DETERMINATION OF SYMMETRIES OF PARTIAL DIFFERENTIAL EQUATIONS.

I. Language and geometry to describe a partial differential equation (pde).

There are two standard, useful methods to describe pde's so that one may then attempt to find their symmetries. The first of these is via jet bundles, and the second via differential forms.

A. Jet Bundles.

Although I will give a general description of the method, let me focus from time to time on some particular pde; I will use **Burgers' equation** for that purpose:

$$u_{xx} + uu_x + u_t = 0 \quad . \tag{1.1}$$

We may characterize the various quantities in the following way

$$\begin{aligned} M &\equiv \{x,t\} \longleftrightarrow \text{independent variables} \quad , \\ N &\equiv \{u\} \longleftrightarrow \text{dependent variables} \quad , \\ J^2 &\equiv \{u_x, u_t, u_{xx}, u_{xt}, u_{tt}\} \longleftrightarrow \text{jet variables} \quad . \end{aligned} \tag{1.2}$$

I have a **base manifold**, M, of independent variables and I attach to each point of M a "fiber" where I may put the values of both a function and as many of its derivatives as I want. If the fiber includes derivatives up to k-th order, I call the space the jet bundle of order k, $J^k(M,N)$, and I may use coordinates in the fiber of

$$\{z, z_x, z_t, z_{xx}, z_{xt}, z_{tt}, ..., z_{x...x}, z_{x...xt}, ..., z_{t...t}\} \quad , \tag{1.3}$$

where I use z, z_x, z_t, etc. instead of u, u_x, u_t, etc. because I want to **emphasize** that these are all **independent** variables, in this larger space, at the moment, which simply provide us with a place to draw graphs of any given function, and to lift it up so as to show off all of its derivatives.

For a given function, $u(x,t)$, we will have at each point of M, a value for all the different coordinates on the fiber, that is, for z, z_x, etc. We therefore define a particular sort of a mapping, called a section:

Defn. For every function $u \in N$, the set $j^k u$ is a lifting up of the base manifold into the entire jet space such that

$$j^k u = \{(x, t, u(x,t), u_x(x,t), u_t(x,t), ...) \mid (x,t) \in M\} \tag{1.4}$$

Returning for a moment to the case of one independent variable, recall that for $u = x^2$, we have $u_x = 2x$, so that, if we use coordinates (x, u, u_x) for points in J^1, the following are all points that belong to $j^1 u$:

$$\text{for } u(x) = x^2, (x, u, u_x) \in J^1 : \quad (0,0,0), (1,1,2), (1/2, 1/4, 1) \quad .$$

On the other hand the point $(1,1,1)$ does NOT lie on this curve, although it might for some other function. More generally, note that the lifted curve at the bottom of page 2 –which, of course is a copy of M, which for this one- dimensional case is just the x-axis– could not have been lifted up so that its height oscillated up and down and still be the section generated by this function. Therefore, we need a way of knowing how we may determine if a particular m-dimensional surface in J^k is actually one generated by a section.

First, however, let me point out that in this language we may now simply describe a pde, of k-th order, as a surface in $J^k(M,N)$. In particular Burgers' equation is the surface in J^2 given by

$$Y^2 : z_{xx} + zz_x + z_t = 0 \quad . \tag{1.5}$$

In the 6-dimensional space called J^2, with coordinates $\{x,t,z,z_x,z_t,z_{xx}\}$, Burgers' equation is the particular 5-dimensional surface which I call Y^2, specified by Eq. (1.5). Moreover, $u = u(x,t)$ is a solution to Burgers' equation if and only if the 2-dimensional surface which is the section j^2u lies completely within this 5-dimensional surface.

Continuing with this language, we may now quickly describe a symmetry of the pde as a mapping that maps solutions into other solutions. Using the geometrical language above, a symmetry is a mapping that maps sections into sections, **all the time staying within the surface, Y^2, that is the pde.** An alternative way of stating this is to say that a symmetry is a mapping that stays within the surface defined by the pde, but one that always maps sections into other sections. It is therefore important to now explain how to determine whether some subspace lies within a given section.

There are 2 distinct, but complementary, ways to determine whether curves within a jet bundle have been generated by sections. The *first* is based on the **contact module**, Ω, which is a set of 1-forms defined over J^k:

$$\Omega \; : \qquad \begin{aligned} \underset{\sim}{\theta} &\equiv dz - z_x dx - z_t dt \quad , \\ \underset{\sim}{\theta}_x &\equiv dz_x - z_{xx} dx - z_{xt} dt \quad , \\ \underset{\sim}{\theta}_t &\equiv dz_t - z_{xt} dx - z_{tt} dt \quad . \end{aligned} \tag{1.6}$$

This is enough for J^2, but more are needed for $k > 2$. Each of these is a 1-form in the cotangent space to J^2 and all (obviously) have the property that they vanish if they are restricted to a section. The converse is in fact also true; that is, any lifting of M into J^2 for which the contact module vanishes is indeed a section for some function, u.

Now, however, we need to also "tell" the contact module about the pde we want to study. That is done by **restricting** it to the surface, Y^k, or, at least, that is one way to do it and the way I choose at the moment! To do this, we must choose new coordinates on the surface that are adapted to it. As an example, I note that if I am using $\{r, \theta, \varphi\}$ as coordinates in 3-dimensional space and I want to restrict myself to the surface of a sphere I simply throw away "r" and use $\{\theta, \varphi\}$. (This is a bit more difficult if I began with $\{x, y, z\}$, but can still be done.)

For Burgers' equation, as given by Y^2, a very reasonable choice of coordinates on Y^2 is $\{x, t, z, z_x, z_{xx}\}$, where I have simply thrown away z_t, because it is always given by solving the equation: $u_t = -u_{xx} - uu_x$. I then want to restrict the contact module to Y^2—so that it "knows" about the pde. Restricting the first 1-form, in Eq. (1.6), is easy:

$$\underset{\sim}{\theta} \ \rightarrow \ \overline{\underset{\sim}{\theta}} = dz - z_x dx + (z_{xx} + z z_x) dt \quad , \tag{1.7a}$$

where the bar over $\underset{\sim}{\theta}$ means that it has been restricted to Y^k. The second one is a little more difficult because we need to replace z_{xt}. We do this, for the moment, by "cheating" just a little. We know that any solution of Burgers' equation has $u_t = -u_{xx} - uu_x$; therefore it must surely be so that

$$u_{tx} = -u_{xxx} - uu_{xx} - u_x^2 \quad . \tag{1.8}$$

It is then reasonable to replace z_{xt} by $-(z_{xxx} + z z_{xx} + z_x^2)$; that is,

$$\underset{\sim}{\theta}_x \ \rightarrow \ \overline{\underset{\sim}{\theta}}_x = dz_x - z_{xx} dx + (z_{xxx} + z z_{xx} + z_x^2) dt \quad . \tag{1.7b}$$

The third element in the contact module is even more troublesome, since we no longer have a variable called z_t—while we are on Y^2. Therefore we should not need it. The truth is that we don't need it, but must eventually use it to carefully define $z_{tt}|_{Y^2}$, etc. so that it is irrelevant. We will come back to this idea later! For now, we simply refer to the set $\bar{\Omega}$ as the set of all linear combinations of $\overline{\underset{\sim}{\theta}}$ and $\overline{\underset{\sim}{\theta}}_x$, which is then the contact module restricted to the cotangent space of the surface which defines the pde. Since $\bar{\Omega}$ both knows about the pde and about sections, the jet bundle approach to symmetries is to look for maps

$$\phi : Y^k \rightarrow Y^k \quad , \text{ such that } \quad \phi^* : \bar{\Omega} \rightarrow \bar{\Omega} \quad . \tag{1.9}$$

In general, I am only going to be interested in such symmetry maps which depend in some continuous way on some parameter. In particular, since the identity map is obviously a symmetry, I want to allow this parameter to vary in such a way that I get the identity and other symmetries that are "near" the identity. This means, since I am mapping solutions into solutions, that I may visualize this mapping as beginning at the identity and then proceeding in **some** direction. We can "define" this direction as

$$\frac{d\phi}{d\tau} \, |_{\phi = identity} \equiv \tilde{v} \quad , \tag{1.10}$$

where $\tilde{v}$ is a vector (field) that gives us this direction, and we denote the constraint, at least rather formally, in the form

$$\mathcal{L}_{\tilde{v}}\bar{C} \in \bar{C},$$

where $\mathcal{L}_{\tilde{v}}$ stands for the Lie derivative in the direction $\tilde{v}$, which will be defined soon.

B. A pde via differential forms.

A somewhat different way to characterize a pde was originally created by Elié Cartan, which utilizes an ideal of differential forms defined on J^{k-1} to uniquely characterize a k-th order pde. The advantage of this approach is that the process of recovering the pde itself involves a decision as to which are the independent and which the dependent variables, so that this characterization of a pde automatically encodes an entire class of pde's that are equivalent in the sense that we could always find a change of variables that would map one into another.

This description begins with the contact module and then appends in a different way the information about the pde. Dealing explicitly with Burgers' equation, we work on J^1 and so do not need (may not write) the other 1-forms one would need if we were working on a higher-order pde, namely $\underset{\sim}{\theta}_x$ and $\underset{\sim}{\theta}_t$, but only the single element in the contact module, $\underset{\sim}{\theta} = dz - z_x dx - z_t dt$. We then append to this a 2-form which defines the particular pde in which we are interested, $\underset{\sim}{\beta}$. That gives us an ideal to study generated by

$$\begin{aligned} \underset{\sim}{\theta} &= dz - z_x dx - z_t dt \quad , \\ \underset{\sim}{\beta} &\equiv dz_x \wedge dt + z dz \wedge dt - dz \wedge dx \quad . \end{aligned} \tag{1.11}$$

At this point it is a good ideal to try to carefully define the important notion of *an ideal* of differential forms. In general, the set ΛM, of all differential forms over any manifold M, is an example of *an algebra.* An algebra is defined as a set of vector spaces with the additional property that a multiplication is defined for them such that the product remains within the algebra. For ΛM, this multiplication is of course the wedge product. An **ideal of an algebra** is then a subset of the algebra which has the property that multiplication of elements of the ideal with <u>any</u> element of the entire algebra stay within that ideal. The common way to describe an ideal is to give a list of *the generators of that ideal*, so that the ideal is then simply the set of all linear combinations of products of anything, from within the algebra, with the generators. For the particular case of an ideal of differential forms, i.e., an ideal of ΛM, we will make this somewhat more precise.

Defn. An ideal, $\mathcal{I} \subseteq \Lambda M$ is to be generated by a set of p-forms, $\{\underset{p}{\underset{\sim}{\alpha}}^{\mathbf{i}}\}$, in general of varying values of p, so that the index directly below the symbol is used to remind us of that value. Then we may describe $\mathcal{I}$ as

$$\mathcal{I} \;\equiv\; \bigcup_{q=1}^{m} \left\{ \underset{q-p}{\underset{\sim}{\eta}_i} \wedge \underset{p}{\underset{\sim}{\alpha}^i} \;\mid\; \underset{q-p}{\underset{\sim}{\eta}_i} \in \Lambda M \right\} \;\subseteq\; \Lambda M \equiv \bigcup_{q=0}^{m} \Lambda^q M \quad ,$$

where m is the dimension of the underlying manifold, and the union in the definition begins with 1, rather than 0, because we do not intend to allow 0-forms within our ideal. (There are some authors who do not exclude 0-forms.)

Returning to our discussion for Burgers' equations, we can now show that the ideal determined by the pair of generators in Eqs. (1.11) has all the information we need. In fact, just two particular 2-forms within that ideal will be sufficient, namely $\underset{\sim}{\gamma}$ and $\underset{\sim}{\delta}$. These contain all the information we need to re-constitute the pde itself, as we now show:

$$\underset{\sim}{\gamma} \equiv \underset{\sim}{\beta} - z\underset{\sim}{\theta} \wedge dt = dz_x \wedge dt + zz_x dx \wedge dt - dz \wedge dx \quad ,$$
$$\underset{\sim}{\delta} \equiv \underset{\sim}{\beta} - z\underset{\sim}{\theta} \wedge dt + \underset{\sim}{\theta} \wedge dx = dz_x \wedge dt + zz_x dx \wedge dt - z_t dt \wedge dx = (dz_x + zz_x dx + z_t dx) \wedge dt \quad .$$

If we had a solution to our pde, then $dz_x \wedge dt$ would be $z_{xx} dx \wedge dt$ and then $\underset{\sim}{\delta} = (z_{xx} + zz_x + z_t)dx \wedge dt$, so that we see that $\underset{\sim}{\delta}$ is actually just the pde, multiplied by the (non-zero) volume form for the independent variables, $dx \wedge dt$. We may therefore reasonably infer that the ideal generated by $\{\underset{\sim}{\theta}, \underset{\sim}{\beta}\}$ knows about sections on J^1, knows about the pde, and makes sense on only J^1 since it doesn't actually use things like z_{xx}. (I note that $\underset{\sim}{\beta}$ is a 2-form because there are 2 independent variables. If there were 5 independent variables, then this approach would insist that $\underset{\sim}{\beta}$ be a 5-form.)

There is, in fact, still too much information in this ideal in the following sense. It might be nicer were they both forms of the same size. In this case that would mean they should both be 2-forms, since one of them has to be. A good choice is just $\{\underset{\sim}{\theta} \wedge dt, \underset{\sim}{\beta}\}$. As an indication that this truly is enough, I use new names for the moment and write

$$\begin{aligned} \underset{\sim}{\theta} \wedge dt &= dz \wedge dt - p dx \wedge dt \quad , \\ \underset{\sim}{\beta} &= dp \wedge dt + z dz \wedge dt - dz \wedge dx \quad , \end{aligned} \tag{1.11'}$$

and I decide, from among $\{x, t, z, p\}$ that $\{x, t\}$ will be the independent variables.

Now, I must use some of the most important properties of the exterior derivative operator, d, and of the wedge product, $\wedge$, so that I will remind you of them here.

Poincaré's Lemma:

If for a p-form $\underset{\sim}{\omega}$, $d\underset{\sim}{\omega} = 0$, then there exists (locally) some (p-1)-form, $\underset{\sim}{\varphi}$, such that $\underset{\sim}{\omega} = d\underset{\sim}{\varphi}$.

Theorem 2.

If two 1-forms, $\underset{\sim}{\alpha}$ and $\underset{\sim}{\beta}$ are such that

$$\underset{\sim}{\alpha} \wedge \underset{\sim}{\beta} = 0 \quad , \tag{1.12a}$$

then there exists a scalar λ such that

$$\underset{\sim}{\alpha} = \lambda \underset{\sim}{\beta} \quad . \tag{1.12b}$$

Therefore, when we cause our two 2-forms, in Eq. (1.11′) to vanish, we have

$$\underset{\sim}{\theta} \wedge dt = (dz - pdx) \wedge dt = 0 \quad , \tag{1.13a}$$

from which we infer the existence of a scalar function q such that $dz - pdx = qdt$. But if x and t are the independent variables, then z, p and q depend on them, so it must also be so that we can write

$$dz = z_x dx + z_t dt \quad , \tag{1.13b}$$

where the indices truly do mean derivatives right now! Therefore we must identify

$$p = z_x \, , \quad q = z_t \quad \longrightarrow \quad \underset{\sim}{\beta} = (z_{xx} + z z_x + z_t) dx \wedge dt \quad , \tag{1.13c}$$

returning our original equation to us!

This particular approach is sufficiently important that I propose to look at another quick example. Consider the equation

$$u_{xx} u_{yy} - u_{xy}^2 = 1 \quad , \tag{1.14}$$

a fairly nasty looking equation. We set the usual contact form on $J^1 : \underset{\sim}{\alpha} = du - pdx - qdy$ and then write a 2-form for the equation itself, namely $\underset{\sim}{\beta} = dp \wedge dq - dx \wedge dy$. Noticing that $\underset{\sim}{\beta}$ has p and q but no u, we use d to make $\underset{\sim}{\alpha}$ a 2-form, so that our ideal is generated by the pair

$$\begin{aligned} -d\underset{\sim}{\alpha} &= dp \wedge dx + dq \wedge dy \quad , \\ \underset{\sim}{\beta} &= dp \wedge dq - dx \wedge dy \quad . \end{aligned} \tag{1.15}$$

As before, a choice of $\{x, y\}$ as independent variables and the use of theorem 1 to retrieve u will bring back our original equation. Please do it! However, instead I want to think of $\{p, y\}$ as a different choice of independent variables, and write $-d\underset{\sim}{\alpha} = d(-xdp + qdy) = 0$, from which we infer the existence of a scalar ω such that $-xdp + qdy = d\omega$, OR, with $\omega = \omega(p, y)$, $x = -\omega_p$, $q = +\omega_y$. Then the other 2-form takes the form

$$\underset{\sim}{\beta} = dp \wedge d\omega_y + d\omega_p \wedge dy = (\omega_{yy} + \omega_{pp}) dp \wedge dy \quad , \tag{1.16}$$

which gives us a new pde,

$$\omega_{yy} + \omega_{pp} = 0 \quad ,$$

which is the 2-dimensional Laplace equation. As it happens, this new equation, that we have traded for, is linear! We have then a mapping between these 2 equations –actually a Legendre transform.

C. Closure of a Differential ideal.

In general, if I am given a system of differential equations, then there may be no solution because there are integrability conditions on the system that cannot be satisfied. Therefore, we must always ensure that all integrability conditions are satisfied. The simplest example of such a system might be $q = \varphi_{,x}, p = \varphi_{,y}$, with given q and p and φ to be found. In general there is no solution, because

$$\varphi_{,xy} = q_{,y} \quad \text{and} \quad \varphi_{,yx} = p_{,x}$$

and then, the equality of mixed partial derivatives causes the need for $q_{,y} = p_{,x}$ –an integrability condition. All of this is much clearer if done with differential forms.

