Symmetries, Conservation Laws, etc., for KP and KdV

O. Some Definitions

We think of all our pde's as defined on an infinite jet bundle, with a set of independent variables, $x^a \equiv \{x, t, y, \ldots\}$, one or more dependent variables, $u^\alpha \equiv \{u, v, \ldots\}$, and their derivatives, and, possibly, some prolonged dependent variables that we think of as potentials, or pseudopotentials, for the problem. We label those pde's by F^μ, which are functions defined on the infinite jet which must vanish, thereby defining the variety which is the solution manifold for the pde's.

A. A conservation law is truly a statement of the existence of a closed 1-form spanned by the derivatives of the independent variables. With only 2 ind. variables and 1 dep. one, we have then the existence of the 1-form

$$\omega \equiv f\,dx + g\,dt \quad \text{such that} \quad d\omega = 0\ . \tag{0.0}$$

This is, however, more usually expressed by an equation of the form

$$f_{,t} = g_{,x} \quad \Longleftrightarrow \quad \overline{D}_t f = \overline{D}_x g\ , \tag{0.1}$$

where f and g are functions defined on this infinite jet bundle and the equivalence is really an equivalence between notations, both meaning the same thing; i.e., when we write, for instance, a t-partial derivative for functions defined on this infinite jet bundle, we really mean a total derivative over the entire bundle, restricted to the variety defined by the pde and its entire set of derivatives, which is what is denoted by the symbol $\overline{D}_t$.
We refer to f as the conserved *density* and g as the *flux*.

i. If f is a local functional of u, i.e., involves only it and its higher-order derivatives, then we call it a *local conserved density*; if g is also local, then we may refer to Eq. (0.1) as a *local conservation law.*

ii. If f does not depend explicitly on the independent variables, then it is a *polynomial conserved density*; likewise, if also g does not depend explicitly on the independent variables, then Eq. (0.1) is a *polynomial conservation law.*

a.) If f vanishes sufficiently rapidly at the ends of some interval of integration, L, then each conservation law yields a *local constant of the motion*:

$$I \equiv \int_L dx\, f \qquad \Longrightarrow \qquad I_{,t} = \int_L dx\, f_{,t} = \int_L dx\, g_{,x} = g_{\partial L} = 0\ . \tag{0.2}$$

b.) Two conserved densities can be *gauge-equivalent* since any x-derivative is trivially a conserved density. Therefore two conserved densities differing by an x-derivative determine the same conservation law. More precisely this means that one may always take a given pair, i.e., a density f and a flux g, and append to them the appropriate total derivatives of some third quantity, say η, such that we acquire another pair, f' and g', which is, however, completely equivalent since they generate the same conservation law:

$$f' \equiv f + \overline{D}_x\eta \quad g' \equiv g + \overline{D}_t\eta\ . \tag{0.3}$$

B. A generator, ϕ, for a symmetry satisfies the Vinogradov equation, $L_F(\phi) = 0$, defined as follows:

$$\begin{aligned} \text{З}_\psi &\equiv \left\{\psi^\nu\partial_{u^\nu} + \{\overline{D}_a(\psi^\nu)\}\partial_{u^\mu_a} + \{\overline{D}_a\overline{D}_b(\psi^\nu)\}\partial_{u^\mu_{ab}} + \ \ldots\ \right\} \equiv \sum_{\sigma=0}^{(\infty)} \{\overline{D}_{(\sigma)}(\psi^\nu)\}\,\partial_{u^\nu_{(\sigma)}}\ , \\ L_{F^\mu}(\phi^\nu) &\equiv \text{З}_{\phi^\nu}(F^\mu)\ . \end{aligned} \tag{0.4}$$

Note that there is a way to "derive" this expression, using the Frechét (or Gateaux) derivative. **First** let the function ϕ be defined on the infinite jet so that we should write it as a function of every one of the coordinates on that jet, and now let us translate each one of them by some very small amount, in some direction, determined by some other function on the jet, say ψ, so that the result would be

$$\begin{aligned}\phi(q, q_a, q_{ab}, q_{abc}, \ \dots\ ) \ \longrightarrow \quad & \phi(q+\epsilon\psi, q_a+\epsilon\overline{D}_a\psi, q_{ab}+\epsilon\overline{D}_a\overline{D}_b\psi, q_{abc}+\epsilon\overline{D}_a\overline{D}_b\overline{D}_c\psi, \ \dots\ ) \\ = \phi(q, q_a, \dots) + & \epsilon\left\{\psi\phi_q + \overline{D}_a\psi\phi_{q_a} + \ \dots\ \right\} = \phi(q,\dots) + \epsilon\mathfrak{Z}_\Psi(\phi) + \dots \ .\end{aligned} \tag{0.5}$$

