right]^2 \frac{k^{2n}}{2n-1} + \ \dots \ , \\
K(k) &\underset{k\to 1}{\longrightarrow} \log(4/\sqrt{1-k^2}) + \tfrac{1}{4}\Big(\log(4/k') - 1\Big)(1-k^2) + O(k'^4) \ , \quad k' \equiv \sqrt{1-k^2} \ .
\end{aligned}
\tag{1.5d}
$$

2. The quartic oscillator, and secular perturbation theory:

We now re-consider the pendulum Hamiltonian, but expand the cosine in a series and maintain only the first three terms; if, as well, we set $\ell = 1$ and throw away the constant, firs term in the expansion of the cosine, we have

$$
\begin{aligned}
H = p^2/2m + \tfrac{1}{2} m\omega_0^2 q^2 + \tfrac{1}{4}\epsilon m q^4 &\equiv H_0 + \epsilon\, H_1 \ , \\
\Longrightarrow \quad \ddot{q} + \omega_0^2\, q + \epsilon\, q^3 \ .
\end{aligned}
\tag{2.1}
$$

Here we have taken the coefficient of q^4 so that it has an explicit form, with $\epsilon = -g/6$. We will use this ϵ as a way of keeping track of terms in our perturbative expansion, and may re-set it back to $g/6$ when that is convenient.

Of course we understand the problem of motion when ϵ is zero, i.e., when we just have the system reacting as the standard simple harmonic oscillator. However, let us see what happens when it is not. We **assume** that the solution may be written as a power series in ϵ:

$$
q = q(t) = q_0(t) + \epsilon\, q_1(t) + \epsilon^2\, q_2(t) + \ \dots \ . \tag{2.2}
$$

It is straightforward to insert this into the equations of motion and separate out quantities term by term:

$$
\begin{aligned}
\ddot{q}_0 + \omega_0^2\, q_0 \ &= 0 \ , \\
\ddot{q}_1 + \omega_0^2\, q_1 \ &= -q_0^3 \ , \\
\ddot{q}_2 + \omega_0^2\, q_2 \ &= -3\, q_0^2 q_1 \ , \\
\ddot{q}_1 + \omega_0^2\, q_1 \ &= -3q_0(q_1^2 + q_0 q_2) \ , \\
\ddot{q}_4 + \omega_0^2\, q_4 \ &= -(6q_0q_1q_2 + 3q_0^2q_3 + q_1^3) \ , \\
\dots \qquad & .
\end{aligned}
\tag{2.3a}
$$

We need boundary conditions in order to solve even this; therefore, let us look for that solution which has value $q(0) = a$ and $\dot{q}(0) = 0$, i.e., it starts out from rest at location a. To make this work with the above series we now take

$$
q_k(0) = \delta_k^0\, a \ , \quad \dot{q}_k(0) = 0 \ . \tag{2.3b}
$$

We then proceed to solve the equations in the most naïve way possible. First, we have the well-understood SHO:

$$
q_0(t) = a\cos\omega_0 t \ . \tag{2.4a}
$$

We insert this value into the second equation, causing it to become inhomogeneous, and find the particular solution that fits the boundary conditions:

$$q_1(t) = -\frac{a^3}{32\omega_0^2}\Big[\cos\omega_0 t - \cos 3\omega_0 t + 12\omega_0 t\,\sin\omega_0 t\Big]\ . \tag{2.4b}$$

It is obvious that this is **not** a good approximation to the behavior of the solution of this equation, at least for long times, since it grows more or less linearly with $\omega_0\, t$. It might well be that this is simply the first term in some sort of a series expansion of a periodic function that should multiply that term, just as the expansion of $\cos\epsilon t$ would have terms like that in it, were it expanded in a power series in its argument. However, this says that this is not a good approach to finding an approximate solution to the dynamics.

Mathematically the reason for the problem is that there is a "resonance" between the frequency of the inhomogeneous term, which can be thought of as a "driving term," and the natural frequency of the homogeneous equation; in this case they are exactly equal, although such a problem would occur even if their frequencies were commensurate, i.e., if their ratio was a ratio of integers. On the other hand, the physical reason for the problem is that we have acted as if the frequency of the exact solution, which we are trying to approximate, was (at least) a harmonic multiple of the original frequency; it is very unlikely that this is the case, as we can verify by thinking on our discussion of the full pendulum equation. From whichever point of view, these terms are bad for our solution; they are usually referred to as *secular terms*, a name which refers to terms which, on average, increase linearly with time.

An approach to dealing with such problems was invented by Lindstedt, in 1882. Since we agree that the frequency must also depend on ϵ in some way; we expand the frequency in powers of ϵ, and we insist that the (desired) solution be periodic with that frequency by considering it, now, as a function of $\tau \equiv \omega\, t$:

$$q = q(\omega t) = q_0(\omega t) + \epsilon\, q_1(\omega t) + \epsilon^2\, q_2(\omega t) + \ \dots\ \ , \qquad \omega = \omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \ \dots\ \ . \tag{2.5}$$

We may then insert both these expansions into our original equation, remembering that the original ω_0 in it, multiplying q, has not changed and also that we use q' to mean $dq/d\tau$. We then collect it out in powers of ϵ, and then use the pair of terms at each level to eliminate terms caused by resonances, i.e., terms that are secular:

$$\begin{aligned}
\omega_0^2\Big[q_0'' + q_0\Big] &= 0\ ,\\
\omega_0^2\Big[q_1'' + q_1\Big] &= -\,q_0^3 - 2\omega_0\omega_1\, q_0''\ ,\\
\omega_0^2\Big[q_2'' + q_2\Big] &= -\,3q_0^2 q_1 - 2\omega_0\omega_1 q_1'' - (\omega_1^2 + 2\omega_0\omega_2) q_0''\ ,\\
&\dots\qquad .
\end{aligned} \tag{2.6a}$$

The solution of the first equation is of course the same as before: $q_0(\tau) = a\cos\tau$, which gives us the differential equation for q_1 in the form

$$\begin{aligned}
\omega_0^2\Big[q_1'' + q_1\Big] &= -a^3\cos^3\tau + 2\omega_0\omega_1\, a\cos\tau \equiv h(\tau) = h[\omega_0 t + \epsilon\omega_1 t + O^2(\epsilon)] = h(\omega_0 t) + O(\epsilon)\\
&\equiv h(\tau_0) = -a^3\cos^3\tau_0 + 2\omega_0\omega_1\, a\cos\tau_0 = [2\omega_0\omega_1 - 3a^2/4]a\cos\tau_0 - (a^3/4)\cos 3\tau_0\ ,
\end{aligned} \tag{2.6b}$$

where the term in ϵ has been dropped in the second line, as required by the approximation. The term $\cos\tau_0$, which has the same frequency as the solution of the homogeneous equation, would be the cause of a secular term in the solution; therefore, we choose ω_1 to cause its coefficient to vanish, and, as above, think of q_1 as a function of τ_0:

$$\text{choose} \quad \omega_1 = \frac{3}{8}\frac{a^2}{\omega_0} \quad \Longrightarrow \quad \omega_0^2\Big[q_1'' + q_1\Big] = -\frac{a^3}{4}\cos 3\tau_0 \ . \tag{2.6c}$$

