**Some Physically-Interesting, Soliton-type pde's**
and characteristics for symmetries for some of them

1. The KdV equation

   for 1-dimensional, unidirectional water waves in a shallow channel,

$$u_{xxx} + u\,u_x = u_t$$

2. The sine-Gordon equation

   for self-induced transparency in nonlinear optical materials,

$$u_{xx} - u_{tt} = \sin u$$

3. The non-linear Schrödinger equation

   for plasma interactions, and many sorts of quantum feedback loops

$$\Psi_{xx} + g|\Psi|^2\Psi = i\,\Psi_t$$

   Note: This is a system of 2 equations, since $\Psi$ is complex-valued.

4. The modified KdV equation

   weakly non-linear lattices

$$u_{xxx} + u^2\,u_x = u_t$$

5. The Liouville equation

   Interesting problems in statistical mechanics, and conformal field theory

$$u_{xy} = e^{-2u}$$

6. The Tzitzeca-Dodd-Bullough equation

   problems varying from fluid flow to quantum field theory

$$u_{xy} = e^{-u} + e^{-2u}$$

7. The classical Heisenberg ferromagnet equation

$$\vec{S}_t = \vec{S} \times \vec{S}_{xx}$$

8. An equation from $\phi^4$ field theory

$$\phi_{xt} = \phi(1 - |\phi_t|^2)^{1/2}$$

9. The Boussinesq equation

   for 2-directional, 1-dimensional water flow in channels

$$u_{tt} - u_{xx} - 3(u^2)_{xx} - u_{xxxx}$$

10. The Kadomtsev-Petviashvilli (KP) equation [for 2-dimensional water flow]

$$(u_t + u\,u_x + u_{xxx})_x + u_{yy} = 0$$

11. A higher-order KdV equation

$$u_{xxxxx} + 15(u_x\,u_{xx} + u\,u_{xxx}) + 45u^2\,u_x = u_t$$

12. (2-dimensional) Toda-lattice equations

    for lattice structures, and conformal quantum field theory

$$u^i_{xy} = e^{A^i{}_j u^j}\ , \qquad \forall i = 1, \ldots, n$$

    for certain classes of matrices whose elements are integers

13. SDiff(2)Toda equation

    for general relativity, and Toda field theory

$$u_{xy} = \partial_s^2\, e^u$$

14. supersymmetric, infinite-growth, 2-dimensional, conformal Toda field theories:

$$u_{xy} = \partial_s\, e^u$$

## Characteristics for Symmetries for various Non-linear pde's

**Burgers' Equation**: $u_{xx} + uu_x - u_t = 0$

$$\Psi^0{}_1 = z_x \,,$$

$$\Psi^1{}_1 = t\, z_x + 1 \,,$$

$$\Psi^0{}_2 = z_{xx} + z\, z_x \,,$$

$$\Psi^1{}_2 = t(z_{xx} + z\, z_x) + \hat{(}xz_x + z) \,,$$

$$\Psi^2{}_2 = t^2(z_{xx} + z\, z_x) + txz_x + tz + x \,,$$

$$\Psi^0{}_3 = z_{xxx} + \tfrac{3}{2} z\, z_{xx} + \tfrac{3}{2}(z_x)^2 + \tfrac{3}{4} z^2 z_x \,,$$

$$\Psi^1{}_3 = tz_{xxx} + \hat{(}3tz + x)z_{xx} + \hat{(}\tfrac{3}{2}tz^2 + xz)z_x + \tfrac{3}{2}t(z_x)^2 + \tfrac{1}{4}z^2 \,,$$

$$\Psi^2{}_3 = t^2 z_{xxx} + \hat{t}(3tz + 2x)z_{xx} + \tfrac{3}{2}t^2(z_x)^2 + \hat{(}\tfrac{3}{2}t^2z^2 + 2txz + \hat{x}^2)z_x + \hat{t}z^2 + \hat{x}z - \tfrac{3}{2} \,,$$