Then the Frechét derivative is to ask for the derivative of all this w.r.t. ϵ, evaluated (afterward) at $\epsilon = 0$, i.e., to look for the contribution to this translation that is linear in ϵ. We see that this linear contribution is just $\mathfrak{Z}_\psi(\phi)$. [Note that the Frechét derivative is often denoted by $\phi'[\psi]$.] When, however, we need to consider some function that also depends on some integrals of jet coordinates, we must define some extension of the $\mathfrak{Z}$ operator, that includes behavior at that level. Indeed, if we were to consider a function that also depended on the very simple quantity, h_0, which is such that $\overline{D}_x(h_0) = u$, then it is straightforward to see that one should extend to

$$\widetilde{\mathfrak{Z}}_\psi\phi = \mathfrak{Z}_\psi\phi + \{\overline{D}_x{}^{-1}\psi\}\phi_{,h_0} \ . \tag{0.6}$$

The real question at the moment is to ask what is the generalization of this to arbitrary first integral quantities?

A relevant example is the quantities h_i, which will be defined below for the KdV equation. We are given two sets of functions defined on the jet, namely η^0_i and f_i, and then we define

$$h_i \equiv \eta^0_i - c_i\,\overline{D}_x{}^{-1}f_{i-1} \ , \tag{0.6}$$

for some constants c_i. When we want to take this linearization, or Frechét derivative in some direction, then the η^0_i terms are no problem, but the ones with the integral operators are no longer simply defined on the jet itself.

C. Such a generator has associated with it a corresponding vector field, on the infinite jet bundle. Any vector field on the bundle would have the generic form:

$$\vec{v} = v^a\partial_{x^a} + v^\alpha\partial_{u^\alpha} + v^\alpha_a\partial_{u^\alpha_a} + v^\alpha_{ab}\partial_{u^\alpha_{ab}} + \ \dots \ . \tag{0.7}$$

When such a vector field is determined by a characteristic, or generator, then we may, instead write it in the following way, where the coefficients v^a are given by $v^a\delta^\mu_\nu = -\phi^\mu_{,z^\nu_a}$:

$$\vec{v}_\phi = \sum_0^\infty\{D^\sigma(\phi^\mu)\}\partial_{u^\mu_{(\sigma)}} - \phi^\mu_{,u^M_a}D_a \ \sim \ \sum_0^\infty\{D^\sigma(\phi^\mu)\}\partial_{u^\mu_{(\sigma)}} \ , \tag{0.8}$$

where two vector fields on the infinite bundle are equivalent if they differ only by a linear combination of total derivatives.

D. Two vector fields of course have a commutator that, when they are described in terms of actual vector fields on the manifold, is just an ordinary Lie bracket. However, since, modulo equivalency, they are completely determined by their characteristics, it is useful to have a specification of that commutator in terms of characteristics. Therefore, we can note that the following is true:

$$[\vec{v}_\phi, \vec{v}_\psi] = \vec{v}_\eta \quad \Leftrightarrow \quad \eta^\mu = \{\phi, \psi\}^\mu \equiv \ \mathfrak{Z}_\phi(\psi^\mu) - \mathfrak{Z}_\psi(\phi^\mu) \ . \tag{0.9}$$

I. Data for the KdV equation

It seems most convenient to take the KdV equation in the form

$$u_{xxx} + u\,u_x = u_t \ . \tag{1.1}$$

On the jet bundle we will often/usually label the higher derivatives with respect to x simply by an integer subscript, where, for instance, u_3 indicates u_{xxx}; occasionally it may be clearer if we use $u_{(\ell)}$, instead, to indicate the ℓ-th derivative of u. We may then label the conserved densities, T, and their associated fluxes

