Continuing with Perturbation Theory

5. Fast Oscillation Periods versus Slowly Varying Parameters: Multiple Time-Scale Averaging: Adiabatic "Invariants"

Beginning with the simplest sort of 1-dimensional problem, we may suppose that we have been given an oscillator with a very-slowly-varying frequency, one could write an equation of motion for it in the simple form

$$\ddot{x} + \omega^2(\epsilon t)\, x = 0 \;, \tag{5.1}$$

where ϵ is simply a (dimensionless) measure of "smallness" that we will use on the frequency, to identify the fact that it is "small," so that we may consider the use of perturbation theory, **and keep track** of the perturbative approach. We will set ϵ back to 1 at the complete end of any calculation; on the other hand, it will not be useful unless we have some sort of actual, physical parameter in the problem that really is small. For instance, the usual frequency for a standard SHO does not depend on time at all; therefore, the obvious way to begin is to make that dependence slow. We also want a dimensionless parameter to describe it; therefore, we would **set the time rate of change of the period** to be very small:

$$\left|\frac{d}{dt}T\right| \equiv 2\pi\epsilon \quad \Longrightarrow \qquad \epsilon = \left|\frac{\dot\omega}{\omega^2}\right| \;. \tag{5.2}$$

The standard approach, **which results in an asymptotic series** for the result, begins by changing variables to this small rate, $\tau \equiv \epsilon t$:

$$\frac{d^2}{d\tau^2}x + \left(\frac{1}{\epsilon}\right)^2 \omega^2(\tau)\, x = 0 \;. \tag{5.3}$$

The equation can be simplified by reducing it to a first-order system, via the (standard) method that moves between linear, second-order ode's and a first-order, (nonlinear) Ricatti equation:

$$y \equiv \frac{1}{x}\frac{d}{d\tau}x = \frac{x'}{x} = \frac{d}{d\tau}\log x \quad \Longrightarrow \quad \epsilon(y^2 + y') + \omega^2 = 0 \;. \tag{5.4a}$$

Inserting an infinite series for y gives us an infinite series of equations to be satisfied:

$$\begin{aligned}
y \equiv y_0/\epsilon + y_1 + \epsilon y_2 + \epsilon^2 y_3 + \;\ldots \;\;, \\
y_0^2 + \omega^2 = 0 \quad \Longrightarrow \quad y_0 = \pm i\omega \;, \\
2y_0y_1 + y_0' = 0 \quad \Longrightarrow \quad y_1 = -y_0'/2y_0 = -\omega'/2\omega \;, \\
2y_0y_2 + y_1^2 + y_1' = 0 \quad \Longrightarrow \quad y_2 = \mp\tfrac{i}{8}\left\{2\frac{\omega''}{\omega^3} - 3\left(\frac{\omega'}{\omega^3}\right)\right\} \;, \\
\ldots \qquad\qquad\qquad .
\end{aligned} \tag{5.4b}$$

We may insert all this information into a form for x, to obtain

$$x = \frac{A}{\sqrt{\omega}}\, e^{\pm i\int dt\,\omega[1-\frac{1}{8}(2\omega''-3(\omega')^2)/\omega^4]} \;, \tag{5.4c}$$

where A is simply a constant of integration, the exterior square root comes from the first-order term, and we have removed the ϵ's by doing the integral over dt rather than $d\tau$. On the other hand, since we are supposing that the period is changing very slowly, the second set of terms in the integral should be much smaller than the beginning 1, so that the integral with only the ω in it is the first-order approximation to the problem, which can then be written in terms of its real form if desired, where δ comes from the phase of A:

$$x = \frac{|A|}{\sqrt{\omega}}\,\cos(\delta + \int dt\,\omega) + O^2(\epsilon)\ . \tag{5.4d}$$

If we now use this form to determine the action, we will find

$$J \equiv \oint dx\,p = m\oint dt\,\dot{x}^2 = \tfrac{1}{2}m\omega|A|^2 + O(\epsilon^2)\ , \tag{5.4e}$$

so that we say that J is "an adiabatic invariant," i.e., constant to within a higher order of perturbation theory than the original variation in the problem.
On the other hand, there is a problem here, as is not unusual with perturbation expansions. This expansion in powers of ϵ is, generically, a divergent, although asymptotic series. It is perhaps therefore appropriate to spend just a few lines summarizing the (mathematical) details of such series:

Denote a **divergent** series as follows, where we also provide a name for the sum of the first n terms:

$$\sum_{k=0}^{\infty} A_k z^{-k}\ , \qquad S_n(z) \equiv \sum_{k=0}^{n} A_k z^{-k}\ . \tag{5.5a}$$

This series is said to be ***an asymptotic expansion of a function*** $\mathbf{f}(\mathbf{z})$ for some given range of values of the argument, z, provided we have the following conditions satisfied:

$$\begin{cases} \lim\limits_{|z|\to\infty} \Big(z^n[f(z) - S_n(z)]\Big) = 0\ , & n \text{ fixed,} \\ \lim\limits_{n\to\infty} \Big|z^n[f(z) - S_n(z)]\Big| = \infty\ , & z \text{ fixed.} \end{cases} \tag{5.5b}$$

Because of this we say that we can always make the quantity $\Big|z^n[f(z) - S_n(z)]\Big|$ arbitrarily small, by choosing $|z|$ large enough. It is common to describe an asymptotic expansion of a function by $f(z) \sim \sum_{k=0}^{\infty} A_k z^{-k}$. One should note that it is legitimate, in the sense that the result is also an asymptotic expansion, to multiply or integrate asymptotic expansions, but **not** to differentiate them, in general.

Example:
Define the function

$$f(x) \equiv \int_x^\infty dt\,\frac{e^{x-t}}{t}\ , \tag{5.6a}$$

for x real and positive, and integration along the real axis. By repeated integration by parts we obtain

$$\begin{aligned} f(x) &= \frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \ \ldots + \ (-1)^{n-1}\frac{(n-1)!}{x^n} + (-1)^n n! \int_x^\infty dt\,\frac{e^{x-t}}{t^{n+1}} \\ &\equiv \sum_{k=1}^{n} (-1)^{k-1}\frac{(k-1)!}{x^{k+1}} + (-1)^n n! \int_x^\infty dt\,\frac{e^{x-t}}{t^{n+1}}\ . \end{aligned} \tag{5.6b}$$

The series is obviously divergent; nonetheless, we see that for large values of x the difference between the finite sum and the function may be made very small.

We may now proceed with the more generic form of such a problem, but also via a Hamiltonian, action-angle formulation, where we suppose that one of the angles, ϕ, varies quite fast, while all the others vary slowly. Therefore, we will, as usual for counting purposes, multiply all those variables by ϵ, intending eventually to set it equal to 1:

$$H \equiv \; H_0(J, \epsilon\eta, \epsilon t) + \epsilon H_1(\phi, J, \epsilon\eta, \epsilon t) + \ldots \quad , \tag{5.7}$$

where $\eta \equiv \{\phi^a_\eta, J_{\eta b}\}$ stands for all the other action- and angle-variables in the problem, and we give names to its action-angle separation. We may then look for a canonical transformation, of type 2, and "near the identity," so that we may say it is small, which will have as new action the variable $\overline{J}$ that will include the action of the perturbation; i.e., J was the associated action when $\epsilon = 0$, while $\overline{J}$ will be the new one when the perturbation is turned on:

$$\begin{gathered} S(\overline{J}, \phi, \epsilon p, \epsilon q, \epsilon t) = \overline{J}\phi + q^a \overline{p}_a + \epsilon\, S_1(\overline{J}, \phi, \overline{p}, q, t) + \ldots \quad , \\ \Longrightarrow \left\{ \begin{aligned} J &= \overline{J} + \epsilon\, \partial S_1/\partial\overline{\phi} \; , \\ \phi &= \overline{\phi} - \epsilon\, \partial S_1/\partial\overline{J} \; , \\ J_{\eta a} &= \overline{J}_\eta a + \epsilon\, \partial S_1/\partial\overline{\phi}^a_\eta \; , \\ \phi^b_\eta &= \overline{\phi}^b_\eta - \epsilon\, \partial S_1/\partial\overline{J}_{\eta b} \; , \\ K(\overline{J}, \epsilon\eta, \epsilon t) &= \epsilon \frac{\partial S}{\partial \epsilon t} + H(J(\overline{J}, \overline{\phi}), \epsilon\overline{\eta}, \epsilon t) \end{aligned} \right. \end{gathered} \tag{5.8}$$