The solution of this that satisfies the boundary conditions is then straightforwardly

$$q_1 \;=\; q_1(\tau) = q_1(\tau_0) = \frac{a^3}{32\omega_0^2}[\cos 3\omega_0 t - \cos\omega_0 t] \ , \tag{2.6d}$$

where again we have dropped excess terms of higher order in ϵ. Therefore, the solution to the problem, at (only) first order in ϵ is given by

$$q(t) = a\cos[(\omega_0 + \epsilon\omega_1)t] + \epsilon\frac{\omega_1}{12\omega_0}[\cos 3\omega_0 t - \cos\omega_0 t] \ . \tag{2.6e}$$

This is a result that is obviously better than the earlier, naïve one which had secular terms increasing off toward infinity for large times. As has already been stated, the problem is one of resonance between the natural frequency and a driving term. Therefore, let us briefly look additionally at a situation where there is a time-dependent driving term, so that we will essentially have two degrees of freedom, which will make it easier to see how the resonant terms cause serious grief. We consider a driven, linear oscillator

$$\ddot{q} + \omega_0^2\, q = g(t) \ . \tag{2.7a}$$

Obviously this problem and the one we have just recently considered are both special cases of a more general situation where we would have taken $g(q,t)$ as a driving force. However, let us spend a little time now on this special case, where we assume the driving term is periodic with frequency Ω. The homogeneous solution is just the standard $A\cos\omega_0 t + B\sin\omega_0 t$. For comparison, then, we Fourier transform our periodic function g:

$$g(t) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}[a_n\cos(n\Omega t) + b_n\sin(n\Omega t)] \ . \tag{2.7b}$$

If we now assume that the **particular solution**, $q_p(t)$, of our problem has the same sort of time dependence, i.e., that it is an infinite sum of terms $q_p = \sum_{n=0}^{\infty} x_{pn}$, where q_{pn} is periodic with frequency $n\Omega$, then the insertion of this into the left-hand side of the differential equation would generate terms of the form $[-(n\Omega)^2 + \omega_0^2]q_{pn}$, so that we would obtain an equation insisting that

$$q_{pn} = \frac{a_n\cos(n\Omega t) + b_n\sin(n\Omega t)}{\omega_0^2 - n^2\Omega^2} \ . \tag{2.7c}$$

The resonance is very easy to see, which causes the solution to "blow up." When the oscillator, unlike this one, also has a nonlinear dependence, or, if you prefer, g depend nonlinearly on q, then all multiples of ω_0 will occur in the general solution, as we can see from our earlier excursion into such phenomena. Therefore, we will always have a difficulty whenever the ratio ω_0/Ω is a rational number. On the other hand, we also saw that, again unlike the SHO itself, for a nonlinear problem the actual frequency, ω, is a function of the initial conditions, and, in particular, the initial amplitude of the motion. **Therefore, by varying the initial conditions**, we can vary this ratio. As the rationals are dense among the real numbers, **the actual sets of initial conditions that causes resonances will also be dense among all possible initial conditions.** This tells us immediately that we can expect serious changes in the topology of the solution manifold in phasespace when we allow sufficiently many degrees of freedom, i.e., as many as two!

On the other hand, there is yet another problem with the secular perturbation theory as we have just described it. The derivation above showed that it is quite difficult to keep careful track of all the terms in ϵ, especially when they are contained within the argument in the form of $\tau = \omega t$. Therefore, a version which used the Hamiltonian formalism to keep better track of these dependencies was invented by Poincaré, in 1892, and pushed further by von Zeipel, in 1916, which is usually referred to as *canonical perturbation theory*. As did the secular perturbation above, it presupposes that the value of ϵ is given small, and fixed, while other things vary. We will describe the first-order version of it soon. On the other hand, more recently a somewhat distinct version of this theory has been created, based on the thoughts of Sophus Lie. It is based on the idea that the Hamiltonian always did depend on the parameter ϵ, and we should follow along a flow vector for which ϵ is the parameter. The canonical approach uses canonical generating functions, which depend on both old and new coordinates to determine the process; therefore the two become rather mixed up when a sequence of such transformations is performed, as is necessary in order to push the canonical process to higher orders in ϵ. On the other hand the Lie transformation process works directly with the ϵ-dependent canonical transformations themselves, rather than the generating functions, so that they give directly the new coordinates, in phase space, in terms of the old ones, and ϵ, thereby rendering multiple transformations much simpler in practice. It was introduced by Hori, in 1966 and Garrido in 1968, with important improvements toward the easier implementation of higher-order terms by Deprit, in 1969, and has become rather popular since then.

3. Canonical Perturbation Theory

We will first consider the canonical perturbation theory, with examples, and then go onward to the Lie transformation approach. We therefore presume that there is a given Hamiltonian which may be divided into a part which we can integrate to the end, and determine equations of motion, etc., and another part which we suppose is "small," at least under appropriate circumstances; we use notation to keep track of this smallness by appending a parameter ϵ, which may, after the end of the calculations, be set to some value which may or may not really be small all the time, but one hopes so. We write this Hamiltonian in the following form, where at least "most of the time," when one writes ξ^α, it is referring to the system having having already been put into the action-angle variable formalism, i.e., $\xi^\alpha \Longrightarrow (\phi^a, J^b)$

$$H(\xi,t,\epsilon) = H_0(\xi,t) + \epsilon\, H_1(\xi,t) + \epsilon^2\, H_2(\xi,t) + \ \dots \ . \tag{3.1}$$

The statement that we know how to resolve the unperturbed Hamiltonian, i.e., that which remains when we set $\epsilon = 0$, is the statement that

$$J_b = J_{b0}\,, \quad \phi^a = \omega_0^a\, t + \phi_0^a\,, \qquad \omega_0^a \equiv \partial H_0/\partial J_a\,, \tag{3.2}$$

where J_{b0}, ω_0^a and ϕ_0^a are all constants, independent of t. We are then going to **seek** a transformation to a new set of variables, $\eta^\alpha = \eta^\alpha(\xi,t,\epsilon) \Longrightarrow (\overline{\phi}^r, \overline{J}_s)$ for which the problem is more "tractable." More precisely, what we want is a transformation to a new set of variables so that the new Hamiltonian, $K = K(\eta,t,\epsilon)$ will in fact depend only upon the new action variables, $\overline{J}_r$, as would surely be the case if we had actually been able to resolve the dynamics of this full Hamiltonian. **This is an assumption, in advance, that this resolution is possible, which of course is certainly not always true!** Therefore, one additional goal of this work is to determine methods to find out when it is not true, or perhaps when it is not even possible. (One should definitely point out that the resolution is indeed possible in the case of motions in only one degree of freedom, as all of those are indeed integrable.)