$$\begin{array}{cc}
f_{-1} = u \ , & g_{-1} = (u_2 + \tfrac{1}{2}u^2) \ ; \\
f_0 = \tfrac{1}{2}u^2 \ , & g_0 = uu_2 - \tfrac{1}{2}u_1^2 + \tfrac{1}{3}u^3 \ ; \\
f_1 = \tfrac{1}{2}u_1^2 - \tfrac{1}{6}u^3 \ , & g_1 = u_1u_3 - \tfrac{1}{2}u_2^2 - \tfrac{1}{2}u^2u_2 + uu_1^2 - \tfrac{1}{8}u^4 \ ; \\
f_2 = \tfrac{1}{2}u_2^2 - \tfrac{5}{6}uu_1^2 + \tfrac{5}{72}u^4 \ , & g_2 = u_2u_4 - \tfrac{1}{2}u_3^2 - \tfrac{5}{3}uu_1u_3 + \tfrac{4}{3}uu_2^2 + \tfrac{5}{6}u_1^2u_2 \\
 & \quad + \tfrac{5}{18}u^3u_2 - \tfrac{5}{4}u^2u_1^2 + \tfrac{1}{18}u^5 \ ; \\
f_3 = \tfrac{1}{2}u_3^2 - \tfrac{7}{6}uu_2^2 + \tfrac{35}{36}u^2u_1^2 - \tfrac{7}{216}u^5 \ , & g_3 = u_3u_5 - \tfrac{1}{2}u_4^2 + 11 \text{ more terms} \ ; \\
f_4 = \tfrac{1}{2}u_4^2 - \tfrac{3}{2}uu_3^2 + 5 \text{ more terms,} & g_4 = u_4u_6 - \tfrac{1}{2}u_5^2 + \ \ldots \ \ ; \\
f_5 = \tfrac{1}{2}u_5^2 - \tfrac{11}{6}uu_4^2 + \ \ldots \ \ , & g_5 = u_5u_7 + \ \ldots \ \ ; \\
f_6 = \tfrac{1}{2}u_6^2 - \tfrac{13}{6}uu_5^2 + \ldots \ \ , & g_6 = u_6u_8 + \ \ldots \ \ ; \\
\ldots & \ldots \\
f_\ell = \tfrac{1}{2}u_{(\ell)}^2 - \tfrac{2\ell+1}{6}uu_{(\ell-1)}^2 + \ \ldots \ \ , \ell \geq 0 & g_\ell = u_{(\ell)}u_{(\ell+2)} + \ \ldots \ \ ;
\end{array} \tag{1.2}$$

where it should be noted that there are (complicated) overall normalization differences between these and the ones given in Ref. 1. (For instance the normalization factor for f_6 and g_6 is 143/3888.)

As pointed out in the introduction, the generators for the conservation laws are ambiguous, as one may always add to f_i the D_x action on some quantity, and to g_i the D_t action on that same quantity. As it turns out, there is a particular such choice that is very useful: We define quantities χ^0_ℓ which are of this form. More specifically, we define them as follows:

$$\begin{aligned}
\chi^0_\ell \quad &= \quad \overline{D}_x\eta^0_\ell - (-1)^\ell\left(\frac{2\ell+1}{3}\right)f_{\ell-1} \\
&= u_{(2\ell)} + \ \ldots \ + c_\ell\, u^{\ell+1} = \overline{D}_x(u_{(2\ell-1)} + \ \ldots \ ) - (-1)^\ell\frac{2\ell+1}{3}\left(\tfrac{1}{2}u_{(\ell)}^2 - \ \ldots \ \right) \ .
\end{aligned} \tag{1.3}$$

These quantities are particularly useful, later on, since they are potentials for the Lie and generalized symmetries:

$$\Psi^0_\ell = \ \overline{D}_x\chi^0_\ell = u_{(2\ell+1)} + \ \ldots \ + (\ell+1)c_\ell\, u^\ell u_1 \ . \tag{1.4}$$

We may now consider those very same symmetry generators/characteristics. We begin with the Lie symmetries, which are those generators for symmetries that involve only the independent variables, the dependent variables, and, perhaps, their first derivatives. They form their own closed Lie algebra, and are 4 for the KdV equation:

$$\begin{array}{c}
\Psi^0_0 = u_x \ , \quad \Psi^1_0 = tu_x + 1 \ , \\
\Psi^0_1 = u_t = u_{xxx} + uu_x \ , \quad \Psi^1_1 = tu_t + \tfrac{1}{3}xu_x + \tfrac{2}{3}u \ ,
\end{array} \tag{1.5}$$

where the subscript, ℓ, says that the leading term is of order $2\ell+1$, while the superscript denotes the power of t that appears, with only 0 and 1 acceptable for KdV.