We now determine the expansion of the new Hamiltonian in powers of ϵ, i.e.,

$$K(\overline{J}, \epsilon\eta, \epsilon t) \equiv K_0(\overline{J}, \epsilon\eta, \epsilon t) + \epsilon K_1(\overline{J}, \overline{\phi}, \epsilon\eta, \epsilon t) + \ldots \quad , \tag{5.9}$$

by inserting all the quantities above, and expanding in Taylor series, where we recall that, following Lichtenberg, we are now writing the following form, $\partial A(q)/\partial\overline{q}$ to mean $(\partial A(q)/\partial q)|_{q=\overline{q}}$:

$$\begin{aligned} K_0(\overline{J}, \epsilon\eta, \epsilon t) &= H_0(\overline{J}, \epsilon\eta, \epsilon t) \; , \\ K_1(\overline{J}, \epsilon q, \epsilon p, \epsilon t) &= \frac{\partial H_0}{\partial \overline{J}} \frac{\partial S_1}{\partial \phi} + H_1(\overline{J}, \overline{\phi}, \epsilon q, \epsilon p, \epsilon t) + \epsilon \left\{ \frac{\partial H_0}{\partial \overline{J}_{\eta a}} \frac{\partial S_1}{\partial \epsilon \overline{\phi}^a_\eta} + \frac{\partial S_1}{\partial \epsilon t} \right\} \; , \\ &\ldots \quad , \end{aligned} \tag{5.10}$$

where we have maintained the second-order terms in the first-order expression for S_1 because we want to see what effect those higher-order terms will have: a calculation we will perform shortly. First, however, we remember that we are insisting that K_1 should be independent of ϕ, so that we arrange for this by averaging the entire equation over ϕ, which gives us the following breakdown of the formulae, where we recall that $\dot{\phi} \equiv \omega = \partial H/\partial J$:

$$\begin{gathered} K_1(\overline{J}, \epsilon q, \epsilon p, \epsilon t) = \langle H_1(\overline{J}, \overline{\phi}, \epsilon q, \epsilon p, \epsilon t) \rangle \equiv \frac{1}{2\pi} \int_0^{2\pi} d\phi \; H_1(\overline{J}, \overline{\phi}, \epsilon q, \epsilon p, \epsilon t) \; , \\ \omega_0 \frac{\partial S_1}{\partial \phi} + \epsilon \left\{ \frac{\partial H_0}{\partial \overline{J}_{\eta a}} \frac{\partial S_1}{\partial \epsilon \overline{\phi}^a_\eta} + \frac{\partial S_1}{\partial \epsilon t} \right\} = \langle H_1(\overline{J}, \overline{\phi}, \epsilon q, \epsilon p, \epsilon t) \rangle - H_1(\overline{J}, \overline{\phi}, \epsilon q, \epsilon p, \epsilon t) \; . \end{gathered} \tag{5.11}$$

As his problem is periodic in all the angular variables, we resolve the differential equation for S_1 by resolving it into Fourier components, which gives us the following result, where $\vec{m}$ is a vector of integers and Ω is the frequency for the temporal oscillation:

$$S_1 = i \sum_{\substack{k,\ell,\vec{m}\\ k\neq 0}} \frac{H_{1,k,\ell,\vec{m}}}{k\omega + \epsilon(\ell\Omega + \vec{m}\cdot\vec{\omega}_\eta)} \, e^{i[k\overline{\phi}+\epsilon(\Omega t + \vec{m}\cdot\vec{\overline{\phi}}_\eta)]} \ . \tag{5.12}$$

