Examples Taking us Toward Secondary Resonances

1. The *Galactic Potential* Problem

There was a lot of work at several times during the twentieth century to try to better understand the problem of motion in a physical, i.e., 3-dimensional potential with cylindrical symmetry. With such a symmetry, the obvious coordinates are $\{\rho, \phi, z\}$, with the potential independent of ϕ, thus creating the symmetry. At least one important motivation for this problem was the belief that it constituted a generalized "galactic potential," in which we could imagine single stars, or star systems, moving in the overall potential generated by the entire galaxy, which generated many papers in *The Astronomical Journal* and *The Astrophysical Journal*; Contopoulos, and some other students and former students of Chandrasekhar were quite involved in this. A somewhat different reason for pushing such a problem forward was simply the thought that it was basically the simplest, physically-motivated, 2-dimensional problem that they could come up with; these were the thoughts, for instance, that motivated Whittaker to study it. From either point of view, the language that comes up is that there are two obvious constants of the motion, the energy and the z-component of the angular momentum. Since there are 3 degrees of freedom, the problem would be resolved, in the sense of Liouville, were one to find yet one more such constant; therefore, many of the papers focus on the question of *the third integral.* Of course, when the problem is reduced to one of only two real degrees of freedom, using the angular momentum to make this reduction, then it appears as one with only two coordinates, and one is then looking for a "second integral." In particular, Michel Hénon and Carl Heiles, in 1964, then at Princeton, created what is today referred to as the Hénon-Heiles potential, and studied it using the computers available at that time:

$$H = \frac{1}{2}(p_x^2 + p_y^2 + x^2 + y^2) + \epsilon(x^2 y - y^3/3) \ , \tag{1.1}$$

where this is indeed a reduction of the obvious problem in 3 dimensions with such a cylindrically-symmetric potential. On the other hand, the particular choice of potential, i.e., the additional term (multiplied by ϵ in our notation, not theirs) was chosen not because it satisfied some particular physical criteria, but, rather, that it was not too complicated, i.e., only of third order in the coordinates, and exhibited what they referred to as very interesting, and certainly unexpected, numerical results. [Of course units have been chosen so that everything is dimensionless, with, in particular, the two fundamental frequencies (of the unperturbed system) set equal to 1, and also to each other.] It turns out that it does NOT have a "third integral," (which would be a second integral in their problem since they have already used the angular momentum integral for the reduction to two degrees of freedom). They showed that this fact was "believable" because of the chaotic nature of the Poincaré sections of the trajectories when the (total) energy was large enough. We will talk in some detail about this Hamiltonian, and some of its properties. However, first I want to diverge both directions from it: first, to a potential that appears to be more complicated, and, moreover, which has this Hamiltonian as its lowest-order limit, but which nonetheless does have a third integral. Secondly, to go the other way and look at some simpler models where we can more immediately determine the behavior, and its unexpected nature. Having done that, we will wander forward into more general theory again, trying to understand in more analytic detail some of the phenomena that we have seen.

Firstly, let's go quickly and look at the Hamiltonian for the Toda lattice for three particles moving on a ring, with exponentially decreasing forces between them [M. Toda, Progress of Theoretical Physics Supplement **45**, p. 174 (1970)]:

$$H = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + e^{-(\phi^1 - \phi^3)} + e^{-(\phi^2 - \phi^1)} + e^{-(\phi^3 - \phi^2)} - 3 \ . \tag{1.2}$$

The energy is a constant; as well, the total momentum, $P_3 \equiv p_1+p_2+p_3$ is a constant of the motion, as is obvious because it is invariant under a rigid rotation of all the angular coordinates. On the other hand, one can easily calculate its time derivative, from Hamilton's equations, and acquire zero. Since that's so, we can use it to again reduce the problem to one in only two degrees of freedom. We change coordinates via a generating function

$$\begin{aligned}
&F^2(q,P) = P_1\phi^1 + P_2\phi^2 + (P_3 - P_1 - P_2)\phi^3 \\
\Longrightarrow\quad &\begin{cases} \quad P_1 = p_1\,,\, P_2 = p_2\,,\; P_3 = p_1+p_2+p_3\,, \\ \Phi^1 = \phi^1 - \phi^3\,,\, \Phi^2 = \phi^2-\phi^3\,,\; \Phi^3 = \phi^3\,, \end{cases} \\
\Longrightarrow\quad &K = \tfrac{1}{2}[P_1^2 + P_2^2 + (P_3 - P_1 - P_2)^2] + e^{-\Phi^1} + e^{-\Phi^2+\Phi^1} + e^{+\Phi^2} - 3\,.
\end{aligned} \tag{1.3a}$$

Since P_3 is now a constant, since Φ^3 is cyclic, we can simply ignore that constant, i.e., set it equal to zero, and, as already mentioned, reduce the problem to one of 2 variables, with Hamiltonian

$$K' \equiv P_1^2 + P_2^2 + P_1P_2 + e^{-\Phi^1} + e^{-\Phi^2+\Phi^1} + e^{+\Phi^2} - 3\,. \tag{1.3b}$$