At any event, we now agree that we consider doing it all up to some order in ϵ, for which we have agreed to solve the problem; we also agree not to ask whether or not the so-created series actually converges. (In many cases the series is actually an asymptotic one, so that it does not converge, but, at least, does give some reasonable approximation if one truncates it in the best way.) There is of course, in principle, a canonical transformation that generates this transformation. We will refer to it as $S(\phi, \overline{J}, t, \epsilon)$, and suppose that it is just the identity transform when $\epsilon = 0$. In principle the transformation it generates, from ξ to the new canonical set, $\eta = \eta(\xi, t, \epsilon)$, should show its t-dependence, as just previously. However, we will drop the explicit presentation of the t-dependence as often as seems feasible, in the interests of keeping the notation less cumbersome:

$$S = \overline{J}_r \phi^r + \epsilon S_1 + \epsilon^2 S_2 + \ \dots \quad , \tag{3.3a}$$

We may then write out the usual CT rules for generation of the opposite pair of canonical sub-variables:

$$\begin{aligned} J_a &= \frac{\partial S}{\partial \phi^a} = \overline{J}_a + \epsilon \, \frac{\partial S_1}{\partial \phi^a}(\phi, \overline{J}) + \epsilon^2 \, \frac{\partial S_2}{\partial \phi^a}(\phi, \overline{J}) + \ \dots \quad , \\ \overline{\phi}^s &= \frac{\partial S}{\partial \overline{J}_s} = \phi^s + \epsilon \, \frac{\partial S_1}{\partial \overline{J}^s}(\phi, \overline{J}) + \epsilon^2 \, \frac{\partial S_2}{\partial \overline{J}^s}(\phi, \overline{J}) + \ \dots \quad , \end{aligned} \tag{3.3b}$$

Having such a presentation we must also include the new Hamiltonian:

$$K(\overline{J}) = H[\phi(\overline{\phi}, \overline{J}), J(\overline{\phi}, \overline{J})] + S_{,t}[\phi(\overline{\phi}, \overline{J}), \overline{J}] \equiv K_0 + \epsilon K_1 + \epsilon^2 K_2 + \ \dots \quad . \tag{3.3c}$$

In this last equation we say that K does not depend on $\overline{\phi}$; however, we will still have to insist that it does not, at each level of the perturbation, rather than simply ignore that dependence as we go along. It will in fact be this insistence that allows us to determine the form for the various levels of the generating function.

To actually determine $K(\overline{J})$, we will need to invert Eqs. (3.3b) to obtain the old variables, $\{\phi, J\}$, in terms of the new. If we did in fact know S_1, S_2, etc., then this would be a coupled set of equations that are not completely trivial, because of the way they are mixed on both sides of the equation. Technically we would do it to some order in ϵ by iteration of the equation on itself. Showing how that iteration needs to work, we may write down the coupled equations in the following form:

$$\begin{aligned} J_a &= \overline{J}_a + \epsilon \, \frac{\partial S_1}{\partial \phi^a}[\phi(\overline{\phi}, \overline{J}), \overline{J}] + \epsilon^2 \, \frac{\partial S_2}{\partial \phi^a}[\phi(\overline{\phi}, \overline{J}), \overline{J}] + \ \dots \quad , \\ \phi^a &= \overline{\phi}^a - \epsilon \, \frac{\partial S_1}{\partial \overline{J}_a}[\phi(\overline{\phi}, \overline{J}), \overline{J}] - \epsilon^2 \, \frac{\partial S_2}{\partial \overline{J}_a}[\phi(\overline{\phi}, \overline{J}), \overline{J}] + \ \dots \quad . \end{aligned} \tag{3.3d}$$

Clearly this equations lend themselves to an iterative approach to solution, which could begin to become tedious after, at least, the second order.

We may now insert all this structure into the equation for K to determine equations for its parts to various levels of ϵ. The unperturbed Hamiltonian already does not depend on the original angle variables, although the higher-order terms do; therefore, in principle we should insert into those terms the dependence of ϕ^a on the set $(\overline{\phi}^r, \overline{J}^r)$. However, since we will be wanting the higher-order terms of the generating function in terms of ϕ^a, rather than $\overline{\phi}^r$, we do not need to make that substitution; instead, it will come out naturally during the process of determining the higher-order terms in the generating function. As well, we insert $J_a = J_a(\phi, \overline{J})$ as it is naturally given by Eqs. (3.3c):

$$\begin{aligned} K(\overline{J}) &= H_0[J(\phi,\overline{J})] + \epsilon\, H_1[\phi, J(\phi,\overline{J})] + \epsilon^2\, H_2[\phi, J(\phi,\overline{J})] + \;\ldots\; + S_{,t}[\phi,\overline{J}] \\ &= H_0(\overline{J} + \epsilon\, S_{1,\phi} + \epsilon^2\, S_{2,\phi} + \ldots\,) + \epsilon\, H_1(\phi, \overline{J} + \epsilon\, S_{1,\phi} + \ldots) \\ &\qquad + \epsilon\, H_2(\phi, \overline{J} + \epsilon\, S_{1,\phi} + \ldots) + \epsilon S_{1,t}(\phi,\overline{J}) + \epsilon^2 S_{2,t}(\phi,\overline{J}) + \ldots \\ &= H_0(\overline{J}) + \epsilon\left\{ H_{0,J_r}|_{J=\overline{J}} S_{1,\phi^r} + H_1(\phi,\overline{J}) + S_{1,t}(\phi,\overline{J})\right\} \\ &\quad + \epsilon^2 \Big\{ H_{0,J_r}|_{J=\overline{J}} S_{2,\phi^r} + H_{1,J_a}|_{J=\overline{J}} S_{1,\phi^a} \\ &\qquad + \tfrac{1}{2} H_{0,J_aJ_b}|_{J=\overline{J}} S_{1,\phi^a} S_{1,\phi^b} + H_2(\phi,\overline{J}) + S_{2,t}(\phi,\overline{J}) \Big\} + O^3(\epsilon)\,, \\ \Longrightarrow \quad K_1(\overline{J}) &= \omega_0^r(\overline{J})\, \frac{\partial S_1[\phi,\overline{J}]}{\partial \phi^r} + H_1[\phi,\overline{J}] + S_{1,t}(\phi,\overline{J})\,, \\ K_2 &= \omega_0^a(\overline{J})\, \frac{\partial S_2}{\partial \phi^a} + \frac{\partial H_1}{\partial J_a}|_{J=\overline{J}} \frac{\partial S_1}{\partial \phi^a} + \tfrac{1}{2} \frac{\partial \omega_0^a}{\partial J_b}|_{J=\overline{J}} \frac{\partial S_1}{\partial \phi^a} \frac{\partial S_1}{\partial \phi^b} + H_2(\phi,\overline{J}) + S_{2,t}(\phi,\overline{J})\,, \end{aligned} \tag{3.3e}$$

where we have used Eqs. (3.2) to introduce $\omega_0^a = \partial H_0/\partial J_a$, which depends on J_a, and then evaluated that in terms of the new variables, and kept terms only to lowest order.