Inserting this value into the expression for $\overline{J}$, our "adiabatic invariant," we obtain

$$\overline{J} = J - \epsilon \frac{\partial S_1}{\partial \phi} = J + \epsilon \sum_{\substack{k,\ell,\vec{m}\\ k\neq 0}} \frac{k\, H_{1,k,\ell,\vec{m}}}{k\omega + \epsilon(\ell\Omega + \vec{m}\cdot\vec{\omega}_\eta)} \, e^{i[k\overline{\phi}+\epsilon(\Omega t + \vec{m}\cdot\vec{\overline{\phi}}_\eta)]} \tag{5.13a}$$

This is then a form for the transformed action, i.e., the area under the perturbed curves of motion, which involves a lot of ϵ's. In the event that we are justified in ignoring, i.e., setting to zero, all the ϵ's in this expression except for the one in front of the sum, then the k's cancel in the numerator and denominator, we may move the ω in front of the sum, and we have simply the standard first-order expression for an adiabatic invariant of the system:

$$\overline{J} = J + \frac{\epsilon}{\omega} \sum_{k\neq 0} H_{1,k}\, e^{ik\overline{\phi}} = J + \frac{\epsilon}{\omega}\Big(H_1 - \langle H_1\rangle\Big) \ . \tag{5.13b}$$

Since we are beginning from action-angle variables for the zeroth order, that part of H_0 that involved J would simply be ωJ, we see that this expression may be re-thought in the form a form that relates the invariant to this order, $\overline{J}$, with the Hamiltonian to this order. This is a reason for choosing this as an adiabatic invariant. It is particularly interesting in this regard to work through the discussion for the slowly-varying harmonic oscillator—our favorite F, G-equation in one degree of freedom—where he shows, both in (2.3.27) and in (2.3.70) that for this problem the $\epsilon = 0$ case has $J = E/\omega$ while the case first-order in ϵ has $\overline{J} = \overline{E}/\omega$, again explaining the notion of an (adiabatic) invariant for this physical quantity.

We could then continue along these lines. This result may be inserted back into the equations for the second-order correction, resolved, and carried forward each time. On the other hand, looking more carefully at Eq. (5.13a), we can see that the sum over all values of ℓ and the components of $\vec{m}$ must surely eventually give rise to values such that their product with ϵ may no longer be ignored relative to $k\omega$, i.e., they will give rise to resonant denominators which may well be very close to the division by zero problems that cause resonances. Of course this will **not occur** if the coefficients $H_{1,k,\ell,\vec{m}}$ are very, very small, or zero, before this happens. However, this is obviously not the generic situation! Therefore, in general, we are ignoring problems at averaging. This may well be alright, since we desire some good approximation for times that are not very, very long. Nonetheless, it is also reasonable that, having ignored those terms, the resulting series, even when carried out to higher orders, is at best asymptotic, i.e., divergent but a useful approximation if handled properly. Under various conditions, there are theorems, due to Arnol'd, that tell us that at least in the limit as $\epsilon \to 0$ the theory behaves well.

6. Removal of (Individual, Primary) Resonances

We assume the standard form of Hamiltonian, where H_0 is solvable in action-angle variables and H_1 is periodic in the angles, and there are only two degrees of freedom, without time dependence:

$$H = H_0(J) + \epsilon \sum_n H_{(n)}(J)\, e^{in_a\phi^a} \ . \tag{6.1}$$

We further consider the case where the initial conditions allow a resonance to exist between the two unperturbed frequencies:

$$r\frac{\partial H_0}{\partial J_1} = r\omega_1 = s\omega_2 = s\frac{\partial H_0}{\partial J_2} \ , \quad r, s \text{ integers} \ . \tag{6.2}$$