Write $\underset{\sim}{\alpha} \equiv d\varphi - qdx - pdy$. The vanishing of this for some section $\varphi = \varphi(x, y)$ is exactly what is meant by finding a solution of the original pair of equations. However,

$$-d\underset{\sim}{\alpha} = dq \wedge dx + dp \wedge dy \quad ,$$

and if $(j^1\varphi)\alpha = 0$ then we need $(j^2\varphi)d\underset{\sim}{\alpha} = 0$, but, with $\{x, y\}$ as independent variables,

$$-d\underset{\sim}{\alpha} = (-q_y + p_x)dx \wedge dy \quad ,$$

as an integrability condition for the existence of such an α as a solution.

We generalize this by saying that when a pde is written in terms of an ideal of forms, that ideal must be closed, with respect to the operator d, in order for it to be useful to us. Therefore, from now on we **modify** the definition of an ideal of differential forms to include the necessity that it be closed. (Some authors then refer to this as a *differential ideal.*) To emphasize this, let us write down an explicit definition for such an ideal to be closed:

Defn. An ideal of differential forms, $\mathcal{I} \subseteq \Lambda M$, is closed $\iff$ for every q-form, $\underset{\sim}{\sigma}_q \in \mathcal{I}$, $d\underset{\sim}{\sigma}$ lies within the ideal. This of course means that there exists a finite set of p-forms $\{\underset{\sim}{\alpha}^i{}_p\} \in \mathcal{I}$ and the same number of $(q+1-p)$-forms within the entire algebra, ΛM, so that the following sum is the same as the differential of $\underset{\sim}{\sigma}_q$:

$$d \underset{q}{\underset{\sim}{\sigma}} = \underset{q+1-p}{\underset{\sim}{\eta}_i} \wedge \underset{p}{\underset{\sim}{\alpha}^i} \quad .$$

Once again we return to the ideal of 2-forms we used to describe Burgers' equation:

$$\begin{aligned} \underset{\sim}{\alpha} &\equiv dz \wedge dt - pdx \wedge dt \quad , \\ \underset{\sim}{\beta} &\equiv dp \wedge dt + zdz \wedge dt - dz \wedge dx \quad . \end{aligned} \tag{1.11''}$$

Then we immediately see that d acting on these generators gives

$$\begin{aligned} d\alpha &= -dp \wedge dx \wedge dt = dx \wedge (\beta - z\alpha) \quad , \\ d\beta &= 0 \quad , \end{aligned} \tag{1.17}$$

showing that $d\{\text{the ideal}\}$ lies within the ideal, i.e., $d\mathcal{I} \subseteq \mathcal{I}$. Since both $\underset{\sim}{\beta}$ and $\underset{\sim}{\alpha}$ vanish when applied to a solution, therefore this linear dependence ensures that $d\underset{\sim}{\alpha}$ and $d\underset{\sim}{\beta}$ will also vanish; so this ideal is closed.

D. Total Derivative Operators.

A concept that will be useful again and again is that of these D_{x^i}. So let me suppose that I have independent variables, $\{x^i\}$, dependent variables, $\{z^\alpha\}$, and jet variables, $\{z^\alpha_i, z^\alpha_{ij}, ..., z^\alpha_{ij...k}\}$, as coordinates on J^k. Then let $\Phi : J^k \to R$ be a function defined on J^k. It follows that

$$d\Phi = \Phi_{,i} dx^i + \Phi_{,\alpha} dz^\alpha + \Phi_{,z^\alpha_i} dz^\alpha_i + \ \dots \ + \Phi_{,z^\alpha_{i...k}} dz^\alpha_{i...k} \quad . \tag{1.18}$$

But we might have wanted to think of our function as it would be seen by a section. For this purpose, we recall that it is the contact module that knows about sections. Therefore, I recall it for you, in this notation,

$$\Omega^k(M,N) \equiv \begin{cases} \theta^\alpha & \equiv dz^\alpha - z^\alpha_i dx^i \quad , \\ \theta^\alpha_i & \equiv dz^\alpha_i - z^\alpha_{ij} dx^j \quad , \\ \vdots & \\ \theta^\alpha_{i...j} & \equiv dz^\alpha_{i...j} - z^\alpha_{i...jk} dx^k \quad . \end{cases} \tag{1.19}$$

Then we may rewrite Eq. (1.18) in the form

$$\begin{aligned} d\Phi =& \Phi_{,i} dx^i + \Phi_{,\alpha}\{\theta^\alpha + z^\alpha_i dx^i\} + \Phi_{,z^\alpha_i}\{\theta^\alpha_i + z^\alpha_{ij} dx^j\} \\ & + \ \dots \ + \Phi_{,z^\alpha_{i...j}}\{\theta^\alpha_{i...j} + z^\alpha_{i...jk} dx^k\} \\ =& \{\Phi_{,i} + z^\alpha_i \Phi_{,\alpha} + z^\alpha_{ij} \Phi_{,z^\alpha_j} + \ \dots \ + z^\alpha_{ij...k} \Phi_{,z^\alpha_{j...k}}\} dx^i \\ & + \text{ terms involving elements of } \Omega^k \\ \equiv& \{D_i \Phi\} dx^i \qquad (\text{mod } \Omega^k) \quad . \end{aligned} \tag{1.18$'$}$$

where $(\text{mod } \Omega^k)$ is an abbreviation for modulo Ω^k, which is supposed to mean the same as written in the line just above that, i.e., that we are ignoring additional terms involving elements of the ideal generated by Ω^k. Since $(j^k u)^* \Omega^k = 0$, i.e., Ω^k vanishes when restricted to every section, the total derivatives tell us everything we want to know about $d\Phi$.

II. Lie Symmetries.

A. Via the closure of the contact module and the pde.

Before wandering into too much more theory I want to give some actual calculations. I begin with **Burgers' equation**, written via the ideal in Eq. (1.11):

$$\begin{aligned} \underset{\sim}{\theta} &= dz - p dx - q dt \quad , \\ \underset{\sim}{\beta} &= dp \wedge dt + z dz \wedge dt + dx \wedge dz \quad . \end{aligned} \tag{1.11}$$

A serious problem with this particular version of the ideal is that it is **not** closed. To solve this, we append the additional 2-form, $d\underset{\sim}{\theta}$, giving us

$$\mathcal{I} \equiv \begin{cases} \underset{\sim}{\theta} = dz - p dx - q dt \quad , \\ -d\underset{\sim}{\theta} = dp \wedge dx + dq \wedge dt \quad , \\ \underset{\sim}{\beta} = (dp + z dz) \wedge dt + dx \wedge dz \quad . \end{cases} \tag{2.1}$$

Since our variables are $\{x, t, z, p, q\}$, we set up a vector field—to be determined—in this 5-dimensional space, which we will envision as the generator of a symmetry transformation of the solution space for this ideal:

$$\tilde{v} = v^x \partial_x + v^t \partial_t + v^z \partial_z + v^p \partial_p + v^q \partial_q \quad , \tag{2.2}$$

and require of our $\tilde{v}$ that

$$\mathcal{L}_{\tilde{v}} \mathcal{I} \subseteq \mathcal{I} \ , \qquad \text{where} \ \ \mathcal{L}_{\tilde{v}} \underset{\approx}{\omega} = d(\tilde{v} \rfloor \underset{\approx}{\omega}) + \tilde{v} \rfloor d\underset{\approx}{\omega} \quad . \tag{2.3a}$$

The step operator is a contraction operator for tensors, i.e., a generalization of the ordinary scalar product. For a vector, $\tilde{v}$, a 1-form, $\underset{\sim}{\alpha}$, and, for instance, a 2-form, $\underset{\sim}{\beta}$, $\widetilde{e}_i$ a basis of vectors, and $\underset{\approx}{\omega}^i$ the reciprocal basis for 1-forms, we have

$$\begin{aligned} &\tilde{v} = v^i \, \widetilde{e}_i \, , \quad \alpha = \alpha_i \, \underset{\approx}{\omega}^i \, , \quad \beta = \tfrac{1}{2} \, \beta_{ij} \, \underset{\approx}{\omega}^i \wedge \underset{\approx}{\omega}^j \, , \\ &\tilde{v} \rfloor \underset{\approx}{\alpha} = v^i \alpha_i \ , \qquad \tilde{v} \rfloor \underset{\sim}{\beta} = \tfrac{1}{2} [v^i \beta_{ij} \underset{\approx}{\omega}^j - v^j \beta_{ij} \underset{\approx}{\omega}^j] \, . \end{aligned} \tag{2.3b}$$

Characteristics for Symmetry Generators:

Before proceeding ahead with Burgers' equation, we want to step aside for a few minutes and first consider the general problem, for **any** arbitrary pde, of what constraints follow on the symmetry generators because they must preserve the contact module. In order to do this, it is much better to use a somewhat more general notation for a while. Therefore, I now define general labels for the coordinates on the jet bundle, J^1:

$$\begin{aligned} &\text{the independent variables, } \{x^i\}, \ i = 1, \ldots m \quad , \\ &\text{the dependent variables, } \{z^\alpha\}, \ \alpha = 1, \ldots n \quad , \\ &\text{and the "true" jet } J^1 \text{ variables, } \{z^\alpha_i\}, \ \alpha = 1, \ldots m \text{ and } i = 1, \ldots n \quad , \end{aligned} \tag{2.4}$$

where these 3 sets of variables generalize the ones we have been using, namely independent variables, $\{x, t\}$, dependent variable, $\{z\}$, and 1st jet variables, $\{p, q\}$, to the case where one can have any number, n, of dependent variables. We may of course now re-write our vector field also in this notation:

$$\tilde{v} = v^i \partial_i + v^\alpha \partial_\alpha + v_i^\alpha \partial_{z_i^\alpha} \quad . \tag{2.2'}$$

We will now construct our ideal so that it always includes the entire set of first 1-forms in the contact module, $\underset{\sim}{\theta}^\alpha$, one for each unknown function to be determined, i.e., one for each dependent variable. The action of $\tilde{v}$ on these 1-forms, $\underset{\sim}{\theta}^\alpha$ is

$$\mathcal{L}_{\tilde{v}} \underset{\sim}{\theta}^\alpha = \mathcal{L}_{\tilde{v}} \{dz^\alpha - z_i^\alpha dx^i\} = d(\tilde{v} \rfloor \underset{\sim}{\theta}^\alpha) + \tilde{v} \rfloor d\underset{\sim}{\theta}^\alpha \quad . \tag{2.5a}$$

As will be clear a little later, it is now very convenient to introduce into our equations the function(s) we will refer to as the *characteristics*, F^α, for the symmetry generators, $\tilde{v}$, one for each dependent variable:

$$F^\alpha \equiv \tilde{v} \rfloor \underset{\sim}{\theta}^\alpha = v^\alpha - z_i^\alpha v^i \quad . \tag{2.6}$$

With this definition, the equations for the **action of $\tilde{v}$ on the $\underset{\sim}{\theta}^\alpha$** can be re-phrased as follows:

$$\begin{aligned} \mathcal{L}_{\tilde{v}} \underset{\sim}{\theta}^\alpha &= dF^\alpha - \tilde{v} \rfloor \{dz_i^\alpha \wedge dx^i\} \\ &= dF^\alpha - v_i^\alpha dx^i + v^i dz_i^\alpha \quad , \end{aligned} \tag{2.5b}$$

We must now impose the requirement, on $\tilde{v}$, that it preserve the contact 1-forms. This means that the Lie-derivative (in the direction $\tilde{v}$) of each $\underset{\sim}{\theta}^\alpha$ must remain in the ideal, so that it must be some linear combination of the generators. However, since the $\underset{\sim}{\theta}^\alpha$'s are only 1-forms, only those generators that are 1-forms, i.e., the $\underset{\sim}{\theta}^\alpha$'s themselves, can be involved in these linear combinations. Rephrased in more mathematical language, this implies that there must exist some set of scalars, i.e., 0-forms, λ_β^α so that the Lie derivative of each $\underset{\sim}{\theta}^\alpha$ will in fact be a linear combination of the generators, where the λ_β^α are the coefficients:

$$\mathcal{L}_{\tilde{v}} \underset{\sim}{\theta}^\alpha = \lambda_\beta^\alpha \, \underset{\sim}{\theta}^\beta \quad , \tag{2.7}$$

Putting together the requirement from Eqs. (2.7) with the form in terms of F^α from Eqs. (2.5b), we may solve the equations for dF^α:

$$dF^\alpha = v_i^\alpha dx^i - v^i dz_i^\alpha + \lambda \{dz^\alpha - z_i^\alpha dx^i\} \quad . \tag{2.8a}$$

However, of course the differential of any function may simply be written out, via the chain rule, in terms of all its derivatives with respect to all the variables on which it depends, giving us this form:

$$dF^\alpha = F_{,i}^\alpha dx^i + F_{,\beta}^\alpha dz^\beta + F_{,z_i^\beta}^\alpha dz_i^\beta \quad . \tag{2.8b}$$

Comparing these last two forms for dF^α, we may read off the equality for each element of the basis for 2-forms, which gives us the following useful relations:

$$\begin{aligned} v^i\delta^\alpha_\beta &= -F^\alpha_{,z^\beta_i} \quad , \\ \lambda^\alpha_\beta &= F^\alpha_{,\beta} \quad , \\ v^\alpha &= F^\alpha + z^\alpha_i v^i \quad , \\ v^\alpha_i &= F^\alpha_{,i} + z^\beta_i F^\alpha_{,\beta} \quad , \end{aligned} \tag{2.9}$$

where we have used the definition of F^α itself, in Eq. (2.6), to generate the equation for v^α.

The very remarkable fact that emerges from these calculations is that any vector field that preserves Ω^1—the contact module on J^1—is automatically generated by these potentials, F^α, with only n functions to be determined instead of $(m+n+mn)$ such functions—namely the entire set of components of $\tilde{v}$. These particular potentials are referred to as the *characteristics* of the symmetry generators.

We now prove even a further result concerning the contact module. In general the contact module, when considered as an ideal in ΛM, is **not closed.** This constrains us, as we have already seen, at Eq. (2.1), that it was necessary to add the additional 2-form, $d\underset{\sim}{\theta}^\alpha$ to our ideal. This of course means that the tangent vector field that would generate a symmetry must also preserve $d\underset{\sim}{\theta}^\alpha$, in addition to just preserving $\underset{\sim}{\theta}^\alpha$. The fascinating result is that a vector field generated by a (set of) characteristics **automatically also preserves** $d\underset{\sim}{\theta}^\alpha$, as will now be shown. In order to do this we first prove a simple property of the Lie derivative.

Lemma The operators d and $\mathcal{L}_{\tilde{v}}$ always commute.

Proof: We begin from the definition of the action of the Lie derivative on p-forms, as given in Eq. (2.3). We may act on that equation with the exterior derivative, giving one result. We may also apply that equation to the exterior derivative of the original $\underset{\sim}{\alpha}$, giving another result:

$$\begin{aligned} d\mathcal{L}_{\tilde{v}}\underset{\sim}{\alpha} &= d\{d(\tilde{v}\rfloor\underset{\sim}{\alpha})\} + d\{\tilde{v}\rfloor(d\underset{\sim}{\alpha})\} = d\{\tilde{v}\rfloor d\underset{\sim}{\alpha}\}\,, \\ \mathcal{L}_{\tilde{v}}d\underset{\sim}{\alpha} &= d(\tilde{v}\rfloor d\underset{\sim}{\alpha}) + \tilde{v}\rfloor d\,d\underset{\sim}{\alpha} = d(\tilde{v}\rfloor d\underset{\sim}{\alpha})\,. \end{aligned}$$

The equality of the right-hand ends of these two lines confirms the desired theorem.

Our desired result now follows quite easily since

$$\mathcal{L}_{\tilde{v}}d\underset{\sim}{\theta}^\alpha = d\mathcal{L}_{\tilde{v}}\underset{\sim}{\theta}^\alpha = d(\lambda^\alpha_\beta\underset{\sim}{\theta}^\beta) = d\lambda^\alpha_\beta\wedge\underset{\sim}{\theta}^\beta + \lambda^\alpha_\beta d\underset{\sim}{\theta}^\beta \quad , \tag{2.10}$$

which is surely a linear combination of $\underset{\sim}{\theta}^\beta$ and $d\underset{\sim}{\theta}^\beta$, and therefore contained within the ideal, as desired.

This tells us that, in general, if we begin with the characteristic(s) of a system of pde's, we have really only to ensure that our generator, $\tilde{v}$, preserves $\underset{\sim}{\beta}$ within the ideal.

Return to Burgers' Equation:

We should now return to our consideration of Burgers' equation. However, it might be a good idea to re-capitulate the important parts of the material in the previous section in the current, simpler notation that we have been using for Burgers' equation. This can easily be done, since the important thing to remember is that

a. Each dependent variable has an independent characteristic. Burgers' equation has only one dependent variable, so that it has only one characteristic, which we call $F = F(x, t, z, p, q)$.

b. The contact module, and its closure, are already preserved under a given symmetry vector, $\tilde{v}$, as defined via Eq. (2.2), provided that its components are determined directly from the characteristic, F, via the following equations:

$$\begin{aligned} v^x &= - F_{,p} \,, \\ v^t &= - F_{,q} \,, \\ v^z &= F - pF_{,p} - qF_{,q} \,, \\ v^p &= F_{,x} + pF_{,z} \,, \\ v^q &= F_{,t} + qF_{,z} \,. \end{aligned} \tag{2.11}$$

If we had, instead, been considering a second-order pde, one would need to use higher-order contact forms, such as $\underset{\sim}{\theta}_x$ and $\underset{\sim}{\theta}_t$, requiring that $\tilde{v}$ be defined on the second jet as well, thereby extending Eq. (2.2). This would then have allowed us to obtain formulae for the coefficients of $\tilde{v}$ in the extra, 2nd jet-directions. Those equations are rather nastier, and lengthier, than the ones above, and, of course, we will not need them now. Nonetheless, they will be needed later. Therefore, I note that the general discussion of the components of $\tilde{v}$ is given in Section E, beginning with Eqs. (2.52). The 2nd jet components, in the format of Eqs. (2.9), are given by Eqs. (2.60b), with the explicit form, analogous to Eqs. (2.11), given at Eqs. (2.64) These are then explicit presentations of $\{v^r, v^s, v^w\}$, where $\{r, s, w\}$ are shorthand for the additional coordinates needed on J^2/J^1, i.e., $\{z_{xx}, z_{xt}, z_{tt}\}$, respectively, just as $\{p, q\}$ are shorthand for the J^1/J^0 coordinates $\{z_x, z_y\}$.