There are then generators for generalized symmetries for the KdV equation, labelled in the same way as the Lie symmetries. However those that involve t explicitly also require certain integrals of u. As it (eventually) turns out, it is most convenient to take these as integrals of the (longer form, above, of the) generators for the conservation laws. We label these integrals as follows, noting that the individual cases follow from knowledge of those potentials, χ_i^0, which are given further below:

$$\begin{aligned}
h_i \equiv \overline{D}_x^{\,-1}\chi_i^0 \ &= \eta_i^0 - (-1)^i \left(\frac{2i+1}{3}\right) \overline{D}_x^{\,-1} f_{i-1} \ , \\
h_0 &= \overline{D}_x^{\,-1} u \ , \\
h_1 &= u_1 - \tfrac{1}{2}\overline{D}_x^{\,-1} u^2 \ , \\
h_2 &= u_3 + \tfrac{5}{3}uu_1 - \tfrac{5}{3}\overline{D}_x^{\,-1}(\tfrac{1}{2}u_1^2 - \tfrac{1}{6}u^3) \ .
\end{aligned} \tag{1.6}$$

We can then immediately write out the following explicit values:

$$\begin{aligned}
\Psi_\ell^j &= t^j u_{(2\ell+1)} + \tfrac{1}{3}[(2\ell+1)t^j u + jt^{j-1}x + \alpha(t)]u_{(2\ell-1)} + \ \ldots \ + (\ell+1)c_\ell\, u^\ell u_1 \ , \\
&\qquad\qquad \ell = 0,1,2,3,\ldots\ , j = 0,1 \ ; \\
\{\Psi_m^j, \Psi_n^k\} &= -\tfrac{1}{3}[(2m+1)k - (2n+1)j]\Psi_{m+n-1}^{j+k-1} \ , \\
&\qquad \text{except that} \quad \{\Psi_0^i, \Psi_0^j\} = 0 \ , \\
\Psi_2^0 \ &= u_5 + \tfrac{5}{3}uu_3 + \tfrac{10}{3}u_1u_2 + \tfrac{5}{6}u^2u_1 \ , \\
\Psi_2^1 &= t\Psi_2^0 + \tfrac{1}{3}x\Psi_1^0 + \tfrac{h_0}{9}\Psi_0^0 + \tfrac{4}{3}(u_2 + \tfrac{1}{3}u^2) \ , \\
\Psi_3^0 \ &= u_7 + \tfrac{7}{3}uu_5 + 7u_1u_4 + \tfrac{35}{3}u_2u_3 + \tfrac{35}{18}u^2u_3 + \tfrac{70}{9}uu_1u_2 + \tfrac{35}{18}(u_1)^3 + \tfrac{35}{54}u^3u_1 \ , \\
\Psi_3^1 &= t\Psi_3^0 + \tfrac{1}{3}x\Psi_2^0 + \tfrac{h_0}{9}\Psi_1^0 + \tfrac{h_1}{9}\Psi_1^0 + 2\left(u_4 + \tfrac{4}{3}uu_2 + \tfrac{17}{18}u_1^2 + \tfrac{4}{27}u^3\right) \ , \\
\Psi_4^0 \ &= u_9 + 3uu_7 + 12u_1u_6 + \left(\tfrac{7}{2}u^2 + 28u_2\right)u_5 + (21uu_1 + 42u_3)u_4 \\
&\quad + \left(\tfrac{161}{6}u_1^2 + 35uu_2 + \tfrac{35}{18}u^3\right)u_3 + \tfrac{35}{6}uu_1^3 + \tfrac{217}{6}u_1u_2^2 + \tfrac{35}{3}u_1u_2u^2 + \tfrac{35}{72}u^4u_1 \ , \\
\Psi_4^1 \ &= t\,\Psi_4^0 + \tfrac{x}{3}\,\Psi_3^0 + \tfrac{h_0}{9}\Psi_2^0 + \tfrac{h_1}{9}\Psi_1^0 + \tfrac{h_2}{9}\Psi_0^0 \\
&\quad + \tfrac{8}{3}\left\{u_6 + 2uu_4 + \tfrac{10}{3}u_2^2 + \tfrac{59}{12}u_1u_3 + \tfrac{17}{9}uu_1^2 + \tfrac{2}{27}u^4 + \tfrac{4}{3}u^2u_2\right\} \ .
\end{aligned} \tag{1.7}$$

One may create the higher-level ones either by using the commutator with a generator with the independent variables involved, i.e., one that has a non-zero superscript, or by using the recursion operator (due to Olver), which must act on something which can be written as a perfect derivative, since it involves a first x-integral, which we denote by $\overline{D}_x^{\,-1}$; it is

$$R \equiv \overline{D}_x^2 + \tfrac{2}{3}u + \tfrac{1}{3}u_x\,\overline{D}_x^{\,-1} \ , \quad R(\Psi_j^k) = \Psi_{j+1}^k = \frac{\{\Psi_2^1, \Psi_j^k\}}{2j+1-5k} \ . \tag{1.8}$$