The method of (approximate) removal is via transformation that arranges for a new set of frequencies, one of which is much smaller than the other, and then an appropriate averaging. We do this by switching to new action variables via a standard Type 2 generating function, and picking labels so that ω^2 is the smaller of the two (unperturbed) frequencies:

$$\begin{aligned} & F^2(\phi, \overline{J}) = (r\phi^1 - s\phi^2)\hat{J}_1 + \phi^2 \hat{J}_2 \ , \\ \Longrightarrow \quad & \begin{cases} J_1 = \dfrac{\partial F^2}{\partial \phi^1} = r\hat{J}_1 \ , \quad J_2 = \dfrac{\partial F^2}{\partial \phi^2} = \hat{J}_2 - s\hat{J}_1 \ , \\ \hat{\phi}^1 = \dfrac{\partial F^2}{\partial \hat{J}_1} = r\phi^1 - s\phi^2 \ , \quad \hat{\phi}^2 = \dfrac{\partial F^2}{\partial \hat{J}_2} = \phi^2 \ . \end{cases} \end{aligned} \tag{6.3}$$

Obviously we have

$$\dot{\hat{\phi}}^2 = r\dot{\phi}^1 - s\dot{\phi}^2 \ , \tag{6.4}$$

which would be exactly zero when exactly on resonance, but, more to the point, is that it is very small when we are near resonance, when the perturbative approach might be expected to give trouble; i.e., this very small frequency is a "measure" of how near we are to exact resonance. Our new perturbing Hamiltonian is then

$$\hat{H}_1 = \sum_{(n_1, n_2)} H_{(n)}(\hat{J})\, e^{i[n_1\hat{\phi}^1 + (n_1 s + n_2 r)\hat{\phi}^2]/r} \ . \tag{6.5}$$

The idea now is to average over the faster variable, which is now $\hat{\phi}^2$, thereby creating an approximate Hamiltonian that depends only on the other angle. (It's worth noting that we originally chose to retain the variable ϕ^2, which in that system had the smaller frequency, so that would retain the lowest-order harmonics, which may well be important at second order in the perturbation theory.) This gives us the new Hamiltonian

$$\overline{H} \equiv \overline{H}_0(\hat{J}) + \epsilon\langle \hat{H}_1(\hat{J}, \hat{\phi})\rangle_{\hat{\phi}^2} = \overline{H}_0(\hat{J}) + \sum_{p=-\infty}^{+\infty} H_{pr,-ps}\, e^{ip\hat{\phi}^1} \ . \tag{6.6}$$

Since the Hamiltonian is independent of $\hat{\phi}^2$, we know that $\hat{J}_2$ is constant. Actually, this is of course only true to within the approximation we have made, which can be looked upon as some sort of *adiabatic approximation.* Therefore, we must in fact come back and study that a bit more than we have so far. However, for now we will go forward with this approach, considering only this approximate Hamiltonian for now. Usually the approximation is quite good, since we after all are really only interested in those cases very close to resonance, so that the variation is very small. Therefore, at least to this order of approximation, we have now reduced the problem to one with only one degree of freedom; all such systems are in fact integrable, and in fact, because of our restriction to cyclic systems, are going to have a structure, at least locally, much like the pendulum, or, perhaps, several copies of it. We proceed ahead, where we will be able to divide systems of this sort into two rather different types.

For a problem in one degree of freedom, we recall that we first look for the stationary points, i.e., those points for which

$$\text{stationary points:}\quad \frac{\partial\overline{H}}{\partial\hat{J}_1}=0\,,\qquad \frac{\partial\overline{H}}{\partial\hat{\phi}^1}=0\,. \tag{6.7a}$$

At the resonance which caused us to look at this problem, i.e., the one where Eq. (6.2) is valid so that $r\omega_1 = s\omega_2$, the fact that the two frequencies are commensurate would tell us that the total motion is actually periodic for those initial conditions, and therefore degenerate in the sense that one may vary other constants of the motion without affecting this. More precisely, the periodicity came because we had, at Eq. (6.2), the fact that $r\omega_1(J) - s\omega_2(J) = 0$, a relation independent of the ϕ-variables, which is the explicit statement of the *degeneracy.*