Because of the relationship of this problem to that of the Hénon-Heiles one, Ford and some co-workers, investigated it, looking for the same chaotic behavior as in the HH-Hamiltonian [Ford, Stoddard and Turner, *Progress in Theoretical Physics* **50**, 1547 (1973)]; they were quite surprised to not find it. Therefore, Hénon found a second integral [Hénon, *Phys. Rev.*__B9__, 1925 (1974)], given by the following **cubic** combination of the variables:

$$I \equiv P_1P_2(P_1+P_2) + P_2e^{-\Phi^1} + P_1e^{+\Phi^2} - (P_1+P_2)e^{\Phi^1-\Phi^2}\,. \tag{1.3c}$$

Calculation of the Poisson bracket with K, with $P_3 = 0$, is straightforward, and indeed does give zero. Having two constants of the motion, the problem is integrable, and every motion should occur on a 2-dimensional surface, in its 4-dimensional phasespace. For instance, we may create the continuous curves in a Poincaré section by, for instance, choosing one of the coordinates, say Φ^1, to have some fixed value, resolving, say $K' = E$, as an equation for P_1 and inserting that value into the equation for I.

On the other hand, we now explain how one puts it in the form that it has the HH Hamiltonian as a limit:

$$\begin{gathered}
\Phi^1 = 2(\sqrt{3}x - y)\,,\; \Phi^2 = 2(\sqrt{3}x + y)\,,\, P_1 = 2(p_x - \sqrt{3}p_y)\,,\; P_2 = 2(p_x + \sqrt{3}p_y)\,, \\
\overline{H} = K'/24 = \tfrac{1}{2}(p_x^2 + p_y^2) + \tfrac{1}{24}[e^{2(y+\sqrt{3}x)} + e^{2(y-\sqrt{3}x)} + e^{-4y} - 3]\,.
\end{gathered} \tag{1.3d}$$

The normalization has been so chosen so that if we now expand this potential function to lowest orders in both x and y, we acquire the Hénon-Heiles Hamiltonian, i.e., Eq. (1.1) with $\epsilon = 1$; i.e., our Hamiltonian in Eq. (1.3d) has the form

$$\overline{H} = \tfrac{1}{2}(p_x^2 + p_y^2 + x^2 + y^2) + x^2y - y^3/3 + \tfrac{1}{2}(x^2+y^2)^2 + x^4y + 2x^2y^3/3 - y^5/3 + O^6(x,y)\,. \tag{1.3e}$$

There is a Maple file showing some of the details of the Hénon-Heiles potential, the third-order truncation of this one, which is in fact now not integrable. It is unbounded for energies greater than 1/6. For energies much less than that it shows well-behaved motions on only slightly-distorted tori. As the energy increases toward this value of 1/6, the motions begin to become chaotic, in complicated ways.

Let's now consider a different Hamiltonian system, which was first presented in an article by G.H. Walker and J. Ford, "Amplitude Instability and Ergodic Behavior for Conservative Nonlinear Oscillator Systems," Phys. Rev. **188**, 416-432 (1969), which is already in action-angle form:

$$\begin{gathered}
H = H_0(J) + \alpha\, H_1(J,\phi) + \beta\, H_2(J,\phi)\ , \\
H_0 = J_1 + J_2 - J_1^2 - 3J_1J_2 + J_2^2\ , \\
H_1 = J_1J_2\cos 2(\phi^1-\phi^2)\ , \qquad H_2 = (J_1)^{3/2}J_2\cos(3\phi^1-2\phi^2)\ , \\
J_i = \frac{1}{2}[(p_i)^2+(q^i)^2]\ , \qquad q^i = \sqrt{2J_i}\cos\phi^i\ ,\ p_i = -\sqrt{2J_i}\sin\phi^i\ .
\end{gathered} \tag{1.4}$$

where we will consider both α and β as perturbation parameters, i.e., "small" like ϵ, but will first consider each of them independently while letting the other one vanish. The relation between these action variables and the original coordinates, on phase space, is given above; that makes it clear that we have made various choices, such as $m\omega_0 = 1$, so as to have dimensionless variables. The Hamiltonian in fact is presented as a special sort of a model, built in a similar way to the famous Henon-Heiles Hamiltonian, also discussed in their paper at some length. That Hamiltonian, in these coordinates, has the form

$$H_{HH} = J_1 + J_2 + f(J_I\cos^2\phi^i)\ ,$$

where the function f is cubic in the q^j's only.

We first proceed with the case with $\beta = 0$ and $\alpha \neq 0$. While H_0 depends only on the actions, so that both the action variables are constant as far as it is concerned; however, when the perturbation is turned on we have

$$\begin{aligned}
\dot J_1 &= -\partial H/\partial\phi^1 = 2\alpha J_1J_2\sin 2(\phi^1-\phi^2) + 3\beta J_1^{3/2}J_2\sin(3\phi^1-2\phi^2)\ , \\
\dot J_2 &= -\partial H/\partial\phi^2 = -2\alpha J_1J_2\sin 2(\phi^1-\phi^2) - 2\beta J_1^{3/2}J_2\sin(3\phi^1-2\phi^2)\ , \\
\omega_1 &\equiv \omega_{10} + \alpha\omega_{11} + \beta\omega_{12} = 1 - 2J_1 - 3J_2 + \alpha J_2\cos 2(\phi^1-\phi^2) + \tfrac{3}{2}\beta J_1^{1/2}J_2\cos(3\phi^1-2\phi^2)\ , \\
\omega_2 &\equiv \omega_{20} + \alpha\omega_{21} + \beta\omega_{22} = 1 - 3J_1 + 2J_2 + \alpha J_1\cos 2(\phi^1-\phi^2) + \beta J_1^{3/2}\ \cos(3\phi^1-2\phi^2)\ .
\end{aligned} \tag{1.5a}$$

We can easily see the **four relevant cases**:

0. When both α and β vanish, then there is no perturbation, and both J_1 and J_2 are constants of the motion;

α. when only α is non-zero, then $J_1 + J_2$ is a constant of the motion, i.e., $\{J_1 + J_2, H\} = 0$;

β. when only β is non-zero, then $2J_1 + 3J_2$ is a constant of the motion;

1. but when both α and β are non-zero there is no combination of the J_i's that is a constant of the motion.