$$\Psi^3{}_3 = t^3 z_{xxx} + \tfrac{3}{2}t^2(tz + x)z_{xx} + \tfrac{3}{2}t^3(z_x)^2 + \hat{(}\tfrac{3}{2}t^3z^2 + 3t^2xz + 6t^2 + \tfrac{3}{2}tx^2)z_x$$
$$+ \tfrac{3}{4}t^2z^2 + \tfrac{3}{2}txz + \tfrac{3}{4}(x^2 + 2t) \,,$$

$$\Psi^0{}_4 = z_{(4)} + 2zz_{xxx} + 5z_x z_{xx} + \tfrac{3}{2}z^2 z_{xx} + 3z(z_x)^2 + \hat{z}^3 z_x \,,$$

$$\Psi^0{}_5 = z_{(5)} + \tfrac{5}{2}zz_{(4)} + \tfrac{15}{2}z_x z_{xxx} + 5(z_{xx})^2 + \tfrac{5}{2}z^2 z_{xxx} + \tfrac{25}{2}zz_x z_{xx}$$
$$+ \tfrac{15}{4}(z_x)^3 + \tfrac{5}{4}z^3 z_{xx} + \tfrac{15}{4}(zz_x)^2 + \tfrac{5}{16}z^4 z_x \,.$$

General Forms for Burgers' Equation:

$$\Psi^j{}_k = t^j z_{(k)} + \hat{(}kt^j z + jt^{j-1}x)z_{(k-1)} + \tfrac{1}{4}\{k(k+1)t^j z_x + \hat{k}(k-1)t^j z^2$$
$$+ j(k-1)t^{j-1}xz + \hat{j}(j-1)t^{j-2}x^2\}z_{k-2} + O(z_{k-3}) \quad ,$$
$$k = 0, 1, 2, \ldots \quad ; \; j = 0, 1, \ldots k \,,$$

$$\{\Psi^i{}_k \,, \Psi^j{}_\ell\} = \hat{(}\ell i - kj)\Psi^{i+j-1}{}_{k+\ell-2} \,,$$

$$\text{except that} \quad \{\Psi^0{}_1 \,, \Psi^1{}_1\} = 0 \,,$$

Recursion Operator:

$$D_x + \hat{z} + \tfrac{1}{2}z_x(D_x)^{-1} \quad .$$

**KdV Equation**: $u_{xxx} + uu_x - u_t = 0$

$$\Psi^0{}_0 = z_x\,,$$

$$\Psi^1{}_0 = t\,z_x + 1\,,$$

$$\Psi^0{}_1 = z_{xxx} + z\,z_x\,,$$

$$\Psi^1{}_1 = 3t(z_{xxx} + z\,z_x) + xz_x + 2z\,,$$

$$\Psi^0{}_2 = z_{(5)} + \tfrac{5}{3}z\,z_{xxx} + \tfrac{10}{3}z_xz_{xx} + \tfrac{5}{6}z^2z_x\,,$$

$$\Psi^0{}_3 = z_{(7)} + \tfrac{7}{3}z\,z_{(5)} + 7z_xz_{xxxx} + \tfrac{35}{3}z_{xx}z_{xxx} + \tfrac{35}{18}z^2z_{xxx} + \tfrac{70}{9}z\,z_x\,z_{xx} + \tfrac{35}{18}(z_x)^3 + \tfrac{35}{54}z^3z_x\,,$$

General Forms for the KdV Equation:

$$\Psi^j{}_\ell = t^j z_{(2\ell+1)} + \tfrac{1}{3}[(2\ell+1)t^jz + jt^{j-1}x + \alpha(t)]\,z_{(2\ell-1)} + O\left(z_{(2\ell-2)}\right) \quad ,$$

$$\ell = 0, 1, 2, \ldots \quad ; \; j = 0, 1\,,$$

$$\{\Psi^j{}_m\,, \Psi^k{}_n\} = \tfrac{1}{3}[(2m+1)k - (2n+1)j]\,\Psi^{j+k-1}{}_{m+n-1}$$

$$\text{except that} \quad \{\Psi^i{}_0\,, \Psi^j{}_0\} = 0\,.$$

Recursion Operator:

$$(D_x)^2 + \tfrac{2}{3}z + \tfrac{1}{3}z_x(D_x)^{-1} \quad .$$

**sine-Gordon equation**: $u_{xy} - \sin u = 0$

Appropriate coordinates on the jet are $\{x, y, z, z_x, z_y, z_{xx}, z_{yy}, z_{xxx}, z_{yyy}, \ldots\}$.