This degeneracy is of course only true for the original Hamiltonian; with our perturbation the degeneracy should be lifted, so that now we are only left with the stationary points described by the previous equation, Eq. (6.7a). In principle, we should now apply that to our complete form for he perturbing Hamiltonian, given in Eq. (6.6). Lichtenberg, on the other hand, suggests that it should often/usually be the case that the coefficients $H_{pr,-ps}$ will be very small except for the first few, so that he truncates that infinite series to just the lowest 3 terms, i.e., the terms where p takes on the values 0 and ±1. Moreover, he insists that one may always arrange it so that $H_{n_1,n_2} = H_{-n_1,-n_2}$. This last of course is a statement about switching the sign of the variables in the original function, before expanding it in the Fourier series. As it is indeed periodic in each of them, presumably this is reasonable that this can be arranged simply by proper choice of the origin, which causes the **truncated, adiabatic approximate of the perturbing Hamiltonian** to be

$$\overline{H}\approx \hat{H}_0(\hat{J})+\epsilon H_{0,0}(\hat{J})+2\epsilon H_{r,-s}(\hat{J})\,\cos\hat{\phi}^1\,. \tag{6.7b}$$

Insertion of this into the stationarity requirements above simply gives the following two equations, where we denote the values of $\hat{J}_1$ and $\hat{\phi}^1$ for which we have a stationary point by an additional subscript, 0:

$$\begin{aligned}\frac{\partial\hat{H}_0}{\partial\hat{J}_{10}}+\epsilon\left\{\frac{\partial H_{0,0}}{\partial\hat{J}_{10}}+2\frac{\partial H_{r,-s}}{\partial\hat{J}_{10}}\cos\phi_0^1\right\}&=0\,,\\ 2\epsilon H_{r,-s}\sin\hat{\phi}_0^1&=0\,.\end{aligned} \tag{6.7c}$$

The second equation tells us immediately, not unexpectedly, that there are two fixed points at $\hat{\phi}_0^1 = 0\,,\,\pi$. Looking at the first equation we first evaluate it without the perturbation:

$$0=\frac{\partial\hat{H}_0}{\partial\hat{J}_{10}}=\frac{\partial J_1}{\partial\hat{J}_1}\frac{\partial\hat{H}_0}{\partial J_{10}}+\frac{\partial J_2}{\partial\hat{J}_1}\frac{\partial\hat{H}_0}{\partial J_{20}}=r\frac{\partial\hat{H}_0}{\partial J_{10}}-s\frac{\partial\hat{H}_0}{\partial J_{20}}=r\omega_1-s\omega_2=0\,. \tag{6.7d}$$

This is of course what was wanted, and expected; the explicit calculation was just a check. Therefore, we now consider the same question when there is a perturbing influence:

$$\frac{\partial\hat{H}_0}{\partial\hat{J}_{10}}+\epsilon\left\{\frac{\partial H_{0,0}}{\partial\hat{J}_{10}}\pm 2\frac{\partial H_{r,-s}}{\partial\hat{J}_{10}}\right\}=0\,. \tag{6.7e}$$

There are now two different possibilities to contend with, depending on the nature of the original resonance, or periodic solution, for the unperturbed problem. For the particular choice of resonance that we are considering, it could be that it would occur for all initial values of J_1 and J_2. This would mean that the unperturbed Hamiltonian, H_0 was a function only of the linear combination $sJ_1 + rJ_2$, as opposed to the more generic case where H_0 depends independently on both J_1 and J_2. In this case the resonance comes only for certain particular values of the two actions, rather than for all of them. We refer to this generic case as **an accidental degeneracy** and describe it by saying that $H_0 = H_0(J_1, J_2)$, while in the more special case, we say that the (unperturbed) system has **an intrinsic degeneracy** and describe it by saying that $H_0 = H_0(sJ_1 + rJ_2)$, only. In this second case, of course we may then say that $\hat{H}_0 = \hat{H}_0(\hat{J}_2)$, only.