The first case is straightforward, and "trivial," being simply two uncoupled harmonic oscillators, so that the motions are all integrable and lie on a 2-torus with each projection being elliptical when they're stable, with a separatrix dividing these from the unbounded motions, etc. The second and third cases are both still integrable, although they have much more complicated structure. Lastly the final case is not integrable and, indeed, does have chaotic motions. We will now concentrate on the second case, when α is non-zero, both β vanishes. Since both the energy E and the sum $J_1 + J_2$ are constants of the motion, the problem is really only two-dimensional, and we should be able to understand it better that way. On the other hand, there will be resonances where perturbation theory will not suffice. In this case we would expect a resonance when there are integers such that $r\omega_1 = s\omega_2$. Because of the unperturbed values for these frequencies, we see that they will truly only occur when

$$n\omega_1 = n\omega_2 \quad \Longrightarrow \quad J_1 = 5\,J_2\ . \tag{1.5c1}$$

On the other hand, when there is a non-zero value of the perturbation we could expect that the behavior is then given by

$$J_1 = J_2 \frac{5 - \alpha\cos(\phi^1 - \phi^2)}{1 - \alpha\cos(\phi^1 - \phi^2)}\ . \tag{1.5c2}$$

We will see below how to properly consider such troublesome points, i.e., points where resonances arise, or, if we prefer, points where there are true periodic orbits, and for points near these singular points.

This is exactly the sort of thing that was described, more generically above. We suppose a canonical transformation to new variables, $\{\theta^1, \theta^2; I_1, I_2\}$, given by

$$I_1 \equiv J_1\ ,\ I \equiv I_2 \equiv J_1 + J_2\ ,\quad ,\quad \theta^1 \equiv \phi^1 - \phi^2\ ,\ \theta^2 \equiv \phi^2\ , \tag{1.5d}$$

which may be verified as a canonical transformation by simply calculating the Poisson brackets $\{I_j, \theta^k\} = \delta_j^k$. The new Hamiltonian is then

$$K = I + I^2 - 5II_1 + 3I_1^2 - \alpha(I_1 - I)I_1\cos 2\theta^1 = E\ , \tag{1.5e}$$

so that θ^2 is a cyclic coordinate, ensuring, as intended, that its conjugate, $I_2 = J_1 + J_2 \equiv I$, is a constant of the motion. In these new coordinates, we may rewrite the various time derivatives:

$$\begin{aligned} \hat{\omega}^1 &= d\theta^1/dt = \partial K/\partial I_1 = 6I_1 - 5I + \alpha(I - 2I_1)\cos 2\theta^1\ ,\\ \hat{\omega}^2 &= d\theta^2/dt = \partial K/\partial I_2 = 1 - 5I_1 + 2I + \alpha I_1\cos 2\theta^1\ ,\\ dI_1/dt &= 2\alpha(I - I_1)I_1\sin 2\theta^1\ ,\quad dI/dt = 0\ . \end{aligned} \tag{1.5f}$$

We now look for the stationary points in the θ^1, I_1-plane, again denoting those points with an additional subscript of 0:

$$\begin{aligned} \frac{dI_1}{dt} = 0 \quad &\Longrightarrow \quad \begin{cases} \theta_0^1 &= n\pi/2\ ,\\ \text{or} & \\ I_{10} &= I\ . \end{cases}\\ \frac{d\theta^1}{dt} = 0 \quad &\Longrightarrow \quad 5I = 6I_{10} + \alpha(I - 2I_{10})\cos 2\theta_0^2\ . \end{aligned} \tag{1.6}$$

Now if we make the second choice for the first of these equations, then the second one requires that $I = 0 = I_{10}$. This is not a particularly interesting choice, so we follow along with the other one instead. Putting the two together, their resolution is given by the following:

$$\text{stationary points:}\ \ \theta_0^1 = n\pi/2\ ,\ I_{10} = \frac{1}{2}I\left(\frac{5 - (-1)^n\alpha}{3 - (-1)^n\alpha}\right) = \frac{5}{6}I\,\left(1 + \tfrac{2}{3}(-1)^n\alpha + O(\alpha^2)\right)\ . \tag{1.7a}$$

It is probably also valuable to write out the location of these stationary points in the original coordinates, which will allow us to notice that they sit squarely on top of the resonances, between the two initial frequencies, described above in Eq. (1.5c2):

$$\text{stationary points:}\;\; \phi_0^1 - \phi_0^2 = n\pi/2\ ,\ J_{10} = J_{20}\left(\frac{5-(-1)^n\alpha}{1-(-1)^n\alpha}\right)\ . \tag{1.7b}$$