$$\Psi_{(0,1)} = z_y\,,$$

$$\Psi_{(1,0)} = z_x\,,$$

$$\Psi_{(1,1)} = xz_x - yz_y\,,$$

$$\Psi_{(3,0)} = z_{xxx} + \hat{(}z_x)^3$$

$$\Psi_{(0,3)} = z_{yyy} + \hat{(}z_y)^3$$

$$\Psi_{(5,0)} = z_{xxxxx} + \tfrac{5}{2}[(z_x)^2z_{xxx} + z_x(z_{xx})^2] + \tfrac{3}{8}(z_x)^5\,,$$

$$\Psi_{(7,0)} = z_{xxxxxxx} + \tfrac{7}{2}\,(z_x)^2z_{xxxxx} + 14\,z_xz_{xx}z_{xxxx} + \tfrac{21}{2}\,z_x(z_{xxx})^2 + \tfrac{35}{2}\,(z_{xx})^2z_{xxx} + \tfrac{35}{8}\,(z_x)^4z_{xxx} + \tfrac{35}{4}\,(z_x)^3(z_{xx})^2 + \tfrac{5}{16}\,(z_x)^7\,.$$

**the non-linear Schrödinger equation**: $u_{xx}+\overline{u}uu+iu_t=0$ and complex-conjugate equation

Appropriate coordinates on the jet are $\{x,t,z,\overline{z},z_x,\overline{z}_x,z_{xx},\overline{z}_{xx}\ \dots\}$.

$$
\begin{aligned}
&{\Psi^0}_1 = z_x\ ,\\
&{\Psi^0}_2 = z_t \equiv i(z_{xx}+\overline{z}zz)\,,\\
&{\Psi^1}_1 = 2tz_x - ixz\ ,\\
&{\Psi^1}_2 = 2it(z_{xx}+\overline{z}zz)+xz_x\,,\\
&{\Psi^0}_3 = z_{xxx}+3\overline{z}zzz_x\\
&{\Psi^0}_4 = z_{xxxx}+4\overline{z}zz_{xx}+3\overline{z}(z_x)^2+2z\overline{z}_xz_x+z^2\overline{z}_{xx}+\tfrac{3}{2}z^3\overline{z}^2\,.
\end{aligned}
$$

**the modified KdV (or MKdV) equation:** $u_{xxx}+u^2u_x-u_t=0$

$$
\begin{aligned}
&{\Psi^0}_0 = z_x\\
&{\Psi^0}_1 = z_{xxx}+z^2z_x\,,\\
&{\Psi^1}_1 = 3\,t(z_{xxx}+z^2z_x)+xz_x+z\,,\\
&{\Psi^0}_2 = z_{(5)}+\tfrac{5}{3}\,z^2z_{xxx}+\tfrac{20}{3}\,z\,z_x\,z_{xx}+\tfrac{5}{3}\,(z_x)^3+\tfrac{5}{6}\,z^4z_x\\
&{\Psi^0}_3 = z_{(7)}+\tfrac{7}{3}\,z^2z_{(5)}+14\,z\,z_x\,z_{xxxx}+21\,(z_x)^2z_{xxx}+\tfrac{70}{3}\,z\,z_{\,xx}\,z_{xxx}+\tfrac{35}{18}\,z^4z_{xxx}\\
&\qquad\qquad +\tfrac{91}{3}\,z_x(z_{xx})^2+\tfrac{140}{9}\,z^3z_x\,z_{xx}+\tfrac{35}{3}\,z^2(z_x)^3+\tfrac{35}{54}\,z^6z_x\,,
\end{aligned}
$$

Recursion Operator:

$$(D_x)^2+\tfrac{2}{3}\,z^2+\tfrac{2}{3}\,z_x(D_x)^{-1}\quad .$$