We want to discuss both these cases, but begin with the first one, **the case of accidental degeneracy**: Using Hamilton's equations, we see that

$$\dot{\hat{J}}_1 = -\partial\overline{H}/\partial\hat{\phi}^1 = 2\epsilon H_{r,-s}\sin\phi^1 = O(\epsilon H_{r,-s})\ , \qquad \dot{\hat{\phi}}_1 = \partial\overline{H}/\partial\hat{J}_1 = \partial\hat{H}_0/\partial\hat{J}_1 = O(1)\ , \tag{6.8a}$$

so that we expect the excursions away from being constant to be much smaller for $\hat{J}_1$ than for the angular variable, $\hat{\phi}^1$. On the other hand the other variables are essentially constant at the moment, so they are being ignored. To understand the behavior in the neighborhood of our stationary point, at $\hat{J}_{10}$, we expand away from there in an ordinary Taylor series, taking the distance away as $\Delta\hat{J}_1 \equiv \hat{J}_1 - \hat{J}_{10}$. We do this for each term defining the complete, approximate Hamiltonian, as given in Eq. (6.7c), expanding to second order and treating $\Delta\hat{J}_1$ and ϵ as of "equal" smallness:

$$\begin{aligned}
\hat{H}_0(\hat{J}_1) &= \hat{H}_0(\hat{J}_{10}) + \frac{\partial\hat{H}_0}{\partial\hat{J}_{10}}\Delta\hat{J}_1 + \tfrac{1}{2}\frac{\partial^2\hat{H}_0}{\partial J_{10}^2}(\Delta\hat{J}_1)^2 + \ \ldots\ \ , \\
\epsilon H_{0,0}(\hat{J}_1) &= \epsilon H_{0,0}(\hat{J}_{10}) + \epsilon\frac{\partial H_{0,0}}{\partial\hat{J}_{10}}\Delta\hat{J}_1 + \ \ldots\ \ , \\
2\epsilon H_{r,-s}(\hat{J}_1)\cos\hat{\phi}^1 &= 2\epsilon H_{r,-s}(\hat{J}_{10})\cos\hat{\phi}^1 + 2\epsilon\frac{\partial H_{r,-s}}{\partial\hat{J}_{10}}\Delta\hat{J}_1\cos\hat{\phi}^1 + \ \ldots\ \ ,
\end{aligned}\tag{6.8b}$$

where it's also worth noting that I am now beginning to use Lichtenberg's simplified convention for derivatives which are evaluated somewhere; i.e., for instance if we have a function $H = H(J_1)$, then we use the simplified notation:

$$\frac{\partial H}{\partial J_{10}} \ \equiv\ \left.\frac{\partial H}{\partial J_1}\right|_{J_1 = J_{10}}\ . \tag{6.8c}$$

Remembering that the sum of the three different lines above, in Eqs. (6.8b), are our approximate Hamiltonian. Therefore, we can add them up, and obtain that (desired) quantity. Looking at the three different lines, we first notice that the first terms of each of the first two lines are simply constant, so that we ignore them, or, if one prefers, we can say that we are now writing the Hamiltonian "away from" those constant terms, since we are interested in motions "away from," but certainly near, the stationary points. Next we notice that the sum of the **second terms of each line** simply gives the left-hand side of Eq. (6.7c), which is required to vanish in order to located the stationary points in question. This leaves us with only the third term of the first line and the first term of the third line, which can be written in the form:

$$\hat{H}(\hat{J}_1) = \tfrac{1}{2}G\,(\Delta\hat{J}_1)^2 - F\,\cos\phi^1\ ; \qquad G \equiv \frac{\partial^2\hat{H}_0}{\partial\hat{J}_{10}^2}\ , \quad F \equiv -2\epsilon H_{r,-s}(\hat{J}_{10})\ . \tag{6.8d}$$

Here both F and G are constants, being relevant functions evaluated at the stationary point; therefore, this is exactly the same form as the simple pendulum equation, that we (usefully) spent time studying earlier! From there we recall that the frequency of oscillation about the stationary point is then given by $\hat{\omega}_1 = \sqrt{FG} = O[\sqrt{\epsilon H_{r,-s}}]$, so that it is rather slow, much smaller than the frequency of oscillation of the currently-being-ignored frequency in the $\hat{J}_2, \hat{\phi}^2$ system, which is of the order of 1. The maximum excursion, on any particular trajectory near the stationary point would be the maximum value of the "momentum variable," $\Delta\hat{J}_1$, which is given by $2\sqrt{F/G}$, which is of the same order of smallness as the frequency noted just above.