At this point, one would like to know something concerning the stability of these singular points. To do that, we may begin by asking for *linear stability* around these stationary points: We may write out

$$\begin{gathered} I_1 = I_{10} + \Delta I_1\ ,\theta^1 = \theta_0^1 + \Delta\theta^1 = n\pi + \Delta\theta^1\ , \\ \frac{d}{dt}\Delta\theta^1 = [6-2\alpha(-1)^n]\Delta I_1\ , \qquad \frac{d}{dt}\Delta I_1 = 4\alpha I_{10}(I-I_{10})(-1)^n\Delta\theta^1\ , \\ \text{or}\quad \frac{d}{dt}\begin{pmatrix}\Delta\theta^1 \\ \Delta I_1\end{pmatrix} = \begin{pmatrix} 0 & 6-2\alpha(-1)^n \\ 4\alpha(-1)^n I_{10}(I-I_{10}) & 0\end{pmatrix}\begin{pmatrix}\Delta\theta^1 \\ \Delta I_1\end{pmatrix}\ . \end{gathered} \tag{1.8a}$$

The solution of such an equation would be of exponential form, and simply depend on the eigenvalues multiplied by the time:

$$\lambda\, t = \pm\sqrt{[6-2\alpha(-1)^n]4\alpha(-1)^n I_{10}(I-I_{10})}\ t\ . \tag{1.8b}$$

Provided that $|\alpha| < 3$, we see that

a.) when n is even, the eigenvalues will both be real, so that the resulting stationary point will be **unstable**, i.e., hyperbolic, and we may consider

$$\theta^1 = 0\ ,\quad J_{10} = J_{20}\frac{5-\alpha}{1-\alpha}\ \ ; \tag{1.9a}$$

b.) while when n is odd, the eigenvalues will both be imaginary, so that the resulting stationary point will be **stable**, i.e., elliptical, and we may consider

$$\theta^1 = \pm\pi/2\ ,\quad J_{10} = J_{20}\frac{5+\alpha}{1+\alpha}\ \ . \tag{1.9b}$$

Since I is a constant, the motion is actually all in one plane, in this particular set of variables, so we don't really even have to consider a section; we can simply consider plots of I_1 versus θ^1 as given by Eq. (1.5e), for various different initial conditions, i.e., by various different values of E and I. Although this equation simply gives I_1 as a quadratic function of $\cos 2\theta^1$ and the constants, there are other reasons why they are more enlightening when presented directly in terms of the original coordinates, i.e., q^1 and p_1. In terms of these variables, we know that the unperturbed problem is just an independent pair of harmonic oscillators, so that the unperturbed orbits are just circles. We can then move out away from this zero value of the perturbation and see what effect it has on those circular motions. In particular, we would like to choose values for the various constants involved so that we may actually view the singular points we have been describing. Therefore, let us first consider for which values of the energy they will exist. This is very similar to the situation for the Hénon-Heiles potential, in the sense that after the energy becomes sufficiently large, the orbits are unbound. In particular, we ask, for the unperturbed motions, which are the allowed values of the energy such that we can have non-negative frequencies at the stationary points; this is simply a way of describing mathematically the physical situation that we want to perturb. For zero perturbation, we simply have the relationship $J_1 = 5J_2$ at the stationary points. Inserting this into the equation for the energy, we find that J_1 and J_2 are imaginary when $E > 3/13$. Therefore, we are interested in values of E less than that. When we allow α to be different from zero, then the expression becomes rather more complicated, but it is still true that one needs to have the energy below some upper bound in order to have real stationary points. In particular, inserting the relation between the two actions, at the singular points, for non-zero α, we may resolve the equation for the energy to better describe the behavior at the stationary points. Because of the differences between the elliptic and the hyperbolic points, as already noted above, we must look at them separately in the two different cases.

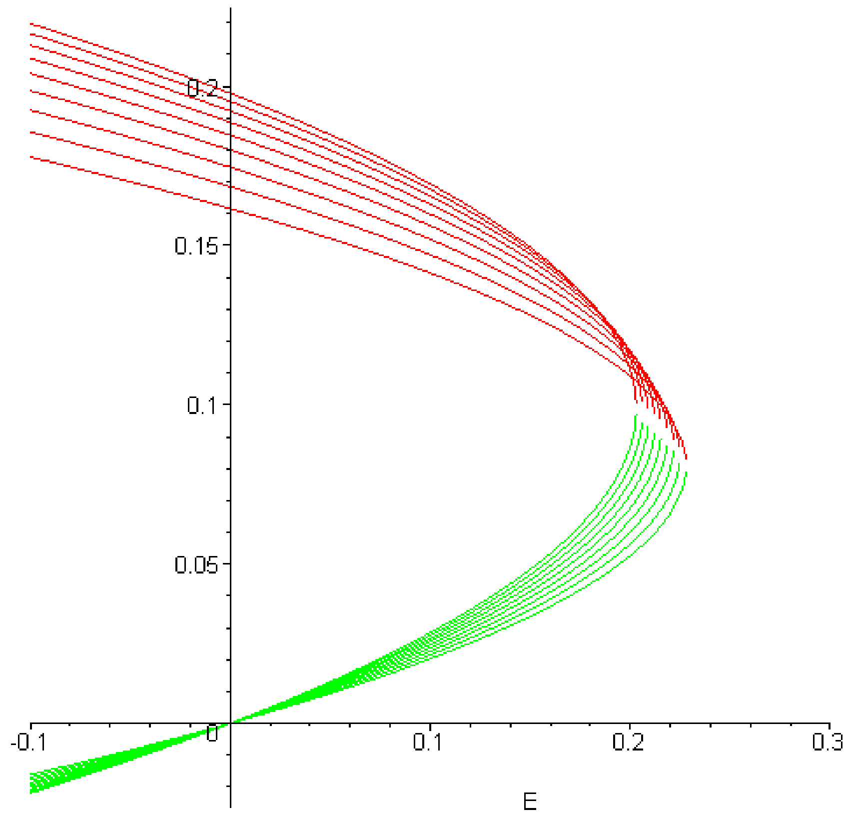
We first consider the case where n is even, so that it may, for instance, be restricted to have the value 0; **these are the unstable stationary points.** Nonetheless, we will also consider the equivalent cases ± 2, which will be helpful because of the way the trigonometric functions behave. Again inserting the relationship between J_{10} and J_{20}, from Eq. (1.9a), into the equation for the energy, we may resolve J_{20} as a function of the energy, and also α, inserting the value $\theta_0^1 = 0$ for the angle, where we note that the important denominator, $D_0 = D_0(\alpha)$ below has no extrema and is negative until its one root at $\alpha = 3$:

$$J_{20} = (\alpha - 1)\frac{3 - \alpha \pm \sqrt{(3-\alpha)^2 + ED_0}}{D_0}\ , \quad J_{10} = \frac{\alpha - 5}{\alpha - 1} J_2\ , \qquad D_0 \equiv \alpha^3 - 9\alpha^2 + 31\alpha - 39 = (\alpha - 3)(\alpha^2 - 6\alpha + 13)\ . \tag{1.10a}$$

It is clear that the energy again has an upper limit, so that

$$J_{20} \ \text{ is real only when } \ E < E_{\max}(\alpha) = (3-\alpha)/(\alpha^2 - 6\alpha + 13)\ , \tag{1.10b}$$

where $E_{\max}$ has the value $3/13$ when $\alpha = 0$, as already noted, and rises slowly to $1/4$ at $\alpha = 1$ and then moves downward to 0 at $\alpha = 3$, beyond that becoming negative, reaching $-1/4$ at $\alpha = 5$ and then slowly rising, remaining negative, to asymptote out at zero for infinite α. When E is in this acceptable range, the function $J_{20}(E, \alpha)$ is double-valued and traces out a parabola, with horizontal symmetry axis and turnaround point at $E = E_{\max}$, where we have $J_{20}(E_{\max}) = (1-\alpha)/(\alpha^2 - 6\alpha + 13)$.



J20 versus E for various values of α between 0 and .9

All these parabolas pass through $E = 0$ at both $J_{20} = 0$ and at twice the value at $E_{\max}$. They all have this general shape so long as $\alpha < 1$. It is also worthwhile, as we will see soon, to consider where these (unstable) stationary points are in terms of the original variables. In order to do this, we must note that they are well-defined with respect to the (current) angular variable, $\theta^1 = \phi^1 - \phi^2$. Relative to the original angles, ϕ^j, we must realize that the θ^1 plane is osculating relative to these original angles, so that this plane, and the associated motion is embedded in a 3-dimensional surface. A 2-dimensional presentation of the motions in terms of the original angles requires us to to look at some Poincaré section. We choose to do this by choosing to look at the so-called 1-plane, i.e., the plane with coordinates q^1 and p_1, taking the sections where $\phi^2 = -\pi/2$. Recalling the relationships, this tells me that $q^2 = 0$ and $p_2 = \sqrt{2J_2} = \sqrt{2(I - I_1)}$. In terms of these coordinates the stationary point then corresponds to $\phi^1 = \mp\pi/2$, where the $\mp$ comes from choosing n even, as either 0 or 2. Therefore at that point $q^1_0 = 0$ and $p_{10} = \pm\sqrt{2J_1}$, which tells us that the Poincaré sections, in the 1-plane, will have these points on the vertical (p_1)-axis, equally spaced above and below. As they are unstable points, this is exactly as one might expect, comparing to the standard graph for the pendulum.

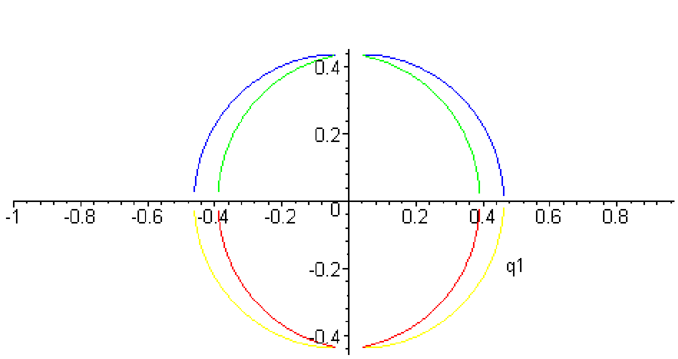
We now go onward and consider **the stable, stationary points**, which correspond to odd values of n, namely $n = \pm 1$. In that case the equations analogous to Eqs. (1.10a), for the stationary points, are given by

$$J_{20} = (\alpha + 1)\frac{3 + \alpha \pm \sqrt{(3+\alpha)^2 - ED_1}}{D_1}\ , \quad J_{10} = \frac{\alpha - 5}{\alpha - 1} J_2\ , \qquad D_1 \equiv \alpha^3 + 9\alpha^2 + 31\alpha + 39 = (\alpha + 3)(\alpha^2 + 6\alpha + 13)\ , \tag{1.10b}$$