Since this “integrability” is necessary in order for the recursion operator to act appropriately, i.e., the generators must have potentials, we now write out several of those potentials:

$$
\begin{aligned}
\chi_0^0 \ &= u \ , \\
\chi_1^0 \ &= u_2 + \tfrac{1}{2}u^2 = \overline{D}_x(u_1) + \tfrac{1}{2}u^2 \ , \\
\chi_2^0 \ &= u_4 + \tfrac{5}{3}uu_2 + \tfrac{5}{6}u_1^2 + \tfrac{5}{18}u^3 \\
&= \overline{D}_x(u_3 + \tfrac{5}{3}uu_1) - \tfrac{5}{3}(\tfrac{1}{2}u_1^2 - \tfrac{1}{6}u^3) \ , \\
\chi_3^0 \ &= u_6 + \tfrac{7}{3}uu_4 + \tfrac{14}{3}u_1u_3 + \tfrac{7}{2}u_2^2 + \tfrac{35}{18}u^2u_2 + \tfrac{35}{18}uu_1^2 + \tfrac{35}{216}u^4 \\
&= \overline{D}_x(u_5 + \tfrac{7}{3}uu_3 + \tfrac{7}{3}u_1u_2 + \tfrac{35}{18}u^2u_1) + \tfrac{7}{3}(\tfrac{1}{2}u_2^2 - \tfrac{5}{6}uu_1^2 + \tfrac{5}{72}u^4) \ , \\
\chi_4^0 \ &= u_8 + 3uu_6 + 9u_1u_5 + (\tfrac{7}{2}u^2 + 19u_2)u_4 + (14uu_1 + \tfrac{23}{2}u_3)u_3 + (\tfrac{77}{6}u_1^2 + \tfrac{21}{2}uu_2 \\
&\qquad + \tfrac{35}{18}u^3)u_2 + \tfrac{35}{12}u^2u_1^2 + \tfrac{7}{72}u^5 \\
&= \overline{D}_x(u_7 + 3uu_5 + 6u_1u_4 + \tfrac{1}{2}(26u_2 + 7u^2)u_3 + 7uu_1u_2 + \tfrac{35}{18}(u_1^2 + u^3)u_1) \\
&\qquad - 3(\tfrac{1}{2}u_3^2 - \tfrac{7}{6}uu_2^2 + \tfrac{35}{36}u^2u_1^2 - \tfrac{7}{216}u^5) \ .
\end{aligned}
\tag{1.8}
$$

For the characteristics involving t and x, we may write

$$
\Psi_\ell^1 = t\,\Psi_\ell^0 + \frac{x}{3}\Psi_{\ell-1}^0 + A_\ell = \overline{D}_x(\chi_\ell^1) \ , \tag{1.9}
$$

where the quantities A_ℓ are independent of t and x. We may then write explicitly

$$
\begin{aligned}
\chi_0^1 &= tu + x = t\chi_0^0 + x \ , \\
\chi_1^1 &= t(u_2 + \tfrac{1}{2}u^2) + \tfrac{x}{3}u + \tfrac{h_0}{3} = t\chi_1^0 + \tfrac{x}{3}\chi_0^0 + \tfrac{h_0}{3} \ , \\
\chi_2^1 &= t\chi_2^0 + \tfrac{x}{3}\chi_1^0 + \tfrac{h_0}{9}\chi_0^0 + \tfrac{h_1}{3} + \tfrac{2}{3}u_1 \ , \\
\chi_3^1 &= t\chi_3^0 + \tfrac{x}{3}\chi_2^0 + \tfrac{h_0}{9}\chi_1^0 + \tfrac{h_1}{9}\chi_0^0 + \tfrac{h_2}{3} + \tfrac{4}{3}\psi_1^0 \ , \\
\chi_4^1 &= t\chi_4^0 + \tfrac{x}{3}\chi_3^0 + \tfrac{h_0}{9}\chi_2^0 + \tfrac{h_1}{9}\chi_1^0 + \tfrac{h_2}{9}\chi_0^0 + \tfrac{h_3}{3} + 2\Psi_2^0 + \tfrac{1}{9}(2uu_3 - 2u_1u_2 + u^2u_1) \ .
\end{aligned}
\tag{1.10}
$$

Appendix II. Soliton Solutions of the KdV Equation, via the τ-function

It is perhaps worthwhile to begin the discussion of the τ-function by explaining Hirota's approach to it via **Hirota's bilinear differential operator**, $\mathcal{D}_x$. It operates on **function pairs**, which is why we refer to it as a bilinear operator. The basic definition is the following, where we use $\mathcal{P}$ to denote an arbitrary polynomial in one quantity, the differential operator, and take σ and τ as arbitrary, differentiable functions of one variable, here referred to as x:

$$\{\mathcal{P}(\mathcal{D}_x)\}(\sigma\cdot\tau)(x) \equiv \Big[\{\mathcal{P}(\partial_y)\}[\sigma(x+y)\tau(x-y)]\Big]\Big|_{y=0} \ . \tag{A2.1}$$