Now let's consider resolving these equations. We begin with the equations only first order in ϵ. Since we have more unknown functions than equations, we must now add some new requirements on the system. Since we are dealing with systems that are periodic, the notion of "averaging" appears in a very natural and intuitive way. Just what is to be averaged, over what curves, is not quite as obvious, however. The equation for $K_1(\overline{J})$, at Eqs. (3.3e), is written as an equation in the variables $(\phi^a, \overline{J}_r)$, which constitute an acceptable set of variables over phasespace. Were the system unperturbed, then a set of values for the actions, $\{J_a\}_1^n$, pick out a set of independent tori, and the motion can be viewed as closed curves on each one, as each of the ϕ^a varies from 0 to 2π; therefore, we can average over them. Of course, the more accurate values for the action are, instead, the transformed actions, $\{\overline{J}_r\}_1^n$. As we have assumed that the perturbed problem also is integrable, i.e., has its own tori, and closed curves on them, then we can imagine following around the closed curves viewed on the tori labelled by the $\overline{J}$. Since this is an acceptable set of coordinates, the $\overline{J}$ must come back to their original values when we do this as well. Therefore, we may, instead project each of those curves onto the $\overline{J}$ tori, and follow along those curves instead. When that happens, the leading term in S will clearly change by $2\pi J$. However, since neither J nor $\overline{J}$ changed, it must be that all of the derivatives of S with respect to ϕ must be periodic functions, so that they do not change. Therefore, each of the S_j's, for $j \geq 1$, must be periodic in ϕ, and perhaps also a term linear in ϕ, only. This last term may then be discarded. The result of all this is then that **the average of $\partial S_j/\partial\phi$ may be chosen to be zero!** In addition, we will **only consider** time-dependencies that have some period, say Ω, in time. Similar arguments allow us to suppose that **we average, also over the temporal period, so that the terms $\partial S_j/\partial t$ vanish.** This allows us to resolve the equations by averaging both sides, which gives us the two useful equations:

$$\begin{aligned} K_1(\overline{J}) = \langle H_1\rangle(\overline{J}) \equiv \langle H_1(\phi,\overline{J})\rangle \equiv \Omega \left(\frac{1}{2\pi}\right)^{n+1} \int_0^{2\pi/\Omega} dt \prod_{a=1}^{n} \left\{ \int_0^{2\pi} d\phi^a \right\}\; H_1(\phi,\overline{J},t)\,, \\ \omega_0^a\, \frac{\partial S_1}{\partial \phi^a} + \frac{\partial S_1}{\partial t} = -H_1(\phi,\overline{J}) + \langle H_1\rangle(\overline{J})\,. \end{aligned} \tag{3.4}$$

The determination of the first perturbation to the Hamiltonian is then rather straightforward! On the other hand, the determination of this first-order correction to the generating function can be a little tricky! Although in principle one might determine the average value in various other ways, a very generic and algorithmic sort of way, for a function that is periodic in all these angles, is to expand it in a Fourier series. Since, by definition, the part desired for the calculation of S_1 has zero average, it will have no constant term in the Fourier series:

$$\langle H_1 \rangle - H_1 \equiv - \sum_{\ell, m \neq 0} H_{1m\ell}(\overline{J}) e^{i(m_a \phi^a + \ell \Omega t)} = \omega_0^a \frac{\partial S_1}{\partial \phi^a} + \frac{\partial S_1}{\partial t} \ . \tag{3.5a}$$

We then expand the generating function in the same way, and equate terms. The result is straightforward:

$$S_1 = i \sum_{\ell, m \neq 0} \frac{H_{1m\ell}(\overline{J})}{\ell \Omega + m_a \omega^a(\overline{J})} e^{i(m_a \phi^a + \ell \Omega t)} \ . \tag{3.5b}$$

This expression is a perfectly legitimate expression for S_1 and does indeed provide a first-order approximation to the exact solution, *provided that none of the denominators vanish!* Obviously the method **does not** provide a solution to the problem if even one of the denominators is zero. This suggests that the entire perturbative procedure is invalid at this point, i.e., that the system's behavior cannot actually be described by action-angle variables that are related through a canonical transformation tot he original, unperturbed set. This will obviously happen for frequencies that are commensurate, unless, for some strange reason, the particular coefficient, $H_{1m\ell}$, were to also vanish. [Do, of course, remember that this problem does **not occur in systems with only one degree of freedom**, since the denominator will not vanish.]

Even a more serious problem is the fact that one may find a set of conditions such that one or more denominator becomes very small; i.e., we are considering initial conditions near to those which caused the denominator to vanish. In that case, the particular term will be very large. This means that multiplying by even your standard "small" value of ϵ does not make that term small; therefore, the "correction" becomes so very large that the perturbative scheme fails, working only for a much more restricted set of values for ϵ. Eventually the KAM theorem will give us some specific methods of finding ways to avoid these degeneracies, and also to know how much accuracy we can expect at "nearby points," in phasespace.

For the moment, let us forget these problems temporarily and return to those terms which determine the approximation to second order in ϵ. Proceeding with an averaging method as before, we obtain the following, where I **consider only the time-independent case**:

$$\begin{aligned} K_2 \ &= \left\langle \frac{\partial H_1}{\partial J_a}\Big|_{J=\overline{J}} \frac{\partial S_1}{\partial \phi^a} + \tfrac{1}{2} \frac{\partial \omega_0^a}{\partial J_b}\Big|_{J=\overline{J}} \frac{\partial S_1}{\partial \phi^a} \frac{\partial S_1}{\partial \phi^b} + H_2 \right\rangle \ , \\ \omega_0^a(\overline{J}) \, \frac{\partial S_2}{\partial \phi^a} \ &= K_2 - \left(\frac{\partial H_1}{\partial J_a}\Big|_{J=\overline{J}} \frac{\partial S_1}{\partial \phi^a} + \tfrac{1}{2} \frac{\partial \omega_0^a}{\partial J_b}\Big|_{J=\overline{J}} \frac{\partial S_1}{\partial \phi^a} \frac{\partial S_1}{\partial \phi^b} + H_2 \right) \end{aligned} \tag{3.6}$$

Now, we recall that we already have an "explicit" expression for S_1, as an infinite Fourier series, and therefore also one for $\partial S_1 / \partial \phi^a$, from our first-order calculation. We may certainly insert that into this, and thereby obtain an expression for the second-order correction to the energy, having only found the generating function to the first order. This is, of course, a very general property of perturbative series: to calculate the correction to the energy to n-th order, one needs to determine the complete correction, including in this case the generating function, only to one earlier order, i.e., to the $n-1$-st order.