Having these details laid out clearly, we may now consider the Lie derivative of $\underset{\sim}{\beta}$, as defined in Eq. (2.1):

$$\begin{aligned} \mathcal{L}_{\tilde{v}}\underset{\sim}{\beta} &= \mu\underset{\sim}{\beta} + \sigma d\underset{\sim}{\theta} + \underset{\sim}{\eta} \wedge \underset{\sim}{\theta} \\ &= \mu\{(dp + zdz) \wedge dt + dx \wedge dz\} - \sigma(dp \wedge dx + dq \wedge dt) \\ &\quad + \eta_x dx \wedge (dz - qdt) + \eta_t dt \wedge (dz - pdx) - \eta_z dz \wedge (pdx + qdt) \\ &\quad + \eta_p dp \wedge (dz - pdx - qdt) + \eta_q dq \wedge (dz - pdx - qdt) \quad . \end{aligned} \tag{2.12a}$$

On the other hand, since $d\underset{\sim}{\beta} = 0$, we may evaluate the left-hand side via $\mathcal{L}_{\tilde{v}}\underset{\sim}{\beta} = d(\tilde{v}\rfloor\underset{\sim}{\beta})$, which gives

$$\begin{aligned}\mathcal{L}_{\tilde{v}}\underset{\sim}{\beta} &= d(v^p dt - v^t dp + zv^z dt - zv^t dz - v^z dx + v^x dz) \\ &= d(v^p + zv^z)\wedge dt - dv^t \wedge dp - dv^z \wedge dx + d(v^x - zv^t)\wedge dz \quad .\end{aligned} \tag{2.12b}$$

Setting the two parts of Eqs. (2.12) equal to each other requires that we choose a basis of 2-forms, write out both sides in terms of that basis, and then require that the coefficients on each side equal that same coefficient on the other side:

$$\begin{aligned}
dp\wedge dq:&\quad v^t_{,q} = 0 \quad , \\
dp\wedge dz:&\quad v^t_{,u} + (v^x - zv^t)_{,p} = \eta_p \quad , \\
dq\wedge dz:&\quad +(v^x - zv^t)_{,q} = \eta_q \quad , \\
dp\wedge dx:&\quad v^t_{,x} - v^z_{,p} = -\sigma - p\eta_p \quad , \\
dq\wedge dx:&\quad v^z_{,q} = p\eta_q \quad , \\
dp\wedge dt:&\quad (v^p + zv^z)_{,p} + v^t_{,t} = \mu - q\eta_p \quad , \\
dq\wedge dt:&\quad (v^p + zv^z)_{,q} = -q\eta_q - \sigma \quad , \\
dz\wedge dx:&\quad (v^x - zv^t)_{,x} + v^z_{,z} = \mu + (\eta_x + p\eta_z) \quad , \\
dz\wedge dt:&\quad (v^p + zv^z)_{,z} + (v^x + zv^t)_{,t} = z\mu - (\eta_t + q\eta_z) \quad , \\
dx\wedge dt:&\quad (v^p + zv^z)_{,x} + v^z_{,t} = p\eta_t - q\eta_x \quad .
\end{aligned} \tag{2.13}$$

The first line says

$$-F_{,qq} = v^t_{,q} = 0 \quad \Rightarrow \quad F = Aq + B \quad , \tag{2.14}$$

where $A,\ B$ depend only on $\{x, t, z, p\}$. Then

$$\begin{aligned}
&v^t = -F_q = -A \quad , \\
&v^x = -F_p = -A_p q - B_p \quad , \\
&v^z = B + pv^x \quad , \\
\Rightarrow \quad &\eta_q = -A_p \ \text{ and } \ v^z_{,q} = -pA_p \quad , \\
\Rightarrow \quad &\eta_p = -A_u - qA_{pp} - B_{pp} + zA_p \quad , \\
&\sigma = qA_p + zpA_p - A_x - pA_z \quad .
\end{aligned} \tag{2.15a}$$

However the equation for $-\sigma - p\eta_p$ implies that

$$2\{A_x + qA_p + pA_u - zpA_p\} = 0 \quad , \tag{2.15b}$$

so that operating with ∂_q gives

$$A_p = 0 \quad \Rightarrow A_x + pA_z = 0 \quad , \tag{2.15c}$$

and operating on this with ∂_p then implies that

$$A_u = 0 \quad \Rightarrow A_x = 0 \quad \Rightarrow \quad A = A(t) \quad . \tag{2.16}$$

At this point we may notice that η_z occurs in the equation only as $\eta_x + p\eta_z$ or $\eta_t + q\eta_z$; even the equation $p\eta_t - q\eta_x = p(\eta_t + q\eta_z) - q(\eta_x + p\eta_z)$ contains only these quantities. Therefore the equations only know about $\eta_x + p\eta_z$ and $\eta_t + q\eta_z$, and do NOT care what η_z is. So we set it equal to zero since we really don't care either!

Now using $v^p = F_{,x} + pF_{,z}$, we determine that

$$\begin{aligned} \mu &= qB_{pp} - A_t - zpB_{pp} + B_{px} + pB_{zp} + B_z \quad , \\ \eta_x &= A_t + (-q + zp)B_{pp} - 2pB_{zp} - 2B_{px} \quad , \\ -\eta_t &= +z(q + zp)B_{pp} - 2zpB_{zp} - zB_{px} - B_{pt} + B - pB_p + B_{zx} + pB_{zz} \quad . \end{aligned} \tag{2.17}$$

So that the last equation now reads

$$\begin{aligned} +B_t + zB_x - zpB_{px} &+ B_{xx} + pB_{zx} \\ &= (q + zp)\{-(q + zp)B_{pp} + 2pB_{zp} + 2B_{xp}\} \\ &\quad - pB_{zx} - p^2 B_{zz} + p^2 B_p - pB - qA_t + pB_{pt} \quad . \end{aligned} \tag{2.18}$$

Acting with ∂_q on this equation implies that

$$-2(q + zp)B_{pp} + 2pB_{zp} + 2B_{xp} + A_t = 0 \quad . \tag{2.19}$$

Acting again with ∂_q implies that $B_{pp} = 0$, which implies the existence of $\{C, E\}$, functions of $\{x, t, z\}$ such that $B = Cp + E$. Inserting this back into Eq. (2.19), we are left with

$$A_t - 2pC_z - 2C_x = 0 \quad . \tag{2.20}$$

Acting on this with ∂_p gives us the implications

$$C_z = 0 \quad \Rightarrow \quad A_t - 2C_x = 0 \quad . \tag{2.21}$$

Inserting these results above, we obtain

$$-pE - pE_{zx} - p^2 E_{zz} + zpC_x = -pC_t + E_t + zE_x + pC_{xx} + E_{xx} + pE_{zx} \quad , \tag{2.22}$$

which is a polynomial in p, allowing us to compare coefficients:

$$\begin{aligned} p^2: &\quad E_{zz} = 0 \quad , \\ p^1: &\quad zC_x - E - 2E_{zx} = -C_t + C_{xx} \quad , \\ p^0: &\quad E_{xx} + zE_x - E_t = 0 \quad . \end{aligned} \tag{2.23}$$

Acting here with ∂_z gives us $C_x - E_z = 0$, from which we may infer that $E = C_x z + G$, where C, G depend only on $\{x, t\}$, allowing the results that

$$\begin{aligned} -G - 2C_{xx} = -C_t + C_{xx} \quad , \\ G_{xx} + zC_{xxx} + zG_x + z^2 C_{xx} + G_t + zC_{tx} = 0 \quad . \end{aligned} \tag{2.24}$$

The last equation is a polynomial in z, so comparing coefficients gives us

$$\begin{aligned} z^2 &: C_{xx} = 0 \quad , \\ z^1 &: C_{tx} = -G_x \quad , \\ z^0 &: -G_t = G_{xx} = C_{tt} \quad . \end{aligned} \tag{2.25}$$

We saw above that $A_t - 2C_x = 0$. Therefore

$$C = \tfrac{1}{2} A_t x + H(t) \quad , \tag{2.26}$$

from which we have that

$$-G = C_t = \tfrac{1}{2} A_{tt} x + H_t \quad , \tag{2.27}$$

which allows the following string of conclusions:

$$\begin{aligned} \Rightarrow G_{xx} = 0 \quad \Rightarrow \quad G_t = \frac{1}{2} A_{ttt} x + H_{tt} = 0 \quad , \\ \Rightarrow \quad A_{ttt} = 0 \text{ and } H = a_0 t + a_1 \quad , \\ \Rightarrow \quad A = \tfrac{1}{2} a_2 t^2 + a_3 t + a_4 \quad , \\ \Rightarrow \quad F = (\tfrac{1}{2} a_2 t^2 + a_3 t + a_4) q + [\frac{1}{2}(a_2 t + a_3) x + a_0 t + a_1] p \\ + [\tfrac{1}{2}(a_2 t + a_3) z - \frac{1}{2} a_2 x - a_0] \quad . \end{aligned}$$

This finally allows us to write down the following results:

$$\begin{aligned} v^x &= -F_{,p} = -C \quad , \\ v^t &= -F_{,q} = -A \quad , \\ v^z &= F - pF_{,p} - qF_{,q} = E \quad , \\ v^p &= F_{,x} + pF_{,z} = \tfrac{1}{2}(a_2 t + a_3) p - \frac{1}{2} a_2 + \tfrac{1}{2} p (a_2 t + a_3) \quad , \\ v^q &= F_{,t} + qF_{,z} = (\frac{3}{2} a_2 t + 2a_3) q + (\tfrac{1}{2} a_2 x + a_0) p + \tfrac{1}{2} a_2 z \quad . \end{aligned} \tag{2.28}$$

Noting that

$$v_i^\alpha = D_i^{(1)} F^\alpha \quad \text{and} \quad F^\alpha = v^\alpha - z_i^\alpha v^i \quad , \tag{2.29}$$

we see that a knowledge of v^α and v^i are sufficient for us to determine v_i^α, so that we don't need to worry about them, and for a while, I give only $\tilde{v} \,|_{J^0}$.

Noting that we have 5 arbitrary constants, we conclude that there are exactly 5 distinct symmetry directions, which can be simply enumerated, by setting, in turn, each $a_i = 1$, and the other ones equal to zero:

a_i	F_i	$\tilde{v}_i$
a_4	q	$-\partial_t$
a_1	p	$-\partial_x$
a_0	$tp - 1$	$-t\partial_x - \partial_z$
a_3	$tq + \frac{1}{2}xp + \frac{1}{2}z$	$-\frac{1}{2}x\partial_x - t\partial_t + \frac{1}{2}z\partial_z$
a_2	$\frac{1}{2}(t^2q + txp + tz - x)$	$-\frac{1}{2}\{tx\partial_x + t^2\partial_t - (tz - x)\partial_z\}$

B. Symmetric mappings from infinitesimal ones.

Now, what do we do with these symmetries?

I first claim that what we have always wanted— with a symmetry—is a map of $J^k \longrightarrow J^k$ that maps solutions into other solutions. Therefore, I propose a one-parameter family of such maps:

$$G_\tau : J^k \longrightarrow J^k, \forall\ \tau\ \in [0, \tau_0] \quad \text{, such that} \quad G_0 \ = \ \text{the identity.} \tag{2.30}$$

An arbitrary point $p \in J^k$ has coordinates $\{x^i, z^\alpha, z_i^\alpha, ...\}$. Since $G_\tau(P) \in J^k$, it must have coordinates $\{x'^i, z'^\alpha, z'^\alpha_i, ...\}$, from which we conclude that a coordinate presentation for Eq. (2.30) is given by specifying the following functions:

$$\begin{aligned} x'^i &= X^i(x^j, z^\alpha, z_j^\alpha, ...; \tau) \quad , \\ z'^\alpha &= Z^\alpha(x^j, z^\beta, z_j^\beta, ...; \tau) \quad , \\ z'^\alpha_i &= Z_i^\alpha(x^j, z^\beta, z_j^\beta, ...; \tau) \quad , \end{aligned} \tag{2.31a}$$

where the $X^i, Z^\alpha, Z_i^\alpha, ...$ are the entire set of functions that are required to specify any such mapping G_τ, and, of course, depend on τ in such a way that

$$\begin{aligned} X^i(x^j, z^\alpha, z_i^\alpha, ...; 0) &= x^i \quad , \\ Z^\alpha(x^j, z^\beta, z_i^\beta, ...; 0) &= z^\alpha \quad , \\ &\text{etc.} \end{aligned} \tag{2.31b}$$

We may now consider expanding these functions in Taylor series in τ, about $\tau = 0$:

$$\begin{aligned} {x'}^i &= x^i + \tau x^i_{,\tau} \mid_{\tau=0} + \ldots \quad , \\ {z'}^\alpha &= z^\alpha + \tau Z^\alpha_{,\tau} \mid_{\tau=0} + \ldots \quad , \\ {z'}^\alpha_i &= z^\alpha_i + \tau Z^\alpha_{i,\tau} \mid_{\tau=0} + \ldots \quad . \end{aligned} \tag{2.32}$$

To be better able to implement these equations, let us first revert to the situation for simply some arbitrary manifold, instead of our fairly complicated jet bundle. This arbitrary manifold we provide with coordinates $q^A, A = 1, ..., a$ and a parametrized path on that manifold, with parameter τ, so that we can write the coordinates on the path as $q^A(\tau)$. Furthermore, let $\tilde{v}(\tau_0)$ be the tangent vector at the point on the path where $\tau = \tau_0$. I claim that what we mean by calling it the tangent vector is that $\tilde{v}(\tau_0) = \frac{d}{d\tau} \mid_{\tau=\tau_0}$, or, more precisely, if $\Phi : M \to R$ is a function on this manifold, then we may write its values as $\Phi(q^A)$ and then the values along the curve as $\Phi[q^A(\tau)]$. Therefore

$$\text{the rate of change of } \Phi \text{ along the curve is } \frac{d}{d\tau}\Phi[q^A(\tau)] \quad . \tag{2.33}$$

However, another way of writing the rate of change along a curve is to take the change along the direction specified by the tangent to that curve; in the old-fashioned notation for 3-dimensional physics, this would simply have been written as $\vec{v} \cdot \nabla\Phi$. But in manifold theory we write the same content as

$$\tilde{v}[\Phi] \mid_{q^A(\tau)} = \frac{d}{d\tau}\Phi[q^A(\tau)] \quad , \tag{2.33$'$}$$

this being the definition of $\tilde{v}$!. But the right-hand side could be written, in even more detail, as

$$\Phi_{,q^A}\frac{dq^A}{d\tau} \mid_{q^B(\tau)} = \frac{dq^A}{d\tau}\frac{\partial}{\partial q^A}\Phi \mid_{q^B(\tau)} = \tilde{v}[\Phi] \mid_{q^B(\tau)} = v^A \frac{\partial}{\partial q^A}\Phi \mid_{q^B(\tau)} \quad . \tag{2.33$''$}$$

In English, this last equation simply says that *the A-component of $\tilde{v}$ is the rate of change, at some point, of the A-coordinate as we progress along the curve!*

Hurrying back to our jet bundle, with coordinates presented as in Eq. (2.31), and our symmetry vectors, as in Eq. (2.32), this implies, for $\tilde{v} = v^i\partial_{x^i} + v^{z^\alpha}\partial_{z^\alpha} + ...$, that

$$\begin{aligned} \frac{dx^i}{d\tau} &= v^i(x^j, z^\beta, z^\beta_j, ...) \quad , \\ \frac{dz^\alpha}{d\tau} &= v^\alpha(x^j, z^\beta, z^\beta_j, ...) \quad , \\ \frac{dz^\alpha_i}{d\tau} &= v^\alpha_i(x^j, z^\beta, z^\beta_j, ...) \quad . \end{aligned} \tag{2.34}$$

Integrating with respect to τ is the process we would use to determine the functions $X^i(x^j, z^\beta, z^\beta_j, ...; \tau)$, etc. We now proceed to do this for each of our symmetries for Burgers' equation.

Referring to the table of symmetries for Burgers' equation, at the end of the previous section, we choose to determine the actual congruence of curves determined by each symmetry vector, one vector at a time. We choose the 5 distinct vectors, labelled by distinct values of a_i, as a basis for any symmetry vector; therefore, we may look at them, in turn, by first choosing a particular a_j to have the value, say, $+1$, and all the others zero. We begin with a very simple one, namely the one determined by a non-zero value for a_4:

$$\begin{aligned} a_4 \quad &\Rightarrow \quad \tilde{v}_4 = -\partial_t = 0\partial_x + (-1)\partial_t + 0\partial_z + \dots \quad , \\ \Longrightarrow \quad &\frac{dx}{d\tau} = 0 = \frac{dz}{d\tau} \quad , \quad \frac{dt}{d\tau} = -1 \quad . \end{aligned} \tag{2.35}$$

These equations integrate rather easily to give

$$x' \equiv x(\tau) = x_0\,, \quad t' \equiv t(\tau) = t_0 - \tau\,, \quad z' \equiv z(\tau) = z_0\,, \tag{2.36a}$$

where the quantities $\{x_0, t_0, z_0\}$ are constants of integration arranged so that they specify the starting point of the curve, i.e., the values when $\tau = 0$. However, having performed the integrations, we should notice that this starting point is quite arbitrary; therefore, we may specify the curve by supposing that it begins with arbitrary choices of $\{x, t, z\}$, and draws a curve onward from there, according to the value of the parameter τ. Specifying the points on the manifold, $J^0(M, N)$, by the points $\{x, t, z\}$, and the flow along the manifold by G_τ, we may re-write Eqs. (2.36a) as

$$\begin{pmatrix} x' \\ t' \\ z' \end{pmatrix} \equiv G_{4\tau} \begin{pmatrix} x \\ t \\ z \end{pmatrix} = \begin{pmatrix} x \\ t - \tau \\ z \end{pmatrix} \quad . \tag{2.36b}$$

We must now specify the meaning of these mappings from the point of view of explicit solutions. For some specific choice of function $u(x, t)$, we say that $z = u(x, t)$ is a solution of our equation, namely Burgers' equation. Then the mapping tells us that $u'(x, t) = u(x, t)$ is also a solution. However, it is $u'(x', t')$ that we expect to satisfy Burgers' equation, in the primed variables! Therefore we re-write the equation relating u' and u, using the expressions for $x = x(x', t')$ and $t = t(x', t')$. In this particular case, this process is very simple; nonetheless, we will re-do this for each of our examples, as they continue to become more complicated. Therefore we re-write everything as follows:

$$\begin{aligned} &\text{given that } u(x,t) \text{ is a solution to } u_{xx} + u\,u_x + u_t = 0\,, \\ G_4(\tau): \qquad &\text{then } u'(x',t') \text{ is a solution to } u'_{x'x'} + u'\,u'_{x'} + u'_{t'} = 0 \\ &\text{where } \; u'(x',t') = u\{x(x',t'), t(x',t')\} = u(x', t' + \tau) \quad . \end{aligned} \tag{2.36c}$$

Surely this is in fact not too exciting, but sometimes useful, and it is the simplest one! Let's proceed onward.