On the other hand, we should now consider the second case, that of **intrinsic degeneracy**, which occurs when the system is such that $\hat{H}_0$ is independent of $\hat{J}_1$. We must then repeat Eqs. (6.8a), determining the rates of change of our variables, which gives us

$$\begin{aligned} \dot{\hat{J}}_1 &= -\partial\overline{H}/\partial\hat{\phi}^1 = 2\epsilon H_{r,-s}\sin\phi^1 = O(\epsilon H_{r,-s}) \ , \\ \dot{\hat{\phi}}_1 &= \partial\overline{H}/\partial\hat{J}_1 = \epsilon\Big(\partial H_{0,0}/\partial\hat{J}_1 + \partial H_{r,-s}/\partial\hat{J}_1\Big) = O(\epsilon H_{0,0}, \epsilon H_{r,-s}) \ , \end{aligned} \tag{6.9a}$$

which tells us that the excursions in the two variables are of the same order. Therefore, we cannot just linearize in one of the variables, alone, but must actually do something with both of them at once. Therefore, now let the stationary point for $\hat{\phi}^1$ be given by $\hat{\phi}_{10}$, and we expand around it as well. The argument given above must now have the cosine function expanded as well, in the difference $\Delta\hat{\phi}^1$, and we find, to second order the result

$$\hat{H} = \tfrac{1}{2}G(\Delta\hat{J}_1)^2 + \tfrac{1}{2}F(\Delta\hat{\phi}^1)^2 \ , \quad G \equiv \frac{\partial^2\hat{H}_0}{\partial\hat{J}_{10}^2} + \epsilon\frac{\partial^2 H_{0,0}}{\partial\hat{J}_{10}^2} + 2\epsilon\frac{\partial^2 H_{r,-s}}{\partial\hat{J}_{10}^2} \ , \ F \equiv -2\epsilon H_{r,-s} \ , \tag{6.9b}$$

where we have kept higher-order terms now, since we have nothing "large" to compare against. In particular, if we are surely in this case of intrinsic degeneracy then the first term in G, above, is zero, so that both F and G are of order ϵ, so that the frequency is of order ϵ as well, but the excursion, i.e., the ratio of the axes of the elliptical motions around the stationary point, being of the order of $\sqrt{F/G}$, is of order unity. Of course, truly, as the transition occurs, in a system, from accidental to intrinsic degeneracy, it is exactly the vanishing of that term in the definition of G above that denotes this occurrence.

Of course at the stationary point at $\hat{\phi}^1 = \pi$, the cosine function has the opposite sign, which causes the approximation to the cosine to be $-1 + \frac{1}{2}(\Delta\hat{\phi}^1)^2 + \dots$, so that our reduced Hamiltonian becomes simply

$$\hat{H} = \tfrac{1}{2}G(\Delta\hat{J}_1)^2 - \tfrac{1}{2}F(\Delta\hat{\phi}^1)^2 \ , \tag{6.9c}$$

and this is the equation of a (pair of) hyperbolas, near that point, as one would expect.

When we averaged over the second angular variable, $\phi^2 = \hat{\phi}^2$, it was quite reasonable as a first approximation; on the other hand, when ϵ is not sufficiently small, there may have been *secondary resonances* in the original Hamiltonian, which may contribute secular terms that modify or destroy the adiabatic invariance of the corresponding action, $\hat{J}_2$. These would be resonances between harmonics of the $\hat{J}_1, \hat{\phi}^1$-oscillations just discussed, and the fundamental frequency ω_2 of the other angular variable. As it turns out, these secondary resonances give rise, in the adiabatic limit after they have been taken into account, to *"island chains"* of trajectories in the Poincaré sections. There are also ways to remove them, which we will talk about.

I also comment that in the handout section on examples there is an example due to Walker and Ford, that specifically describes this sort of Hamiltonian, this removal, and the general effect.

7. Understanding, and Removal of Secondary Resonances