On the other hand, at least at the moment, I will only consider the rather simpler question of further calculation of this form for the second-order correction to the energy in the situation when there is only one degree of freedom. There is no real impediment to doing the more general case, without time dependence, but it is simply messier. In the case of one degree of freedom, Eq. (3.4) takes the form $\omega_0(\partial S_1/\partial\phi) = \langle H_1\rangle - H_1$, so that our equation for K_2 becomes

$$K_2(\overline{J}) = \frac{1}{\omega_0}\left(\langle H_1\rangle\langle\frac{\partial H_1}{\partial J}\rangle - \langle H_1\frac{\partial H_1}{\partial J}\rangle\right) + \frac{\partial\omega_0/\partial J}{2\omega_0^2}\left(\langle (H_1)^2\rangle - \langle H_1\rangle^2\right) + \langle H_2\rangle\ . \tag{3.7}$$

4. Some Examples:

As a reasonable, rather simple, first example, let us look at our approximation to the pendulum, and treat it correct to second order in a small parameter; therefore, I retreat to Eq. (1.1) and take the Hamiltonian there, expand the cosine function in an infinite series, throw away the constant term, keep the quadratic term in H_0, and replace q^2 by ϵq^2, to keep track of the extra terms and the perturbation parameter. At first order, this will be the same as our example of the quartic oscillator, discussed in §2. The small parameter there, which was also called ϵ, we will now call $\eta = -\epsilon\omega_0^2 = -\epsilon FG$. Then we have the following:

$$H = \tfrac{1}{2}Gp^2 + \tfrac{1}{2}Fq^2 - \tfrac{1}{4}\epsilon Fq^4 + \tfrac{1}{20}\epsilon^2 Fq^6 + \ \ldots\ \equiv H_0 + \epsilon\, H_1 + \epsilon^2\, H_2 + \ldots\ ;$$

$$\text{for } \epsilon = 0 \quad \begin{cases} \phi = \tan^{-1}\left(m\omega_0 q/p\right)\ , \quad J = \dfrac{1}{2}\left(p^2/m\omega_0 + m\omega_0 q^2\right)\ , \\[2mm] q = \dfrac{1}{\sqrt{m\omega_0}}\sqrt{2J}\,\sin\phi\ , \quad p = \sqrt{m\omega_0}\sqrt{2J}\,\cos\phi\ , \\[2mm] \qquad\qquad H_0 = \omega_0 J\ , \end{cases}$$

$$\begin{aligned} \Longrightarrow\quad H(\phi, J) &= H_0 + \epsilon\, H_1 + \epsilon^2\, H_2 = \omega_0 J - \epsilon\, G\, J^2\sin^4\phi + \tfrac{2}{5}\epsilon^2\tfrac{G^2}{\omega_0}J^3\,\sin^6\phi \\ &= \omega_0 J - \epsilon\tfrac{1}{8}G\,J^2[3 - 4\cos 2\phi + \cos 4\phi] + \epsilon^2\tfrac{1}{80}\tfrac{G^2}{\omega_0}J^3[10 - 15\cos 2\phi + 6\cos 4\phi - \cos 6\phi]\ . \end{aligned} \tag{4.1}$$

The average of H_1 is immediate, when it has been expressed as a Fourier series, which in this case is finite. Therefore we immediately conclude that

$$\begin{aligned} K_1(\overline{J}) = -\tfrac{3}{8}G\,\overline{J}^{\,2}\ , \quad &\Longrightarrow \quad \omega_1 = -\tfrac{3}{4}G\,\overline{J}\ , \\ \text{so that}\quad \overline{\omega} &= \omega_0 - \epsilon\tfrac{3}{4}G\,\overline{J} + O^2(\epsilon)\ . \end{aligned} \tag{4.2}$$

We may now determine S_1, and the associated canonical transformation:

$$S_1 = -\frac{1}{\omega_0}\int d\phi\left(H_1 - \langle H_1\rangle\right)\Big|_{J=\overline{J}} = -\frac{G\,\overline{J}^{\,2}}{8\omega_0}\int d\phi\left(4\cos 2\phi - \cos 4\phi\right) = -\frac{G\overline{J}^{\,2}}{32\omega_0}[8\sin 2\phi - \sin 4\phi]\ . \tag{4.3}$$

From this we may write out the CT-generated equations for J and $\overline{\phi}$, from Eqs. (3.3c):

$$\begin{aligned} J &= \overline{J} + S_{1,\phi} = \overline{J} - \epsilon\frac{G\overline{J}^{\,2}}{8\omega_0}[4\cos 2\phi - \cos 4\phi] + O^2(\epsilon)\ , \\ \overline{\phi} &= \phi + S_{1,\overline{J}} = \phi - \epsilon\frac{G\overline{J}}{16\omega_0}[8\sin 2\phi - \sin 4\phi] + O^2(\epsilon)\ . \end{aligned} \tag{4.4a}$$

In principle, the solution of this for either set (ϕ, J) or $(\overline{\phi}, \overline{J})$ in terms of the other is rather difficult. On the other hand, it is not fair to solve it more accurately than first-order. In that case we have, at least in one direction, just the following:

$$\begin{aligned} \overline{J} &= J - \epsilon \frac{GJ^2}{8\omega_0}[4\cos 2\phi - \cos 4\phi] + O^2(\epsilon) \ , \\ \overline{\phi} &= \phi - \epsilon \frac{GJ}{16\omega_0}[8\sin 2\phi - \sin 4\phi] + O^2(\epsilon) \ . \end{aligned} \tag{4.4b}$$

Proceeding onward, one should then be able to determine K_2 and the next-order correction to the energy and the frequency.

However, let us stop for now at consider another example. We extend the quartic oscillator to two degrees of freedom, and then couple them, with the the Hamiltonian for this system as

$$H = \tfrac{1}{2}(p_1^2 + p_2^2) + \tfrac{1}{2}(\omega_1^2 q_1^2 + \omega_2^2 q_2^2) + \epsilon(\omega_1\omega_2 q_1 q_2)^2 \equiv H_0 + \epsilon\, H_1 \ . \tag{4.5}$$

The unperturbed action-angle variables are simply two copies of the ones we had above, for one linear oscillator; therefore, we may immediately write

$$H(\phi, J) = \omega_1 J_1 + \omega_2 J_2 + 4\epsilon\omega_1 J_1 \omega_2 J_2 \ \sin^2\phi^1 \ \sin^2\phi^2 \ . \tag{4.6}$$

Therefore the first-order correction to the energy is simply

$$K_1(\overline{J}) = \langle H_1 \rangle = \frac{4\omega_1 \overline{J}_1 \omega_2 \overline{J}_2}{4\pi^2} \int d\phi^1 \ \sin^2\phi^1 \int d\phi^2 \ \sin^2\phi^2 = \omega_1 \overline{J}_1 \omega_2 \overline{J}_2 \ , \tag{4.7}$$

which tells us that the first-order frequencies are

$$\overline{\omega}_1 = \omega_1 + \epsilon\omega_1\omega_2 J_2 \ , \quad \overline{\omega}_2 = \omega_2 + \epsilon\omega_1\omega_2 J_1 \ . \tag{4.8}$$