For the symmetry labelled a_1, which means that $\tilde{v}_1 = -\partial_x$, we obviously have exactly the same sort of thing as the previous one:

$$\begin{aligned} x' &\equiv x(\tau) = x_0 - \tau\,, \\ t' &\equiv t(\tau) = t_0\,, \\ z' &\equiv z(\tau) = z_0\,, \end{aligned} \qquad \Longleftrightarrow \qquad \begin{pmatrix} x' \\ t' \\ z' \end{pmatrix} \equiv G_{1\tau} \begin{pmatrix} x \\ t \\ z \end{pmatrix} = \begin{pmatrix} x - \tau \\ t \\ z \end{pmatrix} \quad . \tag{2.37a}$$

Therefore, we may again state the result in the form

$$G_1(\tau): \qquad \begin{aligned} &\text{given that } u(x,t) \text{ is a solution to } u_{xx} + u\,u_x + u_t = 0\,, \\ &\text{then } u'(x',t') \text{ is a solution to the primed equation } u'_{x'x'} + u'\,u'_{x'} + u'_{t'} = 0 \\ &\text{where } \; u'(x',t') = u\{x(x',t'), t(x',t')\} = u(x'+\tau, t') \quad . \end{aligned} \tag{2.37b}$$

Continuing onward we next consider $a_0 \quad \Rightarrow \quad \tilde{v}_0 = -t\partial_x - \partial_z$, which generates the following differential equations for the curve it generates:

$$\frac{dx}{d\tau} = -t, \quad \frac{dt}{d\tau} = 0, \quad \frac{dz}{d\tau} = -1 \quad , \tag{2.38a}$$

which has a slightly less trivial solution than the previous ones:

$$\begin{aligned} x' &\equiv x(\tau) = x_0 - \tau t_0 \quad , \\ t' &\equiv t(\tau) = t_0 \quad , \\ z' &\equiv z(\tau) = z_0 - \tau \quad , \end{aligned} \qquad \Longleftrightarrow \qquad \begin{pmatrix} x' \\ t' \\ z' \end{pmatrix} = G_{0\tau} \begin{pmatrix} x \\ t \\ z \end{pmatrix} = \begin{pmatrix} x - \tau t \\ t \\ z - \tau \end{pmatrix} \quad . \tag{2.38b}$$

Therefore, we state the result in the form to describe a new solution:

$$G_0(\tau): \qquad \begin{aligned} &\text{given that } u(x,t) \text{ is a solution to } u_{xx} + u\,u_x + u_t = 0\,, \\ &\text{then } u'(x',t') \text{ is a solution to } u'_{x'x'} + u'\,u'_{x'} + u'_{t'} = 0 \\ &\text{where } u'(x',t') = u\{x(x',t'), t(x',t')\} - \tau = u(x'+\tau t', t') - \tau \quad . \end{aligned} \tag{2.38c}$$

We can see that this is actually the earlier solution as seen from a (Galilean) observer's reference frame moving with a constant velocity τ!

Now proceeding to $a_3 \quad \Rightarrow \quad -2\tilde{v}_3 = x\partial_x + 2t\partial_t - z\partial_z$, our ode's take the form:

$$\begin{aligned} &\frac{dx}{d\tau} = x\,, \; \frac{dt}{d\tau} = 2t\,, \; \frac{dz}{d\tau} = -z\,, \\ \Rightarrow \quad &x = x_0 e^{\tau}\,, \; t = t_0 e^{2\tau}\,, \; z = z_0 e^{-\tau} \quad . \end{aligned} \tag{2.39a}$$

For simplicity replacing e^τ by the symbol μ, we may re-write the above in the standard form we have been using:

$$\begin{aligned} x' &\equiv x(\tau) = \mu x_0\,, \\ t' &\equiv t(\tau) = \mu^2 t_0\,, \\ z' &\equiv z(\tau) = z_0/\mu\,, \end{aligned} \qquad \Longleftrightarrow \qquad \begin{pmatrix} x' \\ t' \\ z' \end{pmatrix} \equiv G_{3\tau} \begin{pmatrix} x \\ t \\ z \end{pmatrix} = \begin{pmatrix} \mu x \\ \mu^2 t \\ z/\mu \end{pmatrix} \tag{2.39b}$$

This new symmetry is rephrased as before:

$$G_3(\tau): \quad \begin{aligned} &\text{given that } u(x,t) \text{ is a solution to } u_{xx} + u\,u_x + u_t = 0\,, \\ &\text{then } u'(x',t') \text{ is a solution to } u'_{x'x'} + u'\,u'_{x'} + u'_{t'} = 0\,, \\ &\text{where } u'(x',t') = \frac{1}{\mu} u\{x(x',t'), t(x',t')\} = \frac{1}{\mu} u\left(\frac{x'}{\mu}, \frac{t'}{\mu^2}\right) \quad . \end{aligned} \tag{2.39c}$$

This one may be interesting to sketch the direct proof of this last statement. Beginning with the equation above describing $u'(x',t')$, we may calculate

$$\begin{aligned} &u'_{x'} = \frac{1}{\mu^2} u_x\,, \quad u'_{x'x'} = \frac{1}{\mu^3} u_{xx}\,, \quad u'_{t'} = \frac{1}{\mu^3} u_t \quad , \\ \Longrightarrow\ & u'_{x'x'} + u'\,u'_{x'} + u'_{t'} = \frac{1}{\mu^3}\{u_{xx} + u\,u_x + u_t\} = 0 \quad . \end{aligned}$$

For the last of our symmetries, we consider $a_2 \quad \Rightarrow \quad -2\tilde{v}_2 = tx\partial_x + t^2\partial_t - (tz - x)\partial_z$, which generates the equations to be solved:

$$\frac{dx}{d\tau} = tx\,, \quad \frac{dt}{d\tau} = t^2\,, \quad \frac{dz}{d\tau} = x - tz \quad . \tag{2.40a}$$

The solution of these equations begins with the one involving only t, with the solution

$$t = \frac{t_0}{1 - \tau t_0} \quad , \tag{2.40b}$$

which then permits the solutions of the rest of them:

$$\begin{aligned} \frac{dx}{x} &= \frac{t_0}{1 - \tau t_0} d\tau \quad \Rightarrow \quad x = \frac{x_0}{1 - \tau t_0} \quad , \\ \frac{dz}{x_0 - t_0 z} &= \frac{d\tau}{1 - \tau t_0} \quad \Rightarrow \quad z = (1 - \tau t_0) z_0 + \tau x_0 \quad , \end{aligned} \tag{2.40c}$$

which we re-phrase as

$$\begin{pmatrix} x' \\ t' \\ z' \end{pmatrix} \equiv G_{2\tau} \begin{pmatrix} x \\ x \\ z \end{pmatrix} = \begin{pmatrix} x/(1 - \tau t) \\ t/(1 - \tau t) \\ (1 - \tau t) z + \tau x \end{pmatrix} \quad , \tag{2.40d}$$

which is sufficiently involved that it is worthwhile giving the inverse transformation, obtained by solving for the un-primed independent variables in terms of the primed ones:

$$\begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} x'/(1+\tau t') \\ t'/(1+\tau t') \end{pmatrix} \quad . \tag{2.40d'}$$

It is perhaps worth noting that these inverted equations are exactly what one would have gotten by writing the original equations, Eqs. (2.40c), but changing the sign of τ and switching primed and un-primed variables, i.e., by going *backwards* along the curve. At any event, this allows the by-now usual phraseology:

$$\begin{aligned} &\text{if } u(x,t) \text{ is a solution to } u_{xx} + u\,u_x + u_t = 0 \\ G_2(\tau): \quad &\text{then } u'(x',t') \text{ is a solution to } u'_{x'x'} + u'\,u'_{x'} + u'_{t'} = 0 \quad \text{where} \\ u'(x',t) = [1-\tau t]\,u\{x(x',t'),t(x',t')\} + \tau x(x',t') &= \frac{1}{1+\tau t'}\,u\left(\frac{x'}{1+\tau t'}, \frac{t'}{1+\tau t'}\right) + \frac{\tau x'}{1+\tau t'} \quad , \end{aligned} \tag{2.40e}$$

which is surely **not** something one would have guessed! Again, we will sketch the details of the direct proof that this is a solution, using the symbol q to denote the ubiquitous multiplier $\{1+\tau t'\}^{-1}$:

$$\begin{gathered} u'_{x'} = q^2 u_x + q\tau\,, \;\; u'_{x'x'} = q^3 u_{xx}\,, \\ u'_{t'} = -q^2\tau u - q^3\tau x' u_x + q^2 u_t - q^3\tau t' u_t - q^2\tau^2 x' = -q^2\tau u - q^3\tau x' u_x + q^3 u_t - q^2\tau^2 x'\,, \end{gathered}$$

from whence the statement follows that $u'(x',t')$ satisfies Burgers' equation in the primed variables. As a not very complicated use of it, since a constant c is obviously a solution of Burgers' equation, this transformation allows us to begin with that choice, for u, and to generate the statement that

$$u'(x',t') = q(c+\tau x') \equiv \frac{c+\tau x'}{1+\tau t'} \quad \text{is a 2-parameter family of solutions!} \tag{2.40f}$$

Although I'm not going to do anything with it right now, one should also mention that we may do more than one of these transformations, with different parameters τ_1, , τ_2, etc.

C. Group-Invariant Solutions.

An entirely other way to generate solutions via symmetries is to use them to look for reductions in the number of independent variables. If $\tilde{v}$ is a symmetry of a pde, then $\tilde{v}(\zeta) = 0$ determines combinations of the original variables that are invariant under $\tilde{v}$. These then will be *those* solutions which are left unchanged by the symmetry maps generated by $\tilde{v}$—a subclass of all solutions. When dealing with 2 independent variables, just one symmetry is enough to reduce to an ordinary differential equation (ode), that may be easier to solve.

As an example, I go back to the rather "innocent-looking" scaling symmetry for Burgers' equation

$$-2\tilde{v}_3 = x\partial_x + 2t\partial_t - z\partial_z \quad . \tag{2.41}$$

Then the equation $\tilde{v}_3(\zeta) = 0$ has two independent solutions:

$$y \equiv x/\sqrt{t} \quad \text{and} \quad w \equiv \sqrt{t}z \quad . \tag{2.42}$$

which allows us to decide to look only for solutions of the form

$$u(x,t) = \frac{1}{\sqrt{t}} w(y = x/\sqrt{t}) \quad , \tag{2.43}$$

all of which will be invariant under $\tilde{v}_3$. Then ordinary algebra tells us that

$$\begin{aligned} u_x &= \frac{w'}{t} \quad , \\ u_{xx} &= \frac{w''}{t\sqrt{t}} u_t = -\frac{w}{2t\sqrt{t}} - \frac{xw'}{2t^2} \quad , \end{aligned} \tag{2.44}$$

where u' means $\frac{du}{dy}$. These equalities turn Burgers' equation into the ode:

$$u_{xx} + uu'_x + u_t = \frac{1}{t\sqrt{t}} \{w'' + ww' - \tfrac{1}{2}w - \tfrac{1}{2}yw'\} \quad . \tag{2.45}$$

Notice that the resulting ode has only $y's$ in it, with no extra $t's$, for example. This is because $\tilde{v}_3$ is a symmetry of the original pde!. Therefore solutions of

$$w'' + (w - \tfrac{1}{2}y)w' - \tfrac{1}{2}w = 0 \tag{2.45'}$$

will determine solutions of Burgers' equation for us. However, we now continue playing with this equation a little more time yet, setting

$$\begin{aligned} v &= w - \tfrac{1}{2}y \quad , \\ \Rightarrow \quad v' &= w' - \tfrac{1}{2}, \quad v'' = w'' \quad , \end{aligned}$$

which results in the new form of the equations:

$$\begin{aligned} v'' + vv' - \frac{1}{4}y = 0 &= (v' + \tfrac{1}{2}v^2)' - (\frac{1}{8}y^2)' \quad , \\ \Longrightarrow \quad v' + \tfrac{1}{2}v^2 &= 2C - \frac{1}{8}y^2 \quad , \end{aligned} \tag{2.46}$$

for some constant C. Since this is a Riccati equation, i.e., first-order and quadratic in the unknown variable, we lastly make the substitutions

$$
\begin{aligned}
v = 2\frac{\Psi'}{\Psi} \quad \Rightarrow \quad v' &= 2\frac{\Psi''}{\Psi} - 2(\frac{\Psi'}{\Psi})^2 \quad , \\
\Longrightarrow \quad \Psi'' &= (C - \frac{1}{16}y^2)\Psi \quad ,
\end{aligned} \tag{2.47}
$$

which is a standard, *linear* ode—sometimes ascribed to Weber—with solutions called parabolic cylinder functions. They depend on y and the value of C, and we will refer to any arbitrary solution of the equation by the symbol $Z_C(y) = \Psi$. Then, working backwards, we have that

$$
u = \frac{2}{\sqrt{t}} \left\{ \frac{Z'_C(y)}{Z_C(y)} + \frac{1}{4}y \right\} \Bigg|_{y=x/\sqrt{t}} \tag{2.48}
$$

is a solution of Burgers' equation, with $Z_C(y)$ an arbitrary linear combination of the 2 linearly independent parabolic cylinder functions!

D. Group Properties of the Symmetries.

It is worth noting that one symmetry map following another produces a composite map which also must surely be a symmetry map—since it would indeed map solutions into solutions. The general working out of what symmetry it might be, however, is quite complicated in general. It is therefore usual to do a great deal of the work for this, as previously, on the level of the infinitesimal generators $\tilde{v}$.

Since the various symmetry vector fields, $\tilde{v}_i$, are in fact vector fields, the appropiate operation on pairs of them, to give another one, is the Lie bracket. Therefore, maintaining our example from Burgers' equation we demonstrate all the commutators of those generating vectors, recalling that

$$
\begin{aligned}
[\tilde{v}_1, \tilde{v}_2] &\equiv [v_1^A \partial_{q^A}, v_2^B \partial_{q^B}] \\
&= (v_1^A v^B_{2,q^A} - v_2^A v^B_{1,q^A})\partial_{q^B} \quad .
\end{aligned} \tag{2.49}
$$

Then we find the following table

	$\tilde{v}_4$	$\tilde{v}_1$	$\tilde{v}_0$	$-2\tilde{v}_3$	$-2\tilde{v}_2$
$\tilde{v}_4$	0	0	$-\tilde{v}_1$	$2\tilde{v}_4$	$2\tilde{v}_3$
$\tilde{v}_1$		0	0	$\tilde{v}_1$	$\tilde{v}_0$
$\tilde{v}_0$			0	$-\tilde{v}_0$	0
$-2\tilde{v}_3$				0	$-4\tilde{v}_2$
$-2\tilde{v}_2$					0

where the entries in the table are the commutators of the labels for row and column, and, since $[\tilde{v}_i, \tilde{v}_j] = -[\tilde{v}_j, \tilde{v}_i]$, we only show the entries above the diagonal, the lower ones simply being the negatives of the corresponding upper ones. It is a little prettier if we re-label the vectors according to

$$\begin{gathered} w_0 \equiv -2v_3 \,,\ w_+ = v_0 \,,\ w_{++} = -2v_2 \,, \\ w_- = -v_1 \,, w_{--} = +v_4 \quad . \end{gathered} \tag{2.50}$$

Then, if we think of them as labeled by $\{w_\ell \mid \ell = +2, +1, 0, -1, -2\}$, where ℓ is some sort of a "charge," we may re-write the table in the form

$$\begin{gathered} [w_0\,, w_l] = l w_l \,, \\ [w_+, w_{++}] = [w_-, w_{--}] = 0 \,, \\ [w_+, w_{--}] = -w_- \,,\ \ [w_-, w_{++}] = -w_+ \,, \\ [w_-, w_+] = 0 \,,\ \ [w_{--}, w_{++}] = -w_0 \,. \end{gathered} \tag{2.51}$$

E. Prolongation of symmetry vectors.

So far we have insisted that our vectors $\tilde{v}$ preserve the first of the 1-forms in the contact module, θ^α. This gave us an n-tuple { for n dependent variables} of functions, that generated $\tilde{v}$. This n-tuple is usually referred to as the characteristic of $\tilde{v}$. Let us now learn how to extend $\tilde{v}$ to higher jets. I write

$$\tilde{v} = v^i \partial_i + v^\alpha \partial_{z^\alpha} + v^\alpha_i \partial_{z^\alpha_i} + ... + v^\alpha_{i...k} \partial_{z^\alpha_{i...k}} \quad . \tag{2.52}$$

In general, thinking back to the contact module, Ω^k, the last set of elements may be written as

$$\underset{\sim}{\theta}{}^\alpha_{i...j} \ \equiv\ dz^\alpha_{i...j} - z^\alpha_{i...jk} dx^k \quad , \tag{2.53}$$

so that the term that contracts this with $\tilde{v}$ has the form

$$\tilde{v} \rfloor \theta^\alpha_{i...j} = v^\alpha_{i...j} - z^\alpha_{i...jk} v^k \quad . \tag{2.54}$$