Examples are, for instance

$$\begin{aligned} \mathcal{D}_x(\sigma\cdot\tau) = \sigma_x\tau - \sigma\tau_x \ , \qquad \mathcal{D}_x^2(\sigma\cdot\tau) &= \sigma_{xx}\tau - 2\sigma_x\tau_x + \sigma\tau_{xx} \ , \\ \mathcal{D}_x^p(\sigma\cdot\tau) \ &= \sum_{m=0}^{p}(-1)^m\binom{p}{m}\sigma^{(n-m)}\tau^{(m)} \ , \end{aligned} \tag{A2.2}$$

where the superscripts in parentheses indicate the order of (ordinary) derivatives with respect to x.

This is easily generalized to consider polynomials in more than one unknown, and to then use derivatives with respect to more than one variable, and operate on functions of more than one variable. An example might be

$$\mathcal{D}_x\mathcal{D}_t(\sigma\cdot\tau) = \sigma_{xt}\tau - \sigma_x\tau_t - \sigma_t\tau_x + \sigma\tau_{xt} \ . \tag{A2.3}$$

Following the equation for the m-th bilinear derivative given above, we can easily create the following formula, which will be quite useful later on:

$$\mathcal{D}_x^m(e^{p_1x}\cdot e^{p_2x}) = (p_1-p_2)^m\, e^{(p_1+p_2)x} \ . \tag{A2.4}$$

Using this operator we can rewrite our relationship between u and τ as follows:

$$u = 12D_x\left(\frac{\tau_x}{\tau}\right) = 6\frac{\mathcal{D}_x^2(\tau\cdot\tau)}{\tau^2} \ , \tag{A2.5}$$

and then insert that form into the KdV equation to obtain its expression in terms of the τ-function, and then in terms of Hirota's bilinear differential operator, as well:

$$\begin{aligned} 0 = D_x&\left\{\frac{-\tau\tau_{xt} + \tau_x\tau_t + \tau\tau_{xxxx} - 4\tau_x\tau_{xxx} + 3(\tau_{xx})^2}{\tau^2}\right\} \ , \\ \Longrightarrow \quad &(-\mathcal{D}_x\mathcal{D}_t + \mathcal{D}_x^4)(\tau\cdot\tau) = 0 \ , \end{aligned} \tag{A2.6}$$

where, to obtain the last form, we have integrated once, thrown away the constant of integration, ignored the non-constant overall denominator of τ^2, and divided out by an overall constant multiplier.

We now want to create some τ-functions that are (explicit) solutions of the KdV equation. We first define a set of "phases" that we will need. We are interested in creating an N-soliton solution, where N is simply the number of solitons in the solution. We will then choose $2N$ constants,

$\{k_i, \theta_i^0 \mid i = 1, 2, \ldots, N\}$, and create that many "phases" which will serve as arguments for a set of exponentials:

$$\xi_i \equiv 2k_i(x - (2k_i)^2 t) + \theta_i^0 \equiv \sum_{j=0}^{\infty} (2k_i)^{2j+1} t_{2j+1} + \xi_i^0 \ , \tag{A2.7}$$

where the last relationship inserts the dependence of the phases on the entire infinite collection of "times," i.e., flow parameters along which the infinite collection of Abelian symmetries maps solutions into solutions, instead of just showing the dependence on x and t. (We recall that $t_1 = x$ and $t_3 = -t$, etc., and of course the "constant" changed to account for the explicit introduction of all these "higher" times.) Notice that it is reasonable to describe the speed of these phases as just

$$v_i \equiv (2k_i)^2 \ . \tag{A2.8}$$

although the actual "velocity," namely dx/dt, is the negative of this speed: Because of our sign convention about the direction of time, the motion of the soliton solutions is toward the left, i.e., toward the negative x direction.