Now, however, obtaining the (first-order corrections to the) generating function are where we anticipate trouble from resonances. Therefore, we proceed toward that calculation, i.e., a solution of the equation given in Eq. (3.5a), for our case:

$$\left(\omega_1 \partial_{\phi^1} + \omega_2 \partial_{\phi^2}\right) S_1 = -\Big(\langle H_1 \rangle - H_1\Big) \ . \tag{4.9}$$

As discussed at that point, a simple way to resolve this differential equation is by use of the Fourier transform of the non-constant part of the first-order Hamiltonian, to insert into Eq. (3.5b). Therefore, we first calculate those Fourier coefficients:

$$H_{1,(m_1,m_2)} = \frac{\omega_1\omega_2 J_1 J_2}{(2\pi)^2} \int_0^{2\pi} d\phi^1 \int_0^{2\pi} d\phi^2 (4\sin^2\phi^1 \sin^2\phi^2 - 1)\, e^{-i(m_1\phi^1 + m_2\phi^2)} \ . \tag{4.10a}$$

It is reasonably clear that only a few of these are non-zero; more precisely, those that involve ± 2 and 0. We have the following, where we use Eq. (3.5b), with $\Omega = 0$, to determine the coefficients $S_{1,(m_1,m_2)}$:

$$\begin{gathered} H_{1,(\pm2,\pm2)} = +\tfrac{1}{4}\, K_1 \ , \quad H_{1,(\pm2,0)} = -\tfrac{1}{2}\, K_1 = H_{1,(0,\pm2)} \ , \quad H_{1,(0,0)} = 0 \ , \\ S_{1,(\pm2,\pm2)} = \frac{\pm i/4}{2(\omega_1 + \omega_2)} \ , \ S_{1,(\pm2,\mp2)} = \frac{\pm i/4}{2(\omega_1 - \omega_2)} \ , \ S_{1,(\pm2,0)} = \frac{- \pm i/2}{2\omega_1} \ , \ S_{1,(0,\pm2)} = \frac{- \pm i/2}{2\omega_2} \ . \end{gathered} \tag{4.10b}$$

Therefore, we may insert these back into Eq. (3.5b), with $\Omega = 0$, to obtain the generating function:

$$\begin{aligned} S_1 =& \tfrac{1}{2} J_1 J_2 \left(\omega_2 \sin 2\phi^1 + \omega_1 \sin 2\phi^2 - \frac{\omega_1\omega_2}{2(\omega_1+\omega_2)} \sin 2(\phi^1+\phi^2) - \frac{\omega_1\omega_2}{2(\omega_1-\omega_2)} \sin 2(\phi^1-\phi^2) \right) \\ \equiv& \ \tfrac{1}{2} J_1 J_2 \, F(\phi) \ . \end{aligned} \tag{4.10c}$$

We may then write down the first-order CT, where, to that order, we have replaced the $\overline{J}$'s on the right-hand side of the equation with the older J's:

$$\begin{aligned} \overline{\phi}^1 = \phi^1 + \epsilon J_2 F(\phi) \ , \quad \overline{\phi}^2 = \phi^2 + \epsilon J_1 F(\phi) \ , \\ \overline{J}_1 = J_1 + \epsilon J_1 J_2 \frac{\partial F}{\partial \phi^1} \ , \quad \overline{J}_2 = J_2 + \epsilon J_1 J_2 \frac{\partial F}{\partial \phi^2} \ . \end{aligned} \tag{4.10d}$$

This example gives us our first view of the resonance problem with perturbation theory. Here there is an obvious problem with resonance; as well, it even shows us just why it's called that, since it occurs when the two natural frequencies are equal, i.e., in resonance. On the other hand, this one has a problem **only** when those two frequencies are equal, when one might well expect it. Its Fourier spectrum is sufficiently simple that there are no problems even when the ratio of the two frequencies is any rational number except for the very special case when that number is 1. On the other hand, it is true that more complicated resonances would probably occur at higher orders in perturbation theory. In fact, as we will see later, there are situations where this dynamical system becomes chaotic.

As an example of an alternative sort of a problem, we consider the following problem where we have a time-dependent (external) driving agent, and an explicit magnetic field. The dynamical system is a charged particle moving in a uniform magnetic field and interacting with an additional, oscillating electric field. Let the charged particle have mass m and charge $q \equiv c\beta$, where c is the speed of light. Describe the magnetic field by a vector potential $\vec{A} = -B_0 y \hat{x}$, which causes the magnetic field to be in the $\hat{z}$-direction, $\vec{B} = \nabla \times \vec{A} = +B_0 \hat{z}$, orthogonal to the x, y-plane. We choose the axes in that plane so that the oscillating electric field has wave vector only in the y- and z-directions, i.e., $\vec{k} = k_z \hat{z} + k_y \hat{y}$, and let it be generated by a (time-dependent) potential such that the Hamiltonian may be put in the following form, where ϵ is simply, as usual, an ordering parameter:

$$H = \tfrac{1}{2m} \left| \vec{p} - \beta \vec{A} \right|^2 + \epsilon \beta V_0 \sin(k_z z + k_y y - \omega t) \ . \tag{4.11}$$

We first study the unperturbed system, i.e., the one with $\epsilon = 0$, to put it in the proper form. We recall that the motion in such a situation the particle spirals along the direction of the field and circles in the plane perpendicular to the field, with a constant angular velocity, $\Omega = \beta B_0/m$, around such origin. We take that origin for the circles to have (constant) coordinates in the plane given by (X, Y), and radius ρ, so that we may write the motion in that plane in terms of a single (angular) coordinate ϕ, related in the following way:

$$x = X + \rho \cos\phi \ , \quad y = Y - \rho \sin\phi \ . \tag{4.12a}$$

The unperturbed Hamiltonian has the explicit form

$$2m\, H_0 = (p_x + \beta B_0 y)^2 + (p_y)^2 + (p_z)^2 \ , \tag{4.12b}$$

where we may also write out the Hamiltonian in terms of the velocities as usual:

$$\vec{p} = m\vec{v} + \beta\vec{A} \implies \begin{cases} p_x = m\frac{d}{dt}x - \beta B_0 y = -m\rho \sin\phi\, \dot{\phi} - \beta B_0 y \ , \\ p_y = m\frac{d}{dt}y = -m\rho\cos\phi\,\dot{\phi} \ , \\ p_z = m\frac{d}{dt}z \ . \end{cases} \tag{4.12c}$$