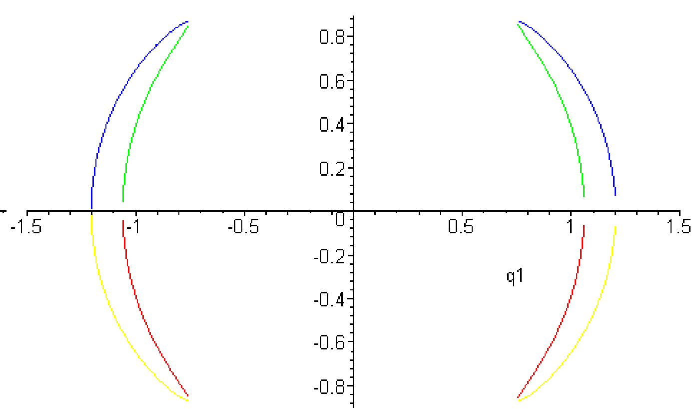
where again D_1 is a cubic polynomial with no extrema, but which is positive for all $\alpha > -3$. Therefore, again it is clear that the energy has an upper limit, so that

$$J_{20} \ \text{ is real only when } \ E < E_{\max}(\alpha) = (3+\alpha)/(\alpha^2 + 6\alpha + 13)\ , \tag{1.10b}$$

where $E_{\max}$ has the value 3/13 when $\alpha = 0$, and then moves downward asymptotically, through always positive values, to 0 as α approaches infinity. When E is in this acceptable range, the function $J_{20}(E, \alpha)$ is double-valued and traces out a parabola, with horizontal symmetry axis and turnaround point at $E = E_{\max}$, where we have $J_{20}(E_{\max}) = (1+\alpha)/(\alpha^2 + 6\alpha + 13)$ at . All these parabolas pass through $E = 0$ at both $J_{20} = 0$ and at twice the value at $E_{\max}$. They all have this general shape, for all $\alpha > -1$. Again, we also consider where these (stable) stationary points are in terms of the original variables. Choosing the same Poincaré section as before, i.e., where $\phi^2 = -\pi/2$, we may then infer that $\phi^1 = \mp\pi/2$, which tells us that these **stable** stationary points have $q^1_0 = \pm\sqrt{2J_1}$ and $p_{10} = 0$, while of course the values related to our (fixed) choice of ϕ^2 did not change; i.e., we still have $q^2 = 0$ and $p_2 = \sqrt{2J_2} = \sqrt{2(I - I_1)}$. Therefore, we expect these stationary points to lie, symmetrically, on the horizontal axis, in our 1-plane sections. To understand these things I first present some graphs that I have created, of level-curves in the Poincaré section already discussed, i.e., in the q^1, p_1-plane, with fixed value of $\phi^2 = -\pi/2$. They are shown for two different values of the constant I, but fixed values of energy, $E = 0.1$, and of $\alpha = 0.1$, which is a fairly small perturbing value. These particular values for I have been chosen because they occur just past and a little ways further past the separation point for pair of circles, so that we see good presentations of the crescents that surround the stable stationary points. The difference in scale is because the left-hand one is for a value of I below the first unstable stationary point, while the right-hand one is at a rather larger value of I which is somewhat above the second unstable stationary point.