We can then create τ-functions as follows:

$$\begin{aligned}
&\tau_1 \equiv 1 + e^{\xi_1} \ , \\
&\tau_2 \equiv 1 + e^{\xi_1} + e^{\xi_2} + A_{12} e^{\xi_1+\xi_2} \ , \\
&\tau_3 \equiv 1 + e^{\xi_1} + e^{\xi_2} + e^{\xi_3} + A_{12} e^{\xi_1+\xi_2} + A_{13} e^{\xi_1+\xi_3} + A_{23} e^{\xi_2+\xi_3} + A_{12}A_{13}A_{23} e^{\xi_1+\xi_2+\xi_3} \ , \\
&\tau_4 \equiv 1 + e^{\xi_1} + e^{\xi_2} + e^{\xi_3} + e^{\xi_4} + A_{12} e^{\xi_1+\xi_2} + A_{13} e^{\xi_1+\xi_3} + A_{14} e^{\xi_1+\xi_4} + A_{23} e^{\xi_2+\xi_3} \\
&\qquad + A_{24} e^{\xi_2+\xi_4} + A_{34} e^{\xi_3+\xi_4} + A_{12}A_{13}A_{23} e^{\xi_1+\xi_2+\xi_3} + A_{12}A_{14}A_{24} e^{\xi_1+\xi_2+\xi_4} \\
&\qquad\quad + A_{13}A_{14}A_{34} e^{\xi_1+\xi_3+\xi_4} + A_{23}A_{24}A_{34} e^{\xi_2+\xi_3+\xi_4} \\
&\qquad\qquad + A_{12}A_{13}A_{14}A_{23}A_{24}A_{34} e^{\xi_1+\xi_2+\xi_3+\xi_4} \ , \\
&\qquad \ldots \ , \\
&\qquad\qquad A_{ij} \equiv \left(\frac{k_i - k_j}{k_i + k_j}\right)^2 \ .
\end{aligned} \tag{A2.8}$$

where the subscript on τ_i indicates the number of solitons that the solution describes. We will than label the associated solution of the KdV equation by u_i.

The construction above continues in what we presume is an obvious way; however, if one wants a more algorithmic approach it may be obtained by calculating the determinant of a matrix, T, with the following elements, where we notice that A_{ii} is always 0:

$$T_{ij} \equiv \delta_{ij} + [1 - A_{ij}]^{1/2} e^{(\xi_i+\xi_j)/2} \ . \tag{A2.9}$$

At this point we can discuss some special properties of these soliton solutions, which are perhaps useful. Looking at the first one, u_1, determined from the τ_1 above, which depends on only one speed parameter, we find that all the different Abelian generalized symmetries are simply proportional to one another:

$$\Psi_j^0\Big|_{u=u_1} = (v_1)^j \, \Psi_0^0\Big|_{u=u1} = (v_1)^j \, \partial_x u_1 = (2k_1)^{2j} \, u_{1,x} \ . \tag{A2.10}$$

To see more detail on this, we first use the bilinear form of the KdV equation, for the τ-function, as given in Eq. (A2.6), and the action on exponentials, as given in Eq. (2.4):

$$\begin{aligned}
\mathcal{D}_x^m\{(1+e^{\xi_1})\cdot(1+e^{\xi_2})\} &= \mathcal{D}_x^m\{1\cdot 1\}\mathcal{D}_x^m\{1\cdot e^{\xi_2}\} + \mathcal{D}_x^m\{e^{\xi_1}\cdot 1\} + \mathcal{D}_x^m\{e^{\xi_1}\cdot e^{\xi_2}\} \\
&= 0 + (-\xi_{2,x})^m\, e^{\xi_2} + (+\xi_{1,x})^m\, e^{\xi_1} + (\xi_{1,x}-\xi_{2,x})^m\, e^{\xi_1+\xi_2}\ , \\
\mathcal{D}_x\mathcal{D}_t\{(1+e^{\xi_1})\cdot(1+e^{\xi_2})\} &= (\xi_{1,x})(\xi_{1,t})\, e^{\xi_1} + (-\xi_{2,x})(-\xi_{2,t})\, e^{\xi_2} \\
&\quad + (\xi_{1,x}-\xi_{2,x})(\xi_{1,t}-\xi_{2,t})\, e^{\xi_1+\xi_2}\ , \\
\Longrightarrow \quad (\mathcal{D}_x^4 - \mathcal{D}_x\mathcal{D}_t)\{(1+e^{\xi_1})\cdot(1+e^{\xi_1})\} &= 2\Big\{\xi_{1,x}^4 - \xi_{1,x}\xi_{1,t}\Big\}e^{\xi_1}\ .
\end{aligned} \qquad (A2.11)$$

It is clear that this (1-soliton) τ-function will satisfy the appropriate equation provided that $\xi_{1,t} = (\xi_{1,x})^3$. This is indeed arranged for by the form of any of the various ξ_i given by Eqs. (A2.7).