Remembering that $\dot{\phi} = \Omega = \beta B_0/m$, we may rewrite the equation for p_x as

$$p_x = -\rho \sin\phi\, \dot{\phi} - \beta B_0 y = -m\Omega(\rho \sin\phi + y) = -m\Omega Y \ , \tag{4.12d}$$

explaining the perhaps-unexpected choice of sign in the relation chosen between x, y, and ϕ. Of course a different way of looking at that sign is that it causes the variable ϕ to progress clockwise around the orbit. As the circle progresses counterclockwise, this is basically a **co-rotating** set of coordinates for the system. We now want to move from the current coordinates, $\{x, y, z; p_x, p_y, p_z\}$, to a new set $\{Y, \phi, z; P_Y, P_\phi, p_z\}$, which we will do via a Type 1 generating function, which maintains the z-oriented variables the same as before:

$$\begin{aligned} &F^1(x, y, z; Y, \phi, z) \equiv m\Omega[\tfrac{1}{2}(y - Y)^2 \cot\phi - xY] \ , \\ \Longrightarrow\ &\begin{cases} p_x = F^1_{,x} = -m\Omega Y \ , \\ p_y = F^1_{,y} = m\Omega(y - Y)\cot\phi = -m\Omega\rho\cos\phi \ , \\ P_Y = \ - F^1_{,Y} = m\Omega[(y - Y)\cot\phi + x] = m\Omega[-\rho\cos\phi + X + \rho\cos\phi] = m\Omega X \ , \\ P_\phi = \ - F^1_{,\pi} = \tfrac{1}{2} m\Omega(y - Y)^2 \csc^2\phi = \tfrac{1}{2} m\Omega\rho^2 \ , \end{cases} \end{aligned} \tag{4.12e}$$

where we notice that the equations for the original momenta are consistent with their earlier versions shown above.

Inserting all this into the equation for H_0, we find, in these variables, that

$$\begin{aligned} K_0 = H_0[old(new)] &= \Big[[m\Omega(Y - y)]^2 + (m\Omega\rho\cos\phi)^2 + (p_z)^2\Big]/(2m) \\ &= \tfrac{1}{2} m\Omega^2\rho^2 + p_z^2/2m = \Omega\, P_\phi + p_z^2/2m \ , \end{aligned} \tag{4.12f}$$

where the form is of course what is expected for a transition to action-angle variables, i.e., terms of the form $J\dot{\phi}$, except for the fact that there are no terms at all that involve Y and its momentum, as both of them are constant! (Of course nothing happened to the z-direction quantities, since we did not involve them in the transformation.)

We may now express H_1, the coefficient of ϵ, in terms of these new coordinates, which we do while using ρ as simply an abbreviation for its value, resolved from the equation for P_ϕ above, namely $\rho \equiv \sqrt{(2P_\phi/M\Omega)}$:

$$K_1 = H_1[old(new)] = \beta V_0 \sin(k_z z - \omega t + k_p \rho \sin\phi) \ . \tag{4.13a}$$

The non-linearity comes from the dependence of ρ on P_ϕ and from the $\sin\phi$. On the other hand, the other two remaining variables, z and t, occur only in the combination $k_z z - \omega t$; therefore, we eliminate the time, explicitly, by another transformation to that one variable, which we refer to as ψ, thinking of it surely as an angular type variable since it only appears as an argument of the sine function. A Type 2 generating function which does that is the following, where the last term, ϕP_ϕ, is simply the identity transformation which leaves those two variables alone:

$$\begin{aligned} &F^2 \equiv (k_z z - \omega t) P_\psi + \phi P_\phi \\ \Longrightarrow\ &\begin{cases} p_z = \dfrac{\partial F^2}{\partial z} = k_z\, P_\psi \ , \quad P_\phi = \dfrac{\partial F^2}{\partial \phi} = P_\phi \ , \\ \psi = \dfrac{\partial F^2}{\partial P_\psi} = k_z z - \omega t \ , \quad \phi = \dfrac{\partial F^2}{\partial P_\phi} = \phi \ , \end{cases} \end{aligned} \tag{4.13b}$$

This generating function depends explicitly on the time, so that the Hamiltonian now has the form

$$\widetilde{K} = \widetilde{K}_0 + \epsilon\Big(+ F^2_{,t} + \widetilde{K}_1\Big) = (k_z P_\psi)^2/2m + \Omega\, P_\phi - \omega P_\psi + \epsilon\beta V_0 \sin(\psi - k_p\rho\sin\phi) \ , \tag{4.13c}$$

where we have not put the multiplier ϵ on the term from the time derivative, $-\omega P_\psi$, since that term is easily included with the rest of the terms and is still integrable.

Since we now have a nonlinearity that is of the form of a sine function of a sine function of a variable, we need to expand this in a Fourier series to see explicitly what sort of resonances may have the possibility of being present. We use a standard identity for such things, that is related to the generating function for (integer-index) Bessel functions, or, if you prefer, we simply expand it all in a Fourier series:

$$\sin(\psi - z\sin\phi) = \sum_{\ell=-\infty}^{+\infty} J_\ell(z)\, \sin(\psi - \ell\phi) \tag{4.14}$$

This gives us the equivalent form for the Hamiltonian with the perturbation in the form of a Fourier series, as we know we need in order to determine the generating function for this perturbed system:

$$\widetilde{K} \equiv \widetilde{K}_0 + \epsilon\widetilde{K}_1 = (k_z P_\psi)^2/2m + \Omega P_\phi - \omega P_\psi + \epsilon\beta V_0 \sum_{\ell=-\infty}^{+\infty} J_\ell[k_p\rho(P_\phi)]\, \sin(\psi - \ell\phi)\ . \tag{4.15a}$$

The unperturbed frequencies are then

$$\omega_\phi = \frac{\partial\widetilde{K}_0}{\partial P_\phi} = \Omega\ , \quad \omega_\psi = \frac{\partial\widetilde{K}_0}{\partial P_\psi} = -\omega + (k_z^2/m)P_\psi\ . \tag{4.15b}$$

From our general discussion of resonances, in particular at Eqs. (3.5b), we know that there will be a problem with the perturbation expansion when the initial conditions put us too close to the places where the ratio of the original frequencies is rational. Since we only have two frequencies here, this is the same as suggesting that

$$0 = -\omega_\psi + m\Omega = -(k_z^2/m)P_\psi + \omega + m\Omega\ . \tag{4.15c}$$

It turns out that the situation when $k_z \neq 0$ is rather more complicated than the other case; physically, of course this turns out to be the case when the incoming electric field is not perpendicular to the existing magnetic field. (It might be expected that these are intrinsically different.) Therefore, we now consider in more detail (at least) the case when $k_z = 0$. To perform this "trick," we first return to our equations, from §3, for the first-order quantities, remembering that what we want is to determine a CT from the old variables to the yet-newer ones, which, as before, we describe with overbars:

$$\begin{aligned}
\overline{K}_0 &= \widetilde{K}_0(new) = (k_z\overline{P}_\psi)^2/2m + \Omega\overline{P}_\phi - \omega\overline{P}_\psi \\
\overline{K}_1 &= \langle K_1\rangle(new) = (\tfrac{1}{2\pi})^2\beta V_0 \int_0^{2\pi} d\psi \int_0^{2\pi} d\phi \sum_{\ell=-\infty}^{+\infty} J_\ell(k_p\rho)\, \sin(\psi - \ell\phi) = 0\ , \\
\Big(\Omega\,\partial_\phi - \omega\,\partial_\psi\Big) S_1 &= -\widetilde{K}_1 = -\beta V_0 \sum_{\ell=-\infty}^{+\infty} J_\ell(k_p\rho)\, \sin(\psi - \ell\phi)\ .
\end{aligned} \tag{4.16}$$