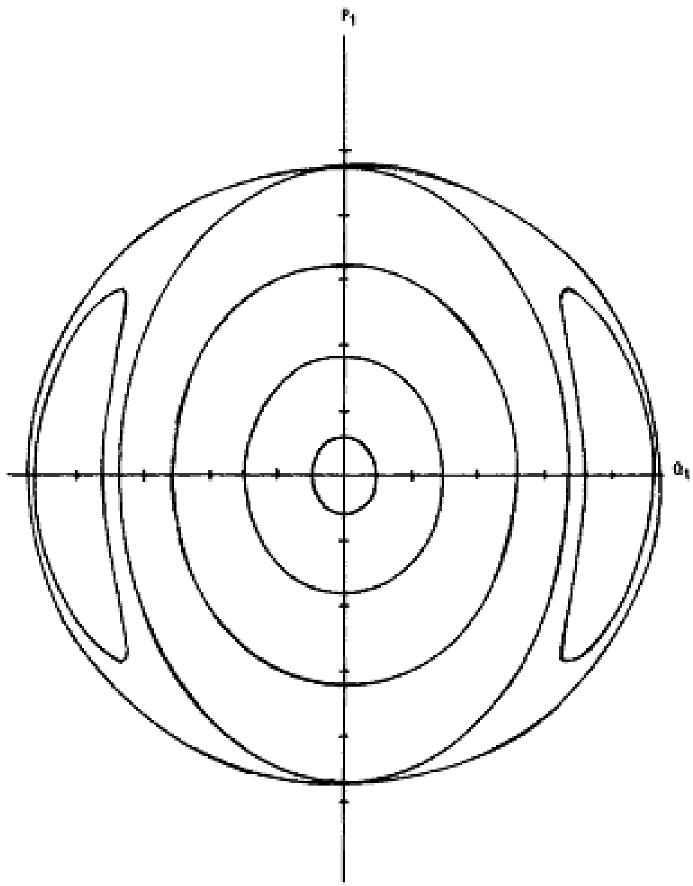


fairly small value for I



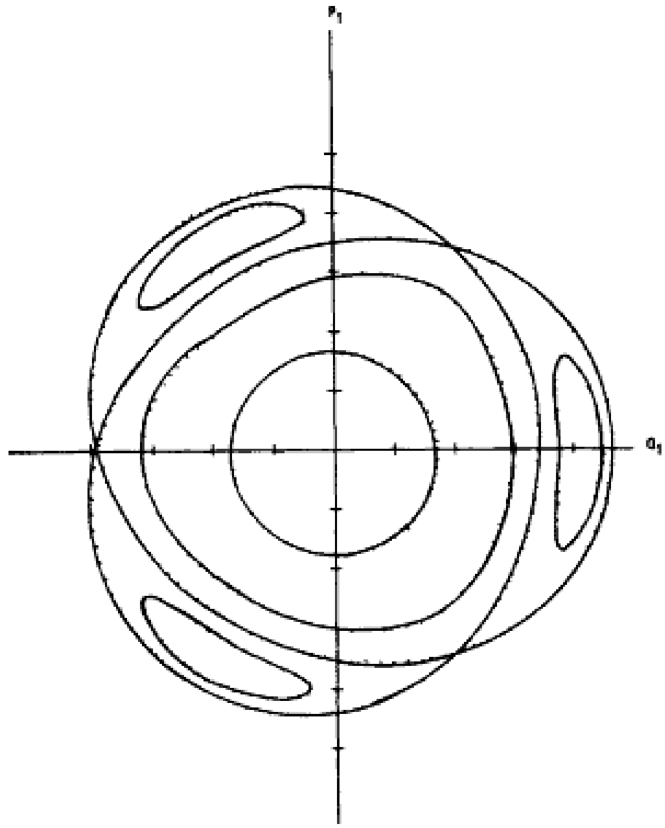
larger value of I, equal to 0.8

On the other hand, the rather better graph shown below, for a considerably larger value of α, is taken from their paper and shows some level curves for this problem, in the (q^1, p_1)-plane, with ϕ^2 set equal to $-\pi/2$, as already described, for some fixed value of $E < 3/13$, and varying values of the constant action, I. For a fixed value of I, the curve is typically a pair of concentric, deformed circles. As I varies from zero they vary as well, and join as the *separatrix value*, $I_0 \equiv I_{10} + I_{20}$, is reached, and then separate into the (lunar) crescents shown. As the value increases further, those crescents become smaller and smaller, shrinking to the stable stationary point in their (concentric) centers. Then as I increases further, typically, there is a region where there are no allowed orbits until, again one comes to a stable stationary point, then allows larger and larger crescents until they then merge, at the other unstable stationary point, and then, with yet increasing values of I, re-separate into a pair of concentric circles. This tells us that the two stationary points, in the centers of the two sets of pairs of crescents correspond to quite different values of I, and therefore to different motions.



for the 2-2 resonance

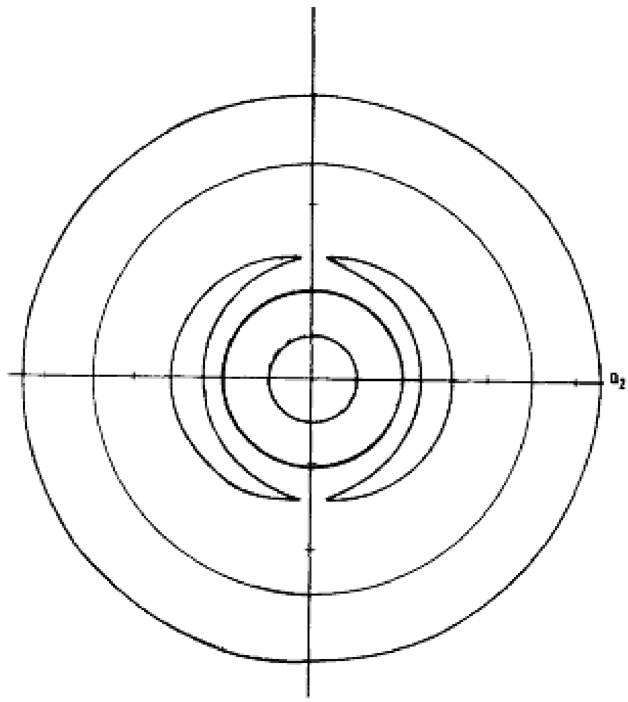
At this point we should have understood the 2-2 resonance situation, for non-zero α fairly well, and we should go forward and study the 3-2 resonance situation, corresponding to non-zero β. I have not taken/found the time to do that. Nonetheless, I go ahead and present their graph here. In this case $2J_1 + 3J_2$ is the second constant of the motion, and the resonance occurs when $2\omega_1 = 3\omega_2$, so that the motions have a somewhat different structure. In this case the motion is still integrable, again in an osculating plane, where the new angular variable is $\theta^1 \equiv 2\phi^1 - 3\phi^2$. It is suggested by those authors that this structure is actually more different than it first appears, since they claim that various stationary points lie on the same motion. I have not myself verified this.