Using the identification between t_3 and $-t$, the first of these relationships is of course just the statement that the function is a solution of the original pde. However, the continuing, infinite, sequence of them is a statement about the form of the τ-function, which depends on the entire infinite sequence of different times with exactly these coefficients, and/or a statement that those times, t_{2j+1}, are indeed the flow parameters for the different Abelian symmetry generators. The correct mathematical/symbolic phraseology for that, using Eqs. (1.7), is simply that

$$u_{(2j+1)} + \ \ldots \quad = \Psi_j^0(u) = \partial_{t_{2j+1}} u\ . \qquad (A2.12)$$

When we go on to larger numbers of solitons, we acquire more than one speed, and the dependencies no longer could be so simple. Of course the 2-soliton solution, u_2, does satisfy the pde itself, so that this says that $\Psi_1^0(u_2)$ must be equal to the negative of $\Psi_0^0(u_2)$. However, there is another relationship, that involves both speeds in an intrinsic way, namely

$$\Psi_{j+1}^0(u_2) - (v_1+v_2)\Psi_j^0(u_2) + (v_1v_2)\Psi_{j-1}^0(u_2) = 0\ , \quad j = 1,2,3,\ldots \quad . \qquad (A2.13)$$

The simplest case, for $j = 1$ was already noted in Eq. (3.44). If, however, we rewrite this relationship using the infinite set of flow parameters, via Eqs. (A2.12), then it becomes linear in u, so that it can then be rephrased in terms of the associated τ-function:

$$\begin{gathered}
\Psi_{j+1}^0(u_2) - (v_1+v_2)\Psi_j^0(u_2) + (v_1v_2)\Psi_{j-1}^0(u_2) = 0 \\
\Longleftrightarrow \\
\partial_{t_{j+1}}u_2 - (v_1+v_2)\partial_{t_j}u_2 + v_1v_2\,\partial_{t_{j-1}}u_2 = 0 \\
\Longleftarrow \\
\partial_{t_{j+1}}\tau_2 - (v_1+v_2)\partial_{t_j}\tau_2 + v_1v_2\,\partial_{t_{j-1}}\tau_2 = 0\ .
\end{gathered} \qquad (\mathrm{A2.14})$$

In fact this sort of "defining" relationship continues for solutions that describe larger numbers of solitons as well. I haven't proven it "generally," but have certainly already shown the 3- and the 4-soliton cases, which may be simply put as they are the solutions that satisfy the relationships:

$$\begin{aligned}
&\partial_{t_{j+1}}u_3 - (v_1+v_2+v_3)\partial_{t_j}u_3 + (v_1v_2+v_2v_3+v_3v_1)\,\partial_{t_{j-1}}u_3 - v_1v_2v_3\,\partial_{t_{j-2}}u_3 = 0\ , \\
&\partial_{t_{j+1}}u_4 - (v_1+v_2+v_3+v_4)\partial_{t_j}u_4 + (v_1v_2+v_2v_3+v_3v_1+v_1v_4+v_2v_4+v_3v_4)\,\partial_{t_{j-1}}u_4 \\
&\qquad -(v_1v_2v_3+v_1v_2v_4+v_1v_3v_4+v_2v_3v_4)\,\partial_{t_{j-2}}u_4 + v_1v_2v_3v_4\,\partial_{t_{j-3}}u_4 = 0\ ,
\end{aligned} \qquad (A2.15)$$

Returning to τ_2 for a moment, we may also calculate how it behaves from the point of view of the solution:

$$\begin{aligned}\{\hat{\mathcal{D}}_x^4 - \mathcal{D}_x\mathcal{D}_t\}(\tau_2\cdot\tau_2) &= \xi_{1,x}[(\xi_{1,x})^3 - \xi_{1,t}]e^{\xi_1} + \xi_{2,x}[(\xi_{2,x})^3 - \xi_{2,t}]e^{\xi_2}\\ &+ A_{12}\xi_{2,x}[(\xi_{2,x})^3 - \xi_{2,t}]e^{2\xi_1+\xi_2} + A_{12}\xi_{1,x}[(\xi_{1,x})^3 - \xi_{1,t}]e^{\xi_1+2\xi_2}\\ &+ (\xi_{1,x} - \xi_{2,x})\Big[(\xi_{1,x} - \xi_{2,x})^3 - (\xi_{1,t} - \xi_{2,t})\Big]e^{\xi_1+\xi_2}\\ &+ A_{12}(\xi_{1,x} + \xi_{2,x})\Big[(\xi_{1,x} + \xi_{2,x})^3 - (\xi_{1,t} + \xi_{2,t})\Big]e^{\xi_1+\xi_2}\ ,\end{aligned} \qquad (A2.16)$$

which then also vanishes when there is this relationship between the x- and t-derivatives, as above, namely that $\xi_{j,t} = (\xi_{j,x})^3$.

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