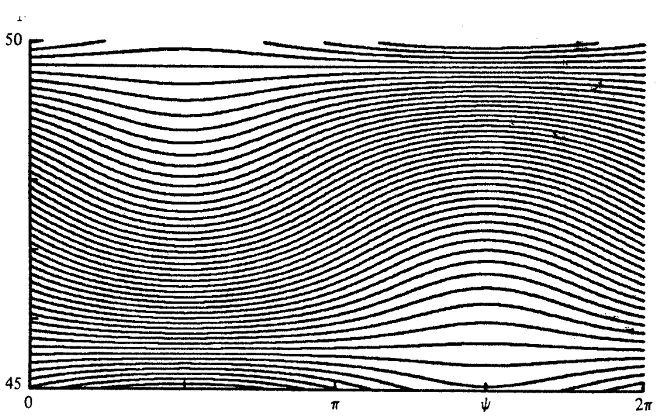
As this is a situation where the first-order correction to the energy vanishes, one might expect that it is more likely to have problems, "sooner." We proceed onward, then, to determine the generating function. The desired solution to the differential equation, i.e., the particular solution, is not hard to determine:

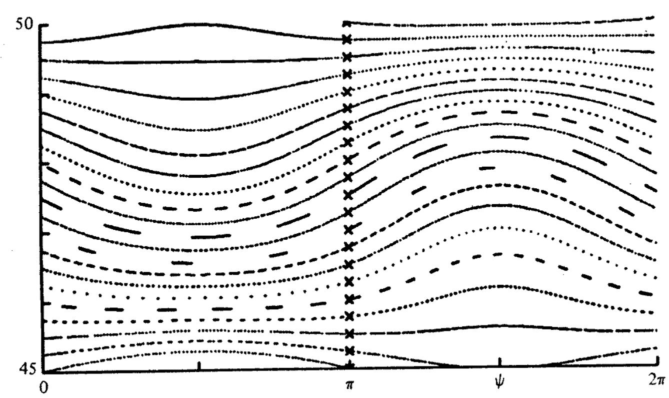
$$S_1 = -\beta V_0 \sum_{\ell=-\infty}^{+\infty} \frac{J_\ell[k_p \rho(P_\phi)]}{\omega + m\Omega} \cos(\psi - \ell\phi) \; . \tag{4.17}$$

We can then determine this CT, to first-order, by using (the inverse of) Eqs. (3.3d):

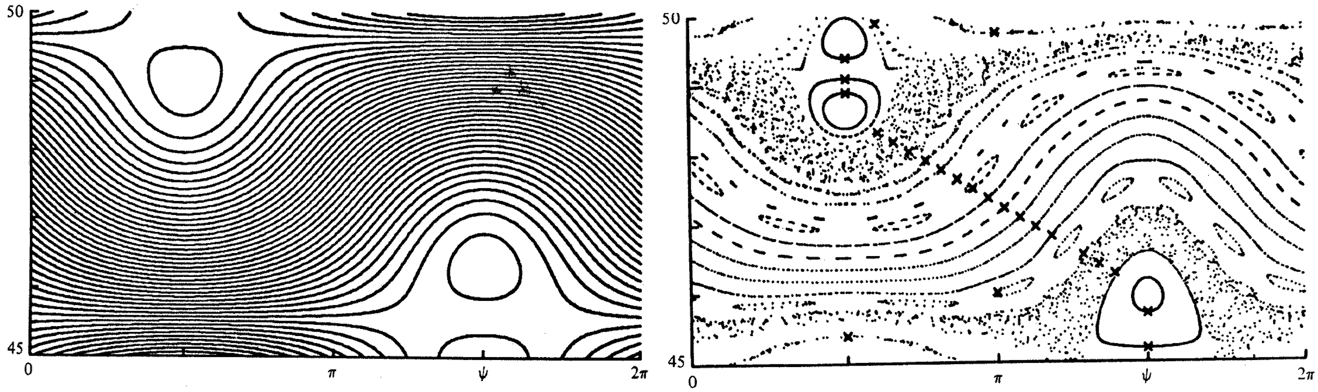
$$\begin{aligned} \overline{P}_\phi &= P_\phi + \epsilon\beta V_0 \sum_{\ell=-\infty}^{+\infty} \ell \frac{J_\ell[k_p \rho(P_\phi)]}{\omega + \ell\Omega} \sin(\psi - \ell\phi) \; , \\ \overline{P}_\psi &= P_\psi - \epsilon\beta V_0 \sum_{\ell=-\infty}^{+\infty} \frac{J_\ell[k_p \rho(P_\phi)]}{\omega + \ell\Omega} \sin(\psi - \ell\phi) \; . \end{aligned} \tag{4.18}$$

Since the Hamiltonian didn't change under this transformation, we can immediately see that it does not depend on any of the transformed angular coordinates; therefore, the transformed momenta, $\overline{P}_\phi$ and $\overline{P}_\psi$, are constant. Therefore these two equations may be conceived as equations giving us the functional form of each of the un-transformed momenta as functions of the two angles. We may therefore easily describe Poincaré sections, in, for instance, the ϕ, P_ψ-plane by plotting the second of Eqs. (4.18) for some fixed value of the other angle, ϕ. For each fixed value of the constant, $\overline{P}_\phi$—dependent on initial conditions—this will give us a curve of P_ϕ versus ψ. We could do this for various values of the ratio of ω versus Ω. It might have seemed more straightforward to have plotted P_ϕ versus ϕ, but surely similar information is provided in this way as well, and Lichtenberg has provided plots for this case, for the ratio $\omega = 30.11\Omega$, where various different values of $\overline{P}_\phi$ are introduced, to produce a variety of distinct motions on the same plot. The plots are given to us for two different values of the driving amplitude, V_0, one referred to as "low" wave amplitude, while the second one has "higher" amplitude. Actually the curves were obtained by C.F.F. Karney, in his (MIT, 1977) thesis studying stochastic heating of ions in a tokamak, induced by RF power. The left-hand curves are produced from the perturbation analysis equations, above, and they are then compared, on the right-hand side, with analogous curves for the same initial conditions, but calculated numerically, and, therefore, presumably rather more accurate. The first set, below, "at low amplitude" for the electric wave, has very good agreement between the perturbative curves and the exact numerical curves:





This second pair, below, with "higher amplitude," do mirror some of the gross features, but the perturbative curves clearly do not reproduce the chaotic structure, nor the "chains of islands" seen in the associated numerical calculations:



I note that there is a related html-file, at a nearby link to this one on the class homepage, that shows these curves and also the results of some additional calculations of my own, for the first-order case.