The statement that $\tilde{v}$ preserves the contact module, Ω^k, is simply that

$$\begin{gathered} \mathcal{L}_{\tilde{v}} \underset{\sim}{\theta}{}^\alpha \subset \Omega^k \ \text{ and }\ \forall\ j = 1, ..., k-1,\ \mathcal{L}_{\tilde{v}} \underset{\sim}{\theta}{}^\alpha_{a_1...a_j} \subset \Omega^k \quad , \\ \text{or }\ \mathcal{L}_{\tilde{v}} \underset{\sim}{\theta}{}^\alpha = 0 \quad (\text{mod } \Omega)^k \ \text{ and }\ \forall\ j = 1, ..., k-1,\ \mathcal{L}_{\tilde{v}} \underset{\sim}{\theta}{}^\alpha_{a_1...a_j} = 0 \quad (\text{mod } \Omega)^k \quad , \end{gathered} \tag{2.55}$$

Taking the forms given in Eq. (2.52-3) above, we compute

$$\begin{aligned} \tilde{v} \rfloor \underset{\sim}{\theta}{}^\alpha_{a_1...a_j} &= v^\alpha_{a_1...a_j} - z^\alpha_{a_1...a_j b} v^b \quad , \\ d(\tilde{v} \rfloor \underset{\sim}{\theta}{}^\alpha_{a_1...a_j}) &= dv^\alpha_{a_1...a_j} - z^\alpha_{a_1...a_j b} dv^b - v^b dz^\alpha_{a_1...a_j b} \quad , \\ \tilde{v} \rfloor d\underset{\sim}{\theta}{}^\alpha_{a_1...a_j} &= -v^\alpha_{a_1...a_j b} dx^b + v^b dz^\alpha_{a_1...a_j b} \quad , \\ \Rightarrow\ \ \mathcal{L}_{\tilde{v}} \underset{\sim}{\theta}{}^\alpha_{a_1...a_j} &= dv^\alpha_{a_1...a_j} - z^\alpha_{a_1...a_j b} dv^b - v^\alpha_{a_1...a_j b} dx^b \quad . \end{aligned} \tag{2.56}$$

However, since, for any function over the jet, $\Phi \in F(J^{k-1})$ we have the identity

$$d\Phi = (D^k_a \Phi) dx^a \qquad (\text{mod } \Omega^k) \quad , \tag{1.18'}$$

this may be re-written in the form

$$\mathcal{L}_{\tilde{v}} \underset{\sim}{\theta}^\alpha_{a_1 \ldots a_j} = \{D^k_b v^\alpha_{a_1 \ldots a_j} - z^\alpha_{a_1 \ldots a_j c} D_b v^c - v^\alpha_{a_1 \ldots a_j b}\} dx^b \quad \text{mod}\Omega^k \quad . \tag{2.56'}$$

Since these are all linearly independent, i.e., $dx^b \wedge \Omega^k \neq 0$ this gives us a **recursion relation** for the components of the symmetry generator, by virtue of requiring that it preserve the contact module:

$$v^\alpha_{a_1 \ldots a_j b} = D^k_b v^\alpha_{a_1 \ldots a_j} - z^\alpha_{a_1 \ldots a_j c} D^k_b v^c \;= D_b (v^\alpha_{a_1 \ldots a_j} - z^\alpha_{a_1 \ldots a_j c} v^c) + v^c z^\alpha_{a_1 \ldots a_j c\, b} \quad . \tag{2.57}$$

For the special case when $j = 0$ we recover the last line of Eq. (2.9):

$$v^\alpha_b = D_b (v^\alpha - z^\alpha_c v^c) + v^c z^\alpha_{b\, c} \quad . \tag{2.58}$$

To see that this indeed has the same appearance as in Eq. (2.9), we first recall that $v^\alpha - z^\alpha_c v^c \equiv F^\alpha$ is the characteristic of our symmetry field, and then re-write the Kronecker delta in an unusual way, generating, thereby, that form:

$$v^c \delta^\alpha_\beta = -F^\alpha_{,z^\beta_c} \quad , \tag{2.59}$$

1st prolongation: $$v^\alpha_b = D_b F^\alpha - z^\beta_{bc} F^\alpha_{,z^\beta_c} = D_b F^\alpha + v^c z^\alpha_{bc} \quad = \quad F^\alpha_{,b} + z^\beta_b F^\alpha_{,\beta} \, . \tag{2.60a}$$

Now, inserting $j = 1$ into our equations, and then $j = 2$, we find the next two prolongation forms:

2nd prolongation:
$$\begin{aligned} v^\alpha_{bc} &= D_c v^\alpha_b - z^\alpha_{bd} D_c v^d \\ &= D_c \{D_b F^\alpha + v^d z^\alpha_{bd}\} - z^\alpha_{bd} D_c v^d \\ &= D_c D_b F^\alpha + z^\alpha_{bdc} v^d \quad , \end{aligned} \tag{2.60b}$$

3rd prolongation:
$$\begin{aligned} v^\alpha_{bcd} &= D_d v^\alpha_{bc} - z^\alpha_{bce} D_d v^e \\ &= D_d D_c D_b F^\alpha + z^\alpha_{bcde} v^e \quad . \end{aligned} \tag{2.60c}$$

See the end of this section, at Eqs. (2.64), for an explicit presentation of the second prolongation.

Switching back and forth between the 2 forms of the equation, we clearly find a fairly nice general form for the prolongation

$$v^\alpha_{a_1...a_j} = D_{a_1}...D_{a_j}F^\alpha + z^\alpha_{a_1...a_j b}v^b \quad . \tag{2.61}$$

However, if we now introduce a so-called *multi-index symbol* σ to indicate any desired sequence of derivatives $\{a_1, ..., a_j\}$, including the option of no derivatives at all by giving σ the value 0, we may write

$$D_\sigma = D_{a_1}...D_{a_j} \quad , \tag{2.62}$$

and use this to re-phrase Eq. (2.61) more briefly:

$$v^\alpha_\sigma = D_\sigma F^\alpha + z^\alpha_{\sigma b}v^b \quad , \tag{2.61$'$}$$

as well as use this notation to take an opportunity to write down simply the entirety of the generating vector $\tilde{v}$:

$$\begin{aligned} \tilde{v} &= \sum_{\sigma=0}^{\infty}(D_\sigma F^\alpha)\partial_{z^\alpha_\sigma} + v^b\{\partial_b + z^\alpha_b\partial_\alpha + z^\alpha_{bc}\partial_{z^\alpha_c} + ...\} \\ &= \sum_{\sigma=0}^{\infty}(D_\sigma F^\alpha)\partial_{z^\alpha_\sigma} + v^b D_b \quad . \end{aligned} \tag{2.63}$$

Remembering Eq. (2.59), namely $v^b\delta^\alpha_\beta = -F^\alpha_{,z^\beta_b}$, we see that there is a 1-1 correspondence between $\tilde{v}$ and its characteristic n-tuple, $F^\alpha \in F(J^k)$, the structure of a symmetry vector being highly constrained by the (necessary) requirement that it preserve sections.

As a continuation of the explicit presentations of a symmetry vector field in terms of its determining characteristic, F, already begun at Eqs. (2.11), I also want to record the explicit forms of the 2nd prolongation of $\tilde{v}$, to J^2. We utilize either of the various, equivalent forms given at Eqs. (2.60b) or Eqs. (2.61$'$). We also explicitly put in the assumption that the characteristic is a function defined only over J^1, so that $F = F(x, t, z, p, q)$. This assumption is true for all Lie symmetries, but no longer valid when one is hunting for "higher symmetries," defined on J^∞, as will be defined in section 4:

$$\begin{aligned} v^r \equiv v^{z_{xx}} &= (D_x)^2 F - z_{xxx}F_{,p} - z_{xxt}F_{,q} \\ &= F_{,xx} + 2\,p\,F_{,xz} + 2\,r\,F_{,xp} + 2\,s\,F_{,xq} + p^2F_{,zz} + 2\,pr\,F_{,zp} + 2\,ps\,F_{,zq} \\ &\qquad + r^2F_{,pp} + 2\,rs\,F_{,pq} + q^2F_{,qq} + r\,F_{,z} \\ v^s \equiv v^{z_{xt}} &= D_tD_xF - z_{xxt}F_{,p} - z_{xtt}F_{,q} = D_xD_tF - z_{xtx}F_{,p} - z_{ttx}F_{,q} \\ &= F_{,xt} + p\,F_{,tz} + q\,F_{,xz} + r\,F_{,tp} + s(F_{,tq} + F_{,xp} + F_{,z}) + w\,F_{,xq} + pq\,F_{,zz} \\ &\qquad + (ps + rq)F_{,zp} + (pw + sq)F_{,zq} + rs\,F_{,pp} + (rw + s^2)F_{,pq} + sw\,F_{,qq}\,, \\ v^w \equiv v^{z_{tt}} &= (D_t)^2F - z_{xtt}F_{,p} - z_{ttt}F_{,q} \\ &= F_{,tt} + 2\,q\,F_{,yz} + 2\,s\,F_{,yp} + 2\,w\,F_{,yq} + w\,F_{,z} \\ &\qquad + q^2F_{,zz} + 2\,qs\,F_{,zp} + 2\,qw\,F_{,zq} + s^2F_{,pp} + 2\,sw\,F_{,pq} + w^2F_{,qq}\,. \end{aligned} \tag{2.64}$$

F. Prolongation of pde's.

Granted the desire to incorporate into our symmetry vectors a dependence on higher derivatives, we must think about the k-th order pde on J^l for $l \geq k$. On J^k, the pde is a surface given by the vanishing of the pde itself, thought of as an algebraic constraint on the coordinates of J^k. One creates J^l for $l > k$ by simply appending more variables among the coordinates in an obvious way. However, it is not true that any values of those new variables–the higher derivatives than k–are allowed, for them to have membership in the surface. More specifically, a solution $u = u(x,t)$ of our equation would of course have all the points of $(j^k u)(x,t)$ lying on this surface, but the additional degrees of freedom in $(j^{k+1}u)(x,t)$ would have additional relations between them. In the way of an example, we expect that if u is such that Burgers' equations is satisfied, i.e., $u_{xx} + uu_x + u_t = 0$, then I expect that $u_{xxx} + uu_x + u_x^2 + u_{xt} = 0$, as well, thus constraining all allowed values of the additional coordinate, u_{xxx}, in $J^{3=2+1}$. Therefore, if a k-th order pde(s) is specified, on J^k, by some equation(s)

$$F^A(x^i, z^\alpha, z^\alpha_\sigma) = 0, \ |\, \sigma \,| \leq k \quad , \tag{2.65a}$$

then on J^{k+l}, it is specified by the set

$$\{D_\sigma F^A = 0 \ | \ 0 \leq |\, \sigma \,| \leq l\} \quad , \tag{2.65b}$$

where by $|\, \sigma \,| = j$, we mean the number of elements in the multi-index, detemined by writing out $\sigma = \{(i_1, ..., i_j)\}$. This additional set of equations then determines, in J^{k+l}, a surface of the same dimension as it was originally in J^k; inside $J^{k+\ell}$, the surface so determined is called the ℓ-th prolongation of the original pde.

G. Evolutionary Symmetry Vectors.

If we view our pde on at least J^{k+1}, we see that $\forall \ i, D_i$ is a symmetry of the pde since $D_i F^A$ is already "a part" of (the prolongation of) the pde. However, of course this is a valid statement about **any** pde, and therefore not very interesting! Such symmetries are called trivial and we will try to ignore trivial symmetries, which means that we will consider equivalent any two symmetries if their difference is a linear combination of trivial ones. Therefore, the general section-preserving symmetry, in Eq. (2.63), is equivalent to the one without the extra term, $v^i D_i$!

Symmetries for which $v^i \equiv 0$ are referred to as evolutionary symmetries and we may see that every (section-preserving) symmetry is equivalent to an evolutionary one, written simply as

$$\tilde{v} = \sum_\sigma (D_\sigma F^\alpha) \partial_{z^\alpha_\sigma} \equiv \mathfrak{Z}_F \quad , \tag{2.66}$$

where the notation with the Russian letter, З, is intended to emphasize that this is an evolutionary symmetry. If one wants to return to the equivalent symmetry where $v^i \neq 0$, since we of course have the characteristic F^α, we may calculate from $v^i\delta^\alpha_\beta = -F^\alpha_{,z^\beta_i}$ and proceed!

On the other hand, it should definitely be noted that $З_F$ usually has infinitely many components and is therefore not as nice to work with, even though the formula looks simpler. For instance, for a simple translation symmetry in the independent variable x, we would have

$$\tilde{v} = -\partial_x \ \Rightarrow F = z_x \, , \Longrightarrow З_{z_x} = z_x\partial_z + z_{xx}\partial_{z_x} + z_{xt}\partial_{z_t} + \dots \ = \ D_x - \partial_x \quad .$$

We resolve most of this difficulty by agreeing to talk about evolutionary symmetries principally in terms of their characteristics, which has the advantages to be described now. In order to do this we need two things. We need a method of determining the commutators via the characteristics, and, even more importantly, we need a method of determining **directly** the appropriate characteristics for a given pde—the characteristics of its symmetries. I will first deal with the easier problem: if $З_\eta$ and $З_\psi$ are two evolutionary symmetries, then their commutator is

$$[З_\eta, З_\psi] = \left[\sum_\sigma (D_\sigma\eta)\partial_{z_\sigma}, \sum_\tau (D_\tau\psi)\partial_{z_\tau}\right] \quad . \tag{2.67}$$

This commutator should of course again be an evolutionary vector field for some characteristic χ. I state without proof that that result is

$$[З_\eta, З_\psi] = З_\chi \quad , \tag{2.68a}$$

where χ is defined by

$$\begin{aligned} \chi^\alpha &\equiv \{\eta, \psi\}^\alpha \equiv З_\eta(\psi^\alpha) - З_\psi(\eta^\alpha) \\ &= \sum_\sigma \left\{ D_\sigma(\eta^\beta)\psi^\alpha_{,z^\beta_\sigma} - D_\sigma(\psi^\beta)\eta^\alpha_{,z^\beta_\sigma} \right\} \quad . \end{aligned} \tag{2.68b}$$

This gives a reasonably simple map between all evolutionary vector fields and the space of n-tuples of functions that maps the Lie bracket of the fields into this Poisson-like bracket of the n-tuples of functions.

H. The Korteweg-deVries equations.

This equation, while very similar to Burgers' equation in functional form, is in fact very different in its physical behavior:

$$u_{xxx} + uu_x + u_t = 0 \quad . \tag{2.69}$$

Since this is a 3rd order equation we may write its appropriate contact ideal as

$$\begin{aligned} \underset{\sim}{\theta} &= dz - pdx - qdt \quad , \\ \underset{\sim}{\theta}_x &= dp - rdx - sdt \quad , \\ \underset{\sim}{\theta}_t &= dq - sdx - vdt \quad , \end{aligned} \tag{2.70a}$$

along with a 2-form that describes the equation

$$\underset{\sim}{\beta} = (dr + zdz) \wedge dt + dx \wedge dz \quad . \tag{2.70b}$$

Since $d\underset{\sim}{\beta}$ is already 0, in principle we must only worry about appending $d\underset{\sim}{\theta}_x$ and $d\underset{\sim}{\theta}_t$ to our set of generators. We then look for $\tilde{v}$, on J^2 since only Ω^2 is needed, such that

$$\mathcal{L}_{\tilde{v}}\underset{\sim}{\beta} = \lambda\underset{\sim}{\beta} + \mu d\underset{\sim}{\theta} + \kappa d\underset{\sim}{\theta}_x + \zeta d\underset{\sim}{\theta}_t + \underset{\sim}{\eta} \wedge \underset{\sim}{\theta} + \underset{\sim}{\rho} \wedge \underset{\sim}{\theta}_x + \underset{\sim}{\xi} \wedge \underset{\sim}{\theta}_t \quad . \tag{2.71}$$

We can actually rather quickly eliminate most of these terms on the right hand side, but will not do so here, but simply list the results. There are 4 independent Lie symmetry vectors:

F_i	$\tilde{v}_i$	G_{τ^i}
z_x	$-\partial_x$	$u(x-\tau,t)$
$z_{xxx} + zz_x$	∂_t	$u(x,t+\tau)$
$1 - tz_x$	$t\partial_x + \partial_z$	$u(x+\tau t,t)+\tau$
$-2z - xz_x - 3tz_t$	$x\partial_x + 3t\partial_t - 2z\partial_z$	$u(\lambda x, \lambda^3 t)/\lambda^2, \qquad \lambda = e^\tau$

As an example of looking for group-invariant solutions, I select $\tilde{v}_3$—which we recognize as dealing with traveling waves. This means that I look for solutions of the form

$$\begin{aligned} u &= w(x-ct) = w(y) \\ &\Rightarrow \quad w''' + ww' - cw' = 0 \quad . \end{aligned} \tag{2.72a}$$

Integrating this equation once is straightforward, resulting in

$$w'' + \tfrac{1}{2}w^2 - cw + k = 0 \quad . \tag{2.72b}$$

By multiplicating by $2w'$ and integrating once again, we reduce the equation to first-order:

$$w'^2 + \frac{1}{3}w^3 - cw^2 + 2kw + \ell = 0 \quad , \tag{2.72c}$$

where c is the speed and k and ℓ are constants of integration. This equation then has been "reduced to quadratures," and we may write the question for the determination of its solution in terms of an integral:

$$\frac{dw}{\sqrt{-\frac{1}{3}w^3 + cw^2 - 2kw - \ell}} = \pm dy \quad , \tag{2.72d}$$

the solution to which is an elliptic function:

$$u(x,t) = \hat{\mathcal{P}}(x - ct + m) \quad , \tag{2.72e}$$

where $\hat{\mathcal{P}}$ stands for some elliptic function, in the most general case, depending on 4 constants of integration!.