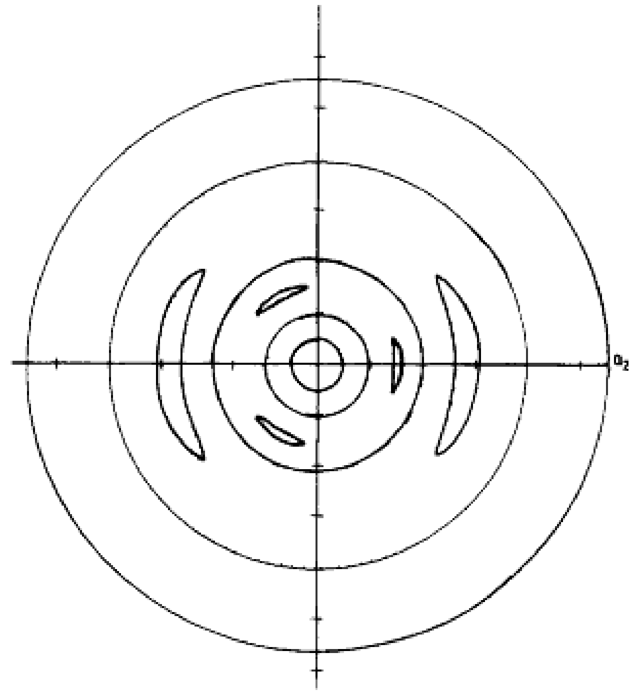
for the 3-2 resonance

Now I present some more graphs, below, from their paper, which are supposed to be appropriate for the final case, where **both** α and β are non-zero. This makes the problem not actually integrable, so that we would expect to have chaos begin to ensue, which can indeed to seen in the graphs presented.

This first pair are for the case when there is no significant overlap in the resonances for the two different integrable cases. In fact, in the one on the left-hand side, the energy is of such a size that we do not yet have any resonances for the $\beta \neq 0$ case, while on the right-hand side the energy has increased so that there are resonances in each case, but they are still far from one another.

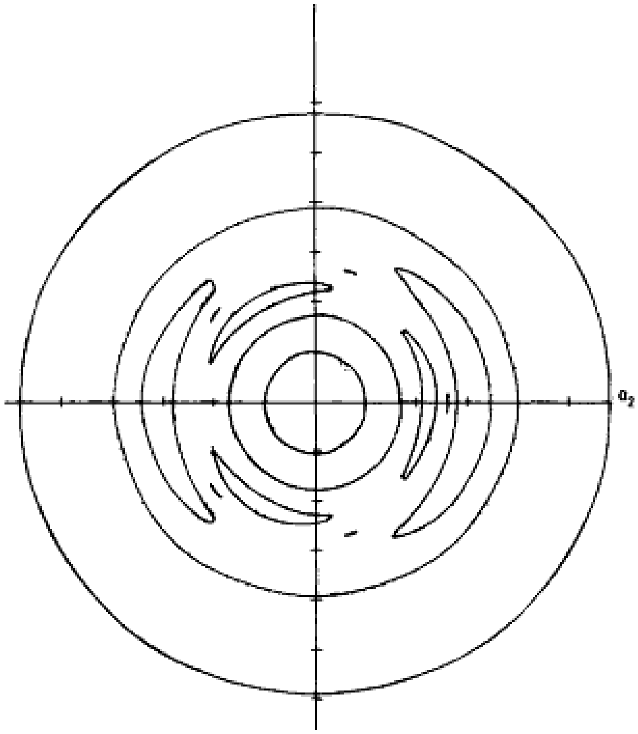


energy below the appearance of the 2-3 resonance

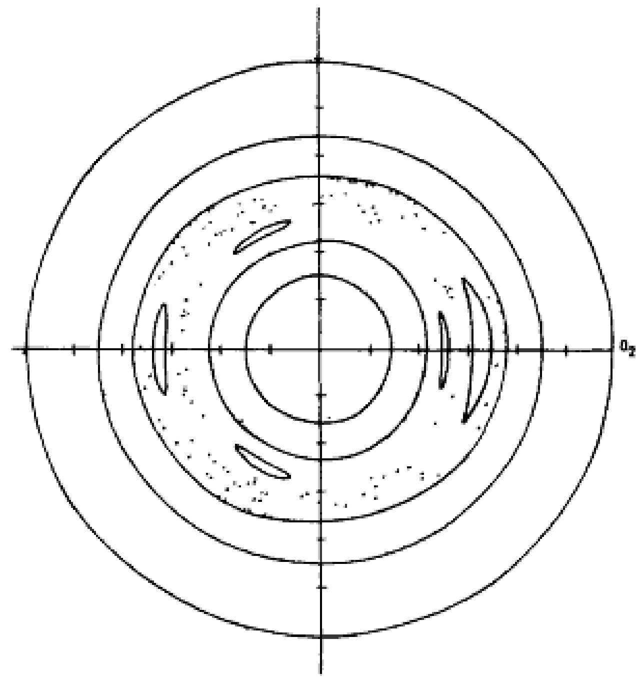


energy where the two resonances are widely separated

We lastly consider, briefly, the two Poincaré sections presented below, with both parts of the perturbing Hamiltonian active, and the energies such that there is overlap. In the left-hand case, the energy is $E = 0.20$, which is just slightly below the predicted overlap energy, and there is a chain of 5 “islands” which can be seen, and it is claimed, in the text, that there is also a chain of 7 that is too small to be shown in this figure. On the right-hand side the energy is what is predicted for the initial overlap of the resonance zones. Here we see some definite chaotic portions of the section.



energy at initiation of overlap



energy with clear overlap of resonances

Some Poincaré sections and actual 3-dimensional orbits are also shown in an accompanying Maple (.html) document, noted on the homepage.