In the special case where $k = \ell = 0$, the elliptic function simplifies to a hyperbolic function, as we show directly by writing, choosing the correct sign:

$$\frac{\frac{dw}{w}}{\sqrt{c - \frac{1}{3}w}} = -dy \quad . \tag{2.73}$$

We know that the integral of this involves hyperbolic functions. The answer may be given as

$$w = A\,\mathrm{sech}^2(ky) \quad , \tag{2.74}$$

with the proof being generated as follows:

$$w' = -2kA\,\mathrm{sech}^2(ky)\ \tanh(ky) = -2kw\sqrt{1 - w/A}$$
$$\Longrightarrow \quad \frac{\frac{dw}{w}}{\sqrt{1 - w/A}} = -2kdy \quad ,$$

where we now have $A = 3c, k = \frac{1}{2}\sqrt{c}$, or

$$u(x,t) = 3c\ \mathrm{sech}^2\left[\tfrac{1}{2}\sqrt{c}(x - ct) + \delta\right] \quad , \tag{2.74$'$}$$

which is the standard, famous **one-soliton solution**.

We can also look very briefly at the symmetries as a Lie algebra

	$\tilde{v}_1$	$\tilde{v}_2$	$\tilde{v}_3$	$\tilde{v}_4$
$\tilde{v}_1$	0	0	0	$\tilde{v}_1$
$\tilde{v}_2$		0	$-\tilde{v}_1$	$3\tilde{v}_2$
$\tilde{v}_3$			0	$-2\tilde{v}_3$
$\tilde{v}_4$				0

It is, hopefully, disappointing to you that the symmetry generation techniques here don't do too much good. The reason is that these techniques have been known for 100 years—discovered mostly by Lie. There are indeed equations for which they do lots of good—simpler ones than we have been considering, although we have also seen some interesting features from Burgers' equation. Nonetheless, if the reasons the KdV equation is famous could have been discovered by these methods, they would have been discovered a very long time ago. Instead, methods involving symmetries with higher derivatives have only been known for about 30 years! We proceed now in that direction.

III. Higher Symmetries on J^∞.

A. The Universal Linearization Operator.

Before defining this operator, let me say that it should seem to you that I have been somewhat remiss in not telling you how to calculate symmetries directly on the jet bundle. I now do so, but only, then, to suggest a better way to do it –via evolutionary operators. Since a k-th order pde is a set, F^A, of functions on J^k, and since vector fields, $\tilde{v}$, map functions into functions, it is straightforward to calculate directly on the jet bundle, without the necessity of involving differential forms at all, although the calculations are usually considerably more tedious:

$$\mathcal{L}_{\tilde{v}} F^A = \tilde{v}(F^A) = v^i F_{,i} + v^\alpha F_{,\alpha} + \ldots \quad , \tag{3.1}$$

and use that definition to attempt to satisfy the requirement for a symmetry, namely that

$$\tilde{v}^{(k)}(F^A) = \lambda^A_B F_B \quad , \tag{3.2}$$

just as it was with forms. The superscript (k) on $\tilde{v}$ is there to remind you that it is important that this $\tilde{v}$ be defined over J^k, i.e., include all terms up through the highest on J^k, namely $v^\alpha_{a_1 \ldots a_k} \partial_{z^\alpha_{a_1 \ldots a_k}}$, where, of course, this higher-order coefficients in $\tilde{v}$ must be determined by the formulas at Eq. (2.61′).

The so-called (universal) linearization operator on J^∞ is an alternative, and simpler, way of determining characteristics for symmetry generators on J^∞. I will restrict myself to dealing with just a single pde—i.e., for a single dependent variable—since the operator seems a good deal more complicated when one has a system. Therefore, we presume we have given some pde, F, on J^k, and that we want to determine whether or not there exist non-trivial symmetries on its prolongation to J^∞. These may be found by first determining their characteristics, $Q = Q(x^i, z_\sigma)$, for any finite value of $\mid \sigma \mid$, by solving the equation

$$\begin{aligned} \bar{L}_F(Q) &= 0 \quad , \\ \text{where} \quad L_F(Q) &\equiv \mathfrak{Z}_Q(F) \quad , \end{aligned} \tag{3.3}$$

and the over-bar means the operator must be restricted to the "surface" Y^∞.

B. Application to Burgers' equation.

The first step in performing such a calculation is a determination of a reasonable choice of coordinates on Y^∞, which should be generated by writing the complete (infinite) sequence of prolongation equations, for the pde. For Burgers' equation, this sequence begins with

$$\begin{aligned} z_t &= -z_{xx} - zz_x \quad , \\ D_x z_t &= -z_{xxx} - zz_{xx} - z_x^2 = z_{xt} \quad , \\ D_x^2 z_t &= -z_{xxxx} - zz_{xxx} - 3z_x z_{xx} = z_{xxt} \quad . \end{aligned} \tag{3.4}$$

This sequence allows us to define the variables $\{z_t, z_{xt}, z_{tt}, \ldots\}$ in J^∞in terms of the following set of variables that we may allow to vary on $\leq infty$: $\{x, t, z, z_x, z_{xx}, z_{xxx}, ...\}$. We will often write z_j (or $z_{(j)}$) to mean $z_{x...x}$, where there are j x's. With this notation we find the restricted total derivatives to be

$$\begin{aligned} \bar{D}_x &= \partial_x + \sum_{j=0}^{\infty} z_{j+1}\partial_{z_{(j)}} \quad , \\ \bar{D}_t &= \partial_t - \sum_{j=0}^{\infty} \{(\bar{D}_x ... \bar{D}_x)(z_{xx} + zz_x)\}\partial_{z_{(j)}} \quad , \end{aligned} \tag{3.5}$$

where there are j copies of $\bar{D}_x$ in the j-th term in the series that describes $\bar{D}_t$.

By the linearization theorem, we want those functions Q such that $\bar{L}_F(Q) = 0$, where, first, we may write

$$\begin{aligned} L_F(Q) = \mathfrak{Z}_Q(F) &= \sum_\sigma \{D_\sigma(Q)\}F_{,z_\sigma} \\ &= \sum_\sigma \{D_\sigma(Q)\}\partial_{z_\sigma}(z_{xx} + zz_x + z_t) \\ &= D_{xx}(Q) + zD_x(Q) + z_x Q + D_t(Q) \\ &= \{D_x^2 + zD_x + z_x + D_t\}Q. \end{aligned} \tag{3.6}$$

Restricting this to Y^∞, we have, finally the equation to be solved:

$$\{\bar{D}_x^2 + z\bar{D}_x + z_x + \bar{D}_t\}Q(x, t, z, z_x, ..., z_{(l)}) = 0 \quad , \tag{3.7}$$

where the integer l is currently arbitrary. Restriction of l to 2 would retrieve the Lie symmetries already found. Any $l > 2$ has a chance of finding more. In order to see how this chance works, I want to retreat just a little bit and generalize Burgers' equation to

$$z_{xx} + f(z)z_x + z_t = 0 \quad , \tag{3.8}$$

whee we will currently maintain f as an arbitrary smooth function of z. Doing this will arrange it that we get non-trivial higher symmetries only for a small class of functions f, and otherwise nothing beyond the Lie symmetries. Then Eq. (3.7) is generalized to

$$\{\bar{D}_x^2 + f(z)\bar{D}_x + f'(z)z_x + \bar{D}_t\}Q(x,t,z,z_x,...,z_{(l)}) = 0 \quad . \tag{3.9}$$

We may now go ahead with the calculation to determine the solutions of the equation in Eq. (3.9). However, as a brief aside, I will note that the differential operator $\bar{D}_i$, as defined, is truly the pull-back of D_i to the subspace defined by the pde, according to the rules of advanced calculus for performing such computations. In Appendix I the details are presented for the calculation for $\bar{D}_x$ for Burgers' equation.

$$\begin{aligned} \bar{D}_x Q \; &= Q_{,x} + z_x Q_{,z} + z_{xx} Q_{,z_x} + ... + z_{(l+1)} Q_{,z_{(l)}} \\ &= Q_{,x} + \sum_{j=0}^{l} z_{(j+1)} Q_{,z_{(j)}} \quad , \\ \Rightarrow \; \bar{D}_x^2 Q = \bar{D}_x(\bar{D}_x Q) &= Q_{,xx} + 2\sum_{j=0}^{l} z_{(j+1)} Q_{,xz_{(j)}} \\ &\quad + \sum_{i=0}^{l} z_{(i+1)} \sum_{j=0}^{l} Q_{,z_{(j)}z_{(i)}} z_{(j+1)} + \sum_{j=0}^{l} z_{(j+2)} Q_{,z_{(j)}} \quad , \end{aligned} \tag{3.10}$$

$$\bar{D}_t Q = Q_{,t} - \sum_{j=0}^{l} \{\bar{D}_x^j (z_{xx} + f(z)z_x)\} Q_{,z_{(j)}} \quad .$$

So our equation reads

$$\begin{aligned} &Q_{,xx} + Q_{,t} + f(z)Q_{,x} + f'(z)z_x Q + 2\sum_{j=0}^{l} z_{(j+1)} Q_{,xz_{(j)}} \\ &+2\sum_{i<j}^{l}\sum_{=1}^{l} z_{(i+1)} z_{(j+1)} Q_{,z_{(i)}z_{(j)}} + \sum_{j=0}^{l} z_{(j+1)}^2 Q_{,z_{(j)}z_{(j)}} \\ &+\sum_{j=0}^{l} \left\{ z_{(j+2)} + f(z)z_{(j+1)} - \bar{D}_x^j (z_{xx} + f(z)z_x) \right\} Q_{,z_{(j)}} = 0 \quad . \end{aligned} \tag{3.11}$$

I first look at this last sum, where the quantity inside the brace is

$$f(z)z_{(j+1)} - (\bar{D}_x)^j(f(z)z_x) \quad . \tag{3.12}$$

It requires some little effort to work out $(\bar{D}_x)^j(f(z)z_x)$, actually. However, the most important terms will be the ones with the highest quantities, $z_{(i)}$, so I will write out carefully the first few. We have

$$(\bar{D}_x)^j(f(z)z_x) = f(z)z_{(j+1)} + (j+1)f'(z)z_x z_{(j)} + \frac{j(j-1)}{2}f'(z)z_{xx}z_{(j-1)} + \ldots$$

so that the quantity in Eq. (3.12) looks like

$$-\{(j+1)f'(z)z_x z_{(j)} + \frac{j(j-1)}{2}f'(z)z_{xx}z_{(j-1)} + \ldots\} \quad . \tag{3.13}$$

Since Q depends only on $z_{(j)}$ up to $z_{(l)}$, we can see that the equation in question is a quadratic polynomial in $z_{(l+1)}$, and therefore we must set the coefficients separately to zero:

$$z^2_{(l+1)} : \quad Q_{,z_{(l)}z_{(l)}} = 0 \quad , \tag{3.14}$$

which implies the existence of $\{A, B\}$, functions that depend on the restricted jet variables only up to and including $z_{(l-1)}$, such that

$$Q = Az_{(l)} + B \quad . \tag{3.15a}$$

Next we set the coefficient of $z^1_{(l+1)}$ to zero:

$$2\bar{D}_x A = 2\{A_{,x} + \sum_{i=1}^{l-1} z_{(i+1)}A_{,z_{(i)}}\} = 0$$

from which we deduce that $A = A(t)$, only. Now, since $B_{,z_{(l)}} = 0$, we have a quadratic polynomial in $z_{(l)}$. Again, comparing coefficients gives us the following:

$$\begin{aligned} z^2_{(l)} &: B_{,z_{(l-1)}z_{(l-1)}} = 0 \\ &\Rightarrow B = Cz_{(l-1)} + D \end{aligned} \quad , \tag{3.15b}$$

where C, D depend on jet variables only through $z_{(l-2)}$. The next lowest powers of $z_{(l)}$ then require that

$$\begin{aligned} z^1_{(l)} \ &: 2\bar{D}_x C + f'(z)Az_x - A' - (l+1)f'(z)z_x A \\ &= 2(C_{,x} + z_x C_{,z} + z_{(2)}C_{,z_x} + \ldots + z_{(l-1)}C_{,z_{(l-2)}}) - A' - lAf'(z)z_x = 0 \end{aligned} \quad , \tag{3.16}$$

which implies the existence of $C = C(x,t,z)$, only, and that

$$2C_{,z} = lAf'(z)\,, \quad 2C_{,x} = A' \quad , \\ \Rightarrow \quad \exists\, \alpha(t) \text{ such that } C = \frac{1}{2}\{lAf(z) + A'x + \alpha\} \quad . \tag{3.17}$$

This sort of procedure clearly continues. For coefficients of $z^2_{(l-1)}$, we then find that

$$D_{,z_{(l-2)}z_{(l-2)}} = 0\,, \quad \Longrightarrow D = Ez_{(l-2)} + F \quad , \tag{3.18}$$

and then the coefficients of $z^1_{(l-1)}$ require that

$$2\bar{D}_x E - \tfrac{1}{2}(l-1)A'f(z) - \frac{1}{2}A''x - \tfrac{1}{2}\alpha' - \tfrac{1}{2}l(l+1)A\{f''z_x^2 - lf'Cz_x + f'Cz_x + f'z_{xx}\} = 0 \quad , \tag{3.19}$$

which implies the existence of $E = E(x,t,z,z_x)$, only. But then the above is a polynomial in z_{xx} and the coefficient of z_{xx} requires that

$$2E_{,z_x} = \tfrac{1}{2}l(l+1)Af'(z) \quad \Rightarrow \quad E = \frac{1}{4}l(l+1)Af'(z)z_x + G(x,t,z) \quad . \tag{3.20}$$

Inserting this back into our equation gives

$$2(G_{,x} + z_xG_{,z}) - \tfrac{1}{2}(l-1)A'f(z) - \tfrac{1}{2}A''x - \frac{1}{2}\alpha' = \tfrac{1}{2}l(l+1)A\{f''z_x^2 - (l-1)f'Cz_x\} \quad . \tag{3.21}$$

Unfortunately Eq. (3.21) is a quadratic polynomial in z_x, and the coefficient of z_x^2, which must vanish, is just

$$A(t)f''(z) \text{ which we must require to be } \quad 0 \quad . \tag{3.22}$$

We have two choices. The first is that $f''(z) = 0$, or $f = \alpha_0 z + \beta_0$, in which case we may then continue on, looking to fully determine Q, but we see that the generalization to general functions f was more or less an irrelevancy. The other choice is that $A \equiv 0$. However, A was the coefficient of the leading term. If A is zero, we simply change l and start over again, and again find $A \equiv 0$.

Therefore, unless $f'' = 0$, there are no higher symmetries.

On the other hand, the apparent constants α_0, β_0, in f are totally illusory. If we are given $u_{xx} + (\alpha_0 u + \beta_0)u_x + u_t = 0$ as an equation to solve, we set $v \equiv \alpha_0 u + \beta_0, v_x = \alpha_0 u_x$, etc. and find that we are back to the original Burgers' equation, namely, $v_{xx} + vv_x + v_t = 0$! Therefore, we find that definitely not all nonlinear pde's have higher symmetries.

From here on out, we obviously take $f(z) = z$, and can continue looking for symmetries. Probably you wonder how to finish, since we are starting at arbitrary l and going down! We are already halfway through computing E. If one goes a little bit further, it is found that

$$\begin{aligned} Q_{(l)} =& az_{(l)} + \tfrac{1}{2}\{lAz + A'x + \gamma(t)\}z_{(l-1)} \\ &+ \frac{1}{4}\{l(l+1)Az_x + \tfrac{1}{2}l(l-1)Az^2 + (l-1)A'x + \frac{1}{2}A''x^2 \\ &+ (l-1)\gamma z + \gamma' x + \zeta(t)\}z_{(l-2)} + \dots \quad . \end{aligned} \tag{3.23}$$

The next step is to calculate the commutator (Poisson bracket) of two such $Q'_{(l)}s$, and use it and a knowledge of the lowest-order higher symmetries to calculate everything. Notice, of course, that we obtain the Lie symmetries simply by setting $l = 2$ and finishing the process.

Writing another such symmetry as $T_{(m)}$, we have

$$T_{(m)} = Bz_{(m)} + \tfrac{1}{2}\{mBz + B'x + \delta(t)\}z_{(m-1)} + \dots \qquad , \tag{3.24}$$

from which it is straightforward to calculate

$$\{Q_{(l)}, T_{(m)}\} = \tfrac{1}{2}(mBA' - lAB')z_{(l+m-2)} + O(l+m-3) \quad . \tag{3.25}$$

Now, a trick to determine the properties of A is as follows. Note that $Q_{(1)} = z_x$ is clearly a solution to this problem –this simply generates the symmetry vector, $-\partial_x$. Therefore z_x satisfies the current equations, with $A = 1$, so one may induce a recursion scheme:

$$\begin{aligned} \{z_x, Q_{(l)}\} &= -\tfrac{1}{2}A'z_{(l-1)} + \dots \qquad , \\ \{z_x, \{z_x, Q_{(l)}\}\} &= (-\tfrac{1}{2})^2 A'' z_{(l-2)} + \dots \quad , \\ &\dots \\ \{z_x, ..., \{z_x, Q_{(l)}\}...\} &= (-\tfrac{1}{2})^{l-1} A^{(l-1)} z_1 \quad . \end{aligned} \tag{3.26}$$

However, we already know all the classical symmetries, and therefore all symmetries that involve no higher derivative than z_x. There are two of them, z_x and $tz_x + 1$. Therefore, knowing that the bracket of any 2 characteristics is either another characteristic or zero, we determine that $A^{(l-1)}$ is a linear polynomial in t, from which we infer that A is an l-th order polynomial in t! This allows us to choose a basis in the linear space of characteristic functions as

$$\{\Psi^i_l \equiv t^i z_{(l)} + O(l-1) \mid \ l = 1, 2, 3, ...; i = 0, 1 \dots, l\} \quad , \tag{3.27}$$

and we see that

$$\{\Psi^i_l, \Psi^j_m\} = \tfrac{1}{2}(mi - lj)\Psi^{i+j-1}_{l+m-2},$$

except for the single lowest order explicit calculation that shows that $\Psi_0^0 = \{\Psi_1^0, \Psi_1^1\} \equiv 0$!

The five classical symmetries found earlier, remembering that in the present notation $p = z_x = z_{(1)}$ and $q = z_t = -z_{(2)} - zz_{(1)}$, have the form

$$\begin{aligned}
\tilde{w}_0 &\leftrightarrow \Psi_2^1 = t(z_{xx} + zz_x) - \tfrac{1}{2}(xz_x + z) \quad , \\
\tilde{w}_{++} &\leftrightarrow \Psi_2^2 = t^2(z_{xx} + zz_x) - t(xz_x + z) - x \quad , \\
\tilde{w}_{--} &\leftrightarrow \Psi_2^0 = z_{xx} + zz_x \quad , \\
\tilde{w}_{+} &\leftrightarrow \Psi_1^1 = tz_x + 1 \quad , \\
\tilde{w}_{-} &\leftrightarrow \Psi_1^0 = z_x. \quad .
\end{aligned} \tag{3.28}$$

With the newly derived commutator for the Ψ_k^i, we see immediately that the set $\{\Psi_2^i \mid i = 0, 1, 2\}$ form a closed subalgebra, in agreement with our previous results, as well as the fact that the set $\{\Psi_k^i \mid \ k = 1, 2, \ i = 0, \dots k\}$ also closes on itself. However

$$\{\Psi_3^i, \Psi_k^j\} \propto \ \Psi_{k+1}^{i+j-1} \quad . \tag{3.29}$$

This says that the quantities Ψ_3^i act like "raising operators" to increase the value of the index k. In fact, given Ψ_3^0, we would be able to use it explicitly as a recursion operator to create new objects:

$$\{\Psi_3^0, \Psi_2^2\} = -3\Psi_3^1\,; \quad \{\Psi_3^1, \Psi_2^2\} = -2\Psi_3^2\,; \quad \{\Psi_3^2, \Psi_2^2\} = -\Psi_3^3 \quad , \tag{3.30}$$

to obtain all the rest of the Ψ_3^i—note that $\{\Psi_3^3, \Psi_2^2\} = 0$, so that it doesn't go any further! Then, however, we could go to the next level via

$$\{\Psi_3^0, \Psi_3^1\} = -\tfrac{3}{2}\Psi_4^0 \quad , \tag{3.31}$$

etc. Therefore the next thing to do in the process of finding all the symmetry generators is to explicitly calculate Ψ_3^0, which we already know is of the form $z_{(3)} + ...$, after which all the rest of the infinite set may be calculated as needed! Some simpler examples are

$$\begin{aligned}
\Psi_3^0 &= z_{xxx} + \tfrac{3}{2}zz_{xx} + \tfrac{3}{2}z_x^2 + \tfrac{3}{4}z^2 z_x \,, \\
\Psi^1{}_3 &= tz_{xxx} + \tfrac{1}{2}(3tz - x)z_{xx} + \tfrac{1}{2}(\tfrac{3}{2}tz^2 - xz)z_x + \tfrac{3}{2}t(z_x)^2 - \tfrac{1}{4}z^2 \,, \\
\Psi^2{}_3 &= t^2 z_{xxx} + \tfrac{1}{2}t(3tz - 2x)z_{xx} + \tfrac{3}{2}t^2(z_x)^2 + \tfrac{1}{2}(\tfrac{3}{2}t^2z^2 - 2txz + \tfrac{1}{2}x^2)z_x - \tfrac{1}{2}tz^2 + \tfrac{1}{2}xz - \tfrac{3}{2} \,.
\end{aligned} \tag{3.32}$$

C. The Korteweg de Vries Equation Again.

It is very interesting to follow through the same mechanism for the KdV equation. We begin with

$$u_{xxx} + uu_x + u_t = 0 \quad ,$$

take $\{x, t, z, z_x, z_{xx}, z_{(3)}, ...\}$ as coordinate on Y^3, set

$$\begin{aligned} -u_t &= u_{xxx} + uu_x \quad , \\ -u_{xt} &= \bar{D}_x(u_{xxx} + uu_x) \quad , \\ &\cdots \quad , \end{aligned}$$

and look for solutions of

$$\{\bar{D}_x^3 + z\bar{D}_x + z_x + \bar{D}_t\}Q(t, x, z, z_x, ..., z_{(l)}) = 0 \quad . \tag{3.33}$$

We quickly find that most of the terms coming from $\bar{D}_t Q$ cancel, leaving us with

$$\begin{aligned} &\{\bar{D}_x^2\partial_x + \sum_{j=0}^{l} z_{(j+1)}\bar{D}_x^2\partial_{z_{(j)}} + 2\sum_{j=0}^{l} z_{(j+2)}\bar{D}_x^2\partial_{z_{(j)}} \\ &+z\partial_x + z_x - \partial_t - (z_{(1)}^2\partial_{z_{(1)}} + 3z_{(1)}z_{(2)}\partial_{z_{(2)}} + ...+ \\ &\quad [(l+1)z_{(1)}z_{(l)} + \tfrac{1}{2}l(l+1)z_{(2)}z_{(l-1)}]\partial_{z_{(l)}})\}Q_{(l)} = 0 \quad . \end{aligned} \tag{3.34}$$

The highest order term is $3z_{(l+2)}z_{(l+1)}Q_{,z_{(l)}z_{(l)}}$ which must vanish separately, giving us the first result:

$$Q = Az_{(l)} + B \quad , \tag{3.35}$$

and the next lowest set of coefficients requires that

$$\bar{D}_x A = 0 \, , \implies \quad A = A(t), \text{ only.} \tag{3.36}$$

Continuing for some little while we find that

$$Q = A(t)z_{(2m+1)} + \frac{1}{3}\{(2m+1)Az + A'x + \alpha(t)\}z_{(2m-1)} + O(2m-2) \quad , \tag{3.37}$$

and $A(t)$ a *linear* (first order) polynomial in t, with

$$\{Q, T\} = \frac{1}{3}\{(2m+1)B'A - (2p+1)A'B\}z_{(2m+2p-1)} + O(2m+2p-2) \quad . \tag{3.38}$$

We notice that one only gets symmetry characteristics at odd order! We can now denote the possible basis for characteristics by

$$\varphi_l^i = t^i z_{(2l+1)} + O(2l-1) \quad , \tag{3.39}$$

with $i = 0$ or 1 **only**. We then label the (original) Lie symmetries according to this labelling of the basis:

$$
\begin{aligned}
\varphi_0^0 &= z_x \quad , \\
\varphi_0^1 &= tz_x - 1 \quad , \\
\varphi_1^0 &= z_{xxx} + zz_x = -z_t \quad , \\
\varphi_1^1 &= t(z_{xxx} + zz_x) - \frac{1}{3}xz_x - \frac{2}{3}z,
\end{aligned}
\tag{3.40}
$$

and use the commutator equation to determine

$$
\{\varphi_l^i, \varphi_m^j\} = \frac{1}{3}[(2l+1)j - (2m+1)i]\varphi_{l+m-1}^{i+j-1} \quad , \tag{3.41}
$$

except that $\{\varphi_0^i, \varphi_0^j\} = 0$.

This tells us that $\{\varphi_1^i, \varphi_1^j\} \propto \varphi_1^{i+j-1}$, and they close on themselves. Again what is needed to generate all of them is to have, say φ_2^1 and φ_3^1. However, explicit calculation, for $l = 5$ and 7 shows that all one gets is φ_2^0 and φ_3^0! Notice that the commutator says that

$$
\{\varphi_m^0, \varphi_l^0\} = 0 \quad ,
$$

i.e., the set with upper index 0 is completely Abelian. Upon calculation we find that there are in fact higher symmetries of the form

$$
\{\varphi_l^0 = z_{(2l+1)} + ... \mid l = 2, 3, 4, ...\},
$$

but none are found of the φ_l^1, for $l > 1$. This leads to two interesting observations!.

D. Hierarchies (of consistent equations).

Continuing with the KdV equation, explicit calculations give one the following results:

$$
\begin{aligned}
\varphi_2^0 =& z_{(5)} + \frac{5}{3}zz_{(3)} + \frac{10}{3}z_x z_{xx} + \frac{5}{6}z^2 z_x \quad , \\
\varphi_3^0 =& z_{(7)} + \frac{7}{3}zz_{(5)} + 7z_{(1)}z_{(4)} + \frac{35}{3}z_{(2)}z_{(3)} + \frac{35}{18}z^2 z_{(3)} \\
&+ \frac{70}{9}zz_x z_{xx} + \frac{35}{18}z_x^3 + \frac{35}{54}z^3 z_x \quad ,
\end{aligned}
\tag{3.42}
$$

and of course an infinite, Abelian hierarchy above that, $\{\varphi_n^0 \mid n = 0, 1, 2, 3, 4, ...\}$. Since all of these commute with one another, we may try to look for simultaneous solutions! This is exactly how one finds the n-soliton solutions for $n > 1$. In particular the 1-soliton solution is obtained by insisting on a solution of the equation that also satisfies a relationship between the characteristics for the symmetries. The lowest-order version of this relationship is just that $\phi_1^0 = v\,\phi_0^0$, where v is the speed of that soliton. This is hardly surprising since $\phi_0^0 = u_x$ and $\phi_1^0 = -u_t$ for a solution of the equation itself, so that this constraint is equivalent to $u_t + vu_x = 0$, which says that the solution should be a function of the one combination of dependent variables, $x - vt$. Nonetheless, it is indeed quite surprising when one learns that this 1-soliton solution has many, related properties, and that this velocity-relationship continues for higher numbers of solitons. The 1-soliton solution is discussed in more detail in Appendix II, where it is noted that

$$\varphi_j^0\Big|_{1-\text{soliton sol'n}} = v^j\,\varphi_0^0\Big|_{1-\text{soliton sol'n}} \quad . \tag{3.43}$$

In addition, also with more discussion in Appendix II, the 2-soliton solution is obtained (originally proven by Lax but noted in Kersten) by looking for solutions of the KdV equation which also simultaneously satisfy

$$\varphi_2^0 = (v_1 + v_2)\varphi_1^0 - v_1 v_2 \varphi_0^0 \,. \tag{3.44}$$

Remembering that the vector field generated by the characteristic for a symmetry is a tangent direction to a curve on the (infinite-dimensional) jet bundle that moves from solutions to solutions on that manifold, and that the generators for our infinite set, $\{\varphi_j^0 \mid j = 0, 1, 2, \ldots\}$, all commute, we may construct a new choice for coordinates on our manifold which correspond to coordinates along these curves. More precisely, for each of our set of Abelian generators, φ_j^0, we label its associated vector field by $\widetilde{v}_j$ and create an associated (new) coordinate, t_{2j+1}, such that we have the following "flow equation":

$$\partial_{t_{2j+1}} = \widetilde{v}_j \quad \Longrightarrow \quad u_{t_{2j+1}} \equiv \partial_{t_{2j+1}} u = \tilde{v}_j u = \sum_0^\infty \{D_\sigma(\varphi_j^0(u))\}\partial_\sigma u = D_0(\varphi_j^0) = \varphi_j^0(u) \,. \tag{3.45}$$

We see that this identifies t_1 with x and t_3 with $-t$, while all the others are new coordinates, thought of as (local) determinations of parameters along the various curves—on the solution manifold—generated by the infinite set of commuting flows. Since the coordinates on the solution manifold have been $\{x, t, z, z_x, z_{xx}, z_{xxx}, z_{xxxx}, \ldots\}$, and there is one Abelian generator for each of $z_{(2j+1)}$, we see that this creation of new coordinates requires some "half" of the original coordinates on the bundle.

There are, however, still other interesting properties of these generators. Each of them is a perfect derivative, which allows us to take each of the equations above and write it as **a conservation law,** i.e., in the form of an equation with a derivative of u on one side and a different derivative, of the same u, on the other side. The first example, of course, is the original KdV equation itself, which may be written in the form

$$-\bar{D}_t u = -u_t = u_{xxx} + u u_x = (u_{xx} + u^2/2)_{,x} = \bar{D}_x(u_{xx} + u^2/2) \; . \tag{3.46}$$

Obviously we could insert the potential w, such that $u = w_x$ into this equation, remove some derivatives and obtain an equation that w must satisfy, which is, however, no easier to solve than the original one:

$$w_{xxx} + \tfrac{1}{2}(w_x)^2 + w_t = 0 \; , \tag{3.47}$$

which is usually referred to as **the KdV-potential, or pKdV, equation**. This process could be continued, infinitely often, since we have the infinite set of compatible equations shown in Eqs. (3.45); we write the first few of them here, in the notation from (3.45):

$$\begin{aligned}
\varphi_0^0 &= \bar{D}_x(z) \quad , \\
\varphi_1^0 &= \bar{D}_x(z_{xx} + \tfrac{1}{2}z^2) \quad , \\
\varphi_2^0 &= \bar{D}_x(z_{(4)} + \frac{5}{3} z z_{(2)} + \frac{5}{6} z_{(2)}^2 + \frac{5}{18} z^3) \quad , \\
\varphi_3^0 &= \bar{D}_x(z_{(6)} + \frac{7}{3} z z_{(4)} + \frac{14}{3} z_{(1)} z_{(3)} + \frac{7}{2} z_{(2)}^2 + \frac{35}{18} z^2 z_{(2)} + \frac{35}{18} z z_{(1)}^2 + \frac{35}{216} z^4) \quad .
\end{aligned} \tag{3.48}$$

We then write each of our characteristics as a perfect derivative, so that we have

$$u_{t_{2j+1}} = \phi_j^0(u) \equiv \bar{D}_x\{\chi_j^0(u)\} \; , \tag{3.49a}$$

each of which then has the form of a conservation law. Because of that, for each of them, we may associate a potential, $u \equiv w^j_{,x}$, so that

$$w^j_{,t_{2j+1}} = \chi_j^0(u)\Big|_{u=w^j_{,x}} \; . \tag{3.49b}$$

One might then hope, since all these equations are indeed compatible, to obtain **a single potential** to define them all. In fact one such quantity does exist, usually referred to as a τ-function, and ascribed to either Hirota or Sato, or both:

$$u = 12\partial_x\left(\frac{\tau_{,x}}{\tau}\right) = 12(\partial_x)^2 \ln\tau \; . \tag{3.50}$$

A more complete discussion of the τ-function, and its usefulness for determining solutions, will also be given in Appendix II.

IV. Non–local Symmetries.

A. Reasons and Motivation.

While we were able to explicitly write down how to determine all of the higher symmetries of Burgers' equation, this was not possible with the KdV equation, because φ_2^1 and φ_3^1 were **not** symmetries of the system. Krasil'shchik resolved this problem in a clever way, perhaps being aware of Olver's earlier, different method of resolution.

While it is true that the higher symmetries of the KdV equation form an Abelian algebra, so that it is not possible to use the Lie bracket of the algebraic structure to generate them all, it is possible to find a "raising operator". There is an "operator" $\mathcal{R}$, first found by Olver, that has the property that

$$\mathcal{R}\varphi_n^0 = \varphi_{n+1}^0 \quad , \tag{4.1}$$

$$\mathcal{R} \equiv \bar{D}_x^2 + \frac{2}{3}z + \frac{1}{3}z_x\bar{D}_x^{-1} \quad , \tag{4.2}$$

where it is "acceptable" that $\mathcal{R}$ involves an inverse derivative operator since it will be acting only on the characteristics, each of which is a perfect derivative. As an example, note that

$$\mathcal{R}\varphi_0^0 = z_{xxx} + \frac{2}{3}zz_x + \frac{1}{3}z_xz = \varphi_1^0 \quad . \tag{4.3}$$

Krasil'shchik's approach was, instead to prolong the J^∞ space by appending to it a new coordinate, which we will call $z_{(-1)}$, which "acts like" a first integral of our dependent variable coordinate, z. Therefore it should have the property that D_x acts on it to give z. Of course the current total derivative operator, D_x would actually give zero; therefore, we must of course also prolong the total derivative operator when we prolong the jet space itself. We want to create a prolonged total derivative operator, which we will call $\tilde{D}_x$ with the property just described; as well, of course, we will need to have a proper behavior under (a prolonged version of) the total derivative operator with respect to t; the obvious requirements are then

$$\tilde{D}_x(z_{(-1)}) = z\,, \qquad \tilde{D}_t\tilde{D}_xz_{(-1)} = z_t = -(z_{(3)} + zz_x) \quad . \tag{4.5}$$

In order for the second requirement to be possible, it is necessary that the quantity $z_{xxx}+zz_x$ be a perfect derivative; as it happens, for our KdV equation, this is true, so that we can indeed define the desired new variable/coordinate in our prolonged jet bundle, and the associated prolongations of the total derivative operator as follows, where the prolonged total derivatives continue to commute:

$$\tilde{D}_x = z\partial_{z_{(-1)}} + \bar{D}_x \;\text{ and }\; \tilde{D}_t = -(z_{xx} + \tfrac{1}{2}z^2)\partial_{z_{(-1)}} + \bar{D}_t\,, \tag{4.6}$$

$$z_t = \tilde{D}_t\tilde{D}_xz_{(-1)} = \tilde{D}_x\tilde{D}_tz_{(-1)} = -\tilde{D}_x(z_{xx} + \tfrac{1}{2}z^2)\,. \tag{4.7}$$

Then, looking for characteristics $Q_l = Q_l(t, x, z_{(-1)}, z, z_{(1)}, z_{(2)}, ..., z_{(l)})$, that satisfy the prolonged version of Eq. (3.36), namely

$$\{\tilde{D}_x^3 + z\tilde{D}_x + z_x + \tilde{D}_t\}Q = \tilde{l}_F(Q) = 0 \quad , \tag{3.36$'$}$$

he found the desired characteristics, φ_2^1, φ_3^1 and in fact he found exactly all (and only all) the quantities

$$\{\varphi_k^i \mid k = 0, 1, 2, ...; i = 0, 1\} \quad ,$$

those needed to make a basis for the set of all Q's. The next one is explicitly given by

$$\begin{aligned} \varphi_2^1 =& tz_{(5)} + \frac{1}{3}(5tz - x)z_{(3)} + \frac{1}{6}(20tz_{(1)} - 8)z_{(2)} \\ &+ \frac{1}{18}(15tz^2 - 6xz - 2z_{(-1)})z_{(1)} - \frac{4}{9}z^2 \quad . \end{aligned} \tag{4.9}$$

Is there a way to understand this "prolongation" in a more serious (or rigorous) way? The answer is yes, but it requires considerable re-tracing of one's steps in order to do so. Therefore, I want to first look at **yet another**, quite different way to find some sort of symmetries of a pde, and then eventually tie the two together.

B. Estabrook-Wahlquist Prolongations.

In 1975 the two authors mentioned decided to try to generalize the notion of finding potentials for a pde. In principle the notion of a symmetry is very different from that of a potential. A potential, however, is rather more like an "integral" of an as–yet–undetermined function. Just for a second I remind you that the Maxwell equations have potentials. The simplest example is to consider the time–independent equations for $\vec{E}$:

$$\nabla \cdot \vec{E} = \rho \,, \quad \nabla \times \vec{E} = 0 \quad . \tag{4.10a}$$

It is easiest to solve these equations by first saying that the second three equations are true if and only if there is a scalar function φ such that $\vec{E} = -\nabla\varphi$, so that φ is "an integral" of the second equation, leaving us only to solve

$$\nabla \cdot \nabla\varphi + \rho = 0 \quad . \tag{4.10b}$$

A very general way to find potentials for a set of equations is to write it as an ideal of differential forms and **note** that, in fact, **all** potentials occur by reason of the existence of some closed differential form, i.e, some form $\underset{\sim}{\omega}$ such that $d\underset{\sim}{\omega} = 0$. For the example of Maxwell's equations, we define a 1-form $\underset{\sim}{E} = adx + bdy + cdz$, where a, b and c are the components of the electric field, which depend on $\{x, y, z\}$. As well we define the dual 2-form which, for my purposes, I may write

$$\underset{\sim}{E} = adx + bdy + cdz \,, \quad \Longleftrightarrow \quad *\underset{\sim}{E} \equiv c\, dx \wedge dy + a\, dy \wedge dz + b\, dz \wedge dx.$$

One then simply checks that the 2 Maxwell's equations are given by the 2-form and 3-form equations,

$$d(*\underset{\sim}{E}) - \rho\, dx \wedge dy \wedge dz = 0 \,, \qquad d\underset{\sim}{E} = 0 \quad . \tag{4.10$'a$}$$

Since $dx \wedge dy \wedge dz$ is a "little piece" of volume, dV, multiplying it by the "charge per unit volume" seems like a very reasonable thing to do. Then, by the theorem mentioned above, namely Poincaré's Lemma, $d\underset{\sim}{E} = 0$ means there exists some 0-form, i.e., some function, φ such that $\underset{\sim}{E} = -d\varphi$. Therefore only one equation remains to be integrated:

$$d(*d\varphi) + \rho dV = 0 \quad . \tag{4.10$'b$}$$

Following this same sort of procedure for **Burgers' equation**, which we now know very well, we may first re-write the equation in the form

$$(u_x + \tfrac{1}{2}u^2)_x = u_{xx} + uu_x = -u_t = (-u)_t \quad , \tag{4.11}$$

from which we easily infer the existence of a potential function Λ such that

$$\begin{aligned} u_x + \tfrac{1}{2}u^2 = \Lambda_t,\ \ -u =&\Lambda_x \quad , \\ \Rightarrow \quad \Lambda_{xx} - \tfrac{1}{2}\Lambda_x^2 + \Lambda_t =&0 \quad , \end{aligned} \tag{4.12}$$

the last one being a "potential" equation, often referred to as the "potential Burgers' equation." Unfortunately, in this case, the existence of the potential Λ does not appear to simplify the finding of a solution! However, what one could do would be now to look for symmetries (or new potentials) of this pair of equations, with jet variables $\{t, x, u, \Lambda, u_x, \Lambda_x, \Lambda_t\}$, and something might be found! This is very similar to Krasil'shchik's addition of $z_{(-1)}$ to the space of variables except that it is on much firmer mathematical ground!

C. Doing it with forms.

To begin such a process, I first recall the contact module and β for Burgers' equation, from Eq. (1.11):

$$\begin{aligned} \underset{\sim}{\theta} &= dz - pdx - qdt \quad , \\ \underset{\sim}{\beta} &= (dp + zdz) \wedge dt + dx \wedge dt \quad , \end{aligned} \tag{1.11}$$

and then restrict the generators so that they are **all pure 2-forms**, by simply "throwing away" the q in θ by keeping only the ideal generated by the following 2-form generators, $\{\underset{\sim}{\alpha}, \underset{\sim}{\beta}\}$:

$$\begin{aligned} \underset{\sim}{\alpha} \equiv \underset{\sim}{\theta} \wedge dt &= (dz - pdx) \wedge dt \quad , \\ \underset{\sim}{\beta} &= (dp + zdz) \wedge dt + dx \wedge dz \quad . \end{aligned} \tag{4.13}$$

We now note that $d\underset{\sim}{\beta} = 0,\ d\underset{\sim}{\alpha} = +dx \wedge \underset{\sim}{\beta} - zdx \wedge \underset{\sim}{\alpha}$, so that the system satisfies Cartan's criterion for being closed, and therefore is—all by itself—sufficient to characterize the original equation.

Of course, the fact that $d\underset{\sim}{\beta} = 0$, as we just discussed, tells us immediately of the existence of a 1-form $\underset{\sim}{\eta}$ such that $\underset{\sim}{\beta} = d\eta$. Of course, $\underset{\sim}{\eta}$ is ambiguous since $\underset{\sim}{\eta} + d\Lambda$ would also generate the same $\underset{\sim}{\beta}$, for any scalar Λ. This Λ we may now take as a new (potential) variable in our system and, in particular, put it into our "jet" space—the now prolonged jet space. To see that this Λ is the same as the one I had above, write

$$\begin{aligned} \underset{\sim}{\beta} &= d(p + \tfrac{1}{2}z^2) \wedge dt - d(z) \wedge dx \\ &= d\{(p + \tfrac{1}{2}z^2)dt - zdx\} \\ &= d\{(p + \tfrac{1}{2}z^2)dt - zdx - d\Lambda\} \end{aligned} \tag{4.14}$$

Since $\beta\ |_{Y^2} = 0$, we of course have that

$$\Rightarrow\ \{(p + \tfrac{1}{2}z^2)dt - zdx - d\Lambda\}\ |_{Y^2} = 0 \quad . \tag{4.15a}$$

However, for $\{x,t\}$ as independent variables, we also have

$$d\Lambda = \Lambda_x dx + \Lambda_t dt \quad . \tag{4.15b}$$

Comparing these two forms of Eq. (4.15), we get exactly the same Λ as in Eq. (4.12). Our prolonged ideal would then consist of

$$\left\{ \begin{array}{l} d\Lambda - (p + \frac{1}{2}z^2)dt + zdx \quad , \\ \alpha \quad , \\ \beta \quad . \end{array} \right. \tag{4.16}$$

on the space $\{t, x, z, p, \Lambda\}$. We would then try to look for yet another closed 2–form in this ideal and find yet another potential, etc., etc. The new one would of course have the option of depending on our Λ, and the next one on the previous one, etc.

Estabrook and Wahlquist proposed a method to short–circuit this infinite sequence by looking for the entire set, however many there might be, all at once, and letting them all depend on all of them. This process complicates the arithmetic, but leads to much more interesting results. Following their procedure, we do *assume* the existence of some as-yet-unknown number of new potentials, w^μ, for our system of forms, and append to our ideal the following extra set of 1–forms, which are contact 1-forms for these new potentials:

$$\underset{\sim}{\omega}^\mu \equiv -dw^\mu + F^\mu dx + G^\mu dt \quad , \tag{4.17}$$

where the F^μ and G^μ in principle depend on all the variables in the prolonged space: $\{x^i, z^\alpha_\sigma, w^\mu\}$. It actually turns out that it is safe—almost all the time—to ignore their dependence on the $\{x^i\}$, which we will do. The trick, of course is that, in the example of the determination of a potential that I gave earlier, the closure of the new 1–form was simply the closed 2–form that we had already discovered. This time we are doing it in advance, so we don't know which, if any, of the forms is closed, so we have to look for them at the same time. However, all that is truly necessary is that the new ideal be closed, i.e., that should exist as-yet-unknown functions $\lambda^{i\mu}$, ρ^μ_A, and 1-forms, $\underset{\sim}{\eta}^\mu_\nu$, such that the ideal is still closed after these new forms are appended:

$$\Rightarrow \; d\underset{\sim}{\omega}^\mu = \lambda^{i\mu}\underset{\sim}{\alpha}_i + \rho^\mu_A \underset{\sim}{\beta}^A + \underset{\sim}{\eta}^\mu_\nu \wedge \underset{\sim}{\omega}^\nu \quad . \tag{4.18}$$

These are a set of equations for the dependence of F^μ and G^μ on their variables, and also for the determination of the Lagrange-multiplier type objects, λ^i, μ_A, and $\underset{\sim}{\eta}_\mu$.

I will first give a quick, reasonably-abstract description of the general process just described above, and, then, go through it explicitly and in detail for an example, which I will take to be the KdV equation. Therefore, let us begin by recalling that a k-th order pde is a variety within the k-th order jet bundle, $J^k(M,N)$, with coordinates $\{x^a, z^\mu, z^\mu_a, \ \ldots \}$. Now, in addition we prolong this space by "adjoining" a space, N' for the desired "potentials" to live; these are additional functions of the underlying coordinates $x^a \in M$. The Estabrook-Wahlquist process may then be described as searching for a map of the following kind:

$$\Upsilon : J^{k-1}(M,N) \times N' \ \longrightarrow \ J^1(M,N'),$$

with the following coordinate presentation:

$$\Upsilon : \begin{cases} x^a = x^a \ , \\ z^\mu = z^\mu \ , \\ z'^A = z'^A \\ z'^A_a = Z^A_a(x, z_\sigma, z') \ . \end{cases} \tag{4.19}$$

It is worth making some comments concerning the choices made above:

1.) If we allowed the new independent variables, x^a to depend on all the variables, i.e., to allow $x^a = X^a(x, z_\sigma, z')$, this would be acceptable, but more complicated than needed for quasi-linear pde's.

2.) If we allowed z'^A to vary it would complicate the problem too much, since they are really part of the input to the problem, unless, perhaps, the problem was of some very special sort that insisted on that. It is therefore better to do it this way, and then contemplate further transformations later, of the sort $\widetilde{z}'^R = F^R(x, z, z')$.

3.) If there were no dependence on z'^A in our functions Z^A_r then this should properly be called a potential; however, in this more general case, *pseudopotential* is the standard word.

Now the pullback of Υ maps the cotangent bundles, so it is reasonable that we insist that it map the contact modules appropriately:

$$\begin{aligned} \Upsilon^* : \Lambda^p(J'^1) &\longrightarrow \Lambda^p(J^{k-1} \times N') \ , \\ \Upsilon^*(\theta'^A) \ &= dz'^A - Z^A_a \, dx^a \ . \end{aligned} \tag{4.19}$$

We must therefore insist that it be closed in the combined contact module, which creates (serious) requirements on the functions Z^A_r:

$$\begin{aligned} d\{\Upsilon^*(\theta'^A) \equiv 0 \quad (\text{mod } \Omega)^{(k)} + \Upsilon^* \Omega'^{(1)} \equiv \Omega_\Upsilon \ , \\ \Longrightarrow \ \ \widetilde{D}_{[b} Z^A_{a]} \, dx^a \wedge dx^b \equiv 0 \quad (\text{mod } \Omega)_\Upsilon \ , \\ \widetilde{D}^{(k)}_a \equiv D^{(k)}_a + Z^A_a \, \partial_{z'^A} \ . \end{aligned} \tag{4.20}$$

We insist that these coefficients are functions over the original space, i.e., over $J^k(M,N)\times N'$. In point of fact that they should end up just being the original pde! Therefore, the entire construction "looks like" a curvature that vanishes when the pde is satisfied, i.e., a curvature that vanishes when restricted to the variety defined by the pde itself.

Now we will try to explain all this by showing you how to deal with an explicit example. This time **I will work with the KdV equation** as our sample equation, namely Eq. (2.69). Therefore we choose an ideal of 2-forms, that lies within the ideal generated by those forms presented already in Eq. (2.70):

$$\left\{\begin{array}{l} \underset{\sim}{\alpha} \equiv (dz - p dx)\wedge dt \quad , \\ \underset{\sim}{\alpha}_x \equiv (dp - r dx)\wedge dt \quad , \\ \underset{\sim}{\beta} \equiv (dr + z dz)\wedge dt + dx\wedge dz \quad . \end{array}\right. \tag{4.19}$$

with variables $\{t, x, z, p, r\}$. Inserting the definition of $\underset{\sim}{\omega}^{\mu}$, the defining equations are

$$\begin{aligned} dF^{\mu}\wedge dx + dG^{\mu} =& \lambda_1^{\mu}(dz - p dx)\wedge dt + \lambda_2^{\mu}(dp - r dx)\wedge dt \\ &+ \rho^{\mu}[(dr + z dz)\wedge dt + dz\wedge dz] - \underset{\sim}{\eta}^{\mu}{}_{\nu}\wedge dw^{\nu} \\ &+ \underset{\sim}{\eta}^{\mu}{}_{\nu}\wedge(F^{\nu}dx + G^{\nu}dt) \quad , \end{aligned} \tag{4.20}$$

where that $dF^{\mu}(z, p, r, w^{\mu})$ must be expanded in the usual way,

$$dF^{\mu} = F^{\mu}_{,z}dz + F^{\mu}_{,p}dp + F^{\mu}_{,r}dr + F^{\mu}_{,w^{\xi}}dw^{\xi} \quad , \tag{4.21}$$

deliberately ignoring any dependence on $\{t, x\}$, and the 1-form $\underset{\sim}{\eta}$ has its coefficients labelled as follows:

$$\underset{\sim}{\eta}^{\mu}{}_{\nu} = \eta^{\mu}_{\nu x}dx + \eta^{\mu}_{\nu t}dt + \eta^{\mu}_{\nu z}dz + \eta^{\mu}_{\nu p}dp + \eta^{\mu}_{\nu r}dr + \eta^{\mu}_{\nu\lambda}dw^{\lambda} \quad . \tag{4.22}$$

The first thing we notice is that all the terms on the left–hand side have either a dx or a dt in them. Therefore any terms on the right-hand side without a dx or a dt in them **must surely vanish!** Since there are only the terms $\underset{\sim}{\eta}^{\mu}{}_{\nu}\wedge dw^{\nu}$ that meet this criterion, we may immediately say that all components of $\underset{\sim}{\eta}^{\mu}{}_{\nu}$ must vanish except for those in the direction of dx or dt. Note that this is true whenever the pde is quasilinear, i.e., linear in its highest derivatives! Secondly, another general property is found by looking at the coefficients of

$$\begin{aligned} dw^{\nu}\wedge dx: \ F^{\mu}_{,w^{\nu}} &= +\eta^{\mu}_{\nu x} \quad , \\ dw^{\nu}\wedge dt: \ G^{\mu}_{,w^{\nu}} &= +\eta^{\mu}_{\nu t} \quad . \end{aligned} \tag{4.23}$$

This allows us to state that any quasilinear pde has these Lagrange 1-forms given as

$$\underset{\sim}{\eta}^{\mu}_{\nu} = F^{\mu}_{,w^{\nu}}dx + G^{\mu}_{,w^{\nu}}dt \quad . \tag{4.24}$$

Evaluation now proceeds by comparing the rest of the coefficients of the basis of 2–forms. However, it is best to do this by holding the coefficients of $dx \wedge dt$ until the end, looking at all the others first. Therefore we first consider the following comparisons of the coefficients of the two sides of the equation:

$$\begin{aligned} dr \wedge dx: &\; F^\mu_{,r} = 0 \quad , \\ dp \wedge dx: &\; F^\mu_{,p} = 0 \quad , \\ dz \wedge dx: &\; F^\mu_{,z} = -\rho^\mu \quad , \\ dr \wedge dt: &\; G^\mu_{,r} = \rho^\mu \quad , \\ dp \wedge dt: &\; G^\mu_{,p} = \lambda^\mu_2 \quad , \\ dz \wedge dt: &\; G^\mu_{,z} = \lambda^\mu_1 + z\rho^\mu \quad , \end{aligned} \tag{4.25}$$

from which we conclude that F^μ depends only on $\{z, w^\nu\}$ while G^μ must depends on all these variables! It is in fact quite important to emphasize that **the final solution must have non–zero $\rho, \lambda_1, \lambda_2$ in order for "the mathematics" to remember the original pde**! However, in addition, the Eqs. (4.25) tell us that there is a constraint on the solutions:

$$F^\mu_{,z} + G^\mu_{,r} = 0 \quad . \tag{4.26}$$

Applying some of the dependence statements for F^μ to this equation informs us of the following:

$$\begin{aligned} \partial_r \quad \Rightarrow \quad & G^\mu_{,rr} = 0 \quad , \\ \partial_p \quad \Rightarrow \quad & G^\mu_{,rp} = 0 \quad , \\ \Rightarrow \quad G^\mu = & A^\mu_z(z)r + B^\mu(z,p) \quad , \end{aligned} \tag{4.27}$$

where, since the coefficient of r was an arbitrary function of z, I wrote it as $\frac{\partial A}{\partial z}$ for reasons that will be obvious almost immediately. We may now consolidate our information so far:

$$\begin{aligned} F^\mu_{,z} = -G^\mu_{,r} = -A^\mu_{,z} \qquad , \\ \Rightarrow \; F^\mu = -A^\mu(z), \quad G^\mu = +A^\mu_z r + B^\mu(z,p) \quad , \end{aligned} \tag{4.28}$$

and proceed onward to write down the coefficient of $dx \wedge dt$:

$$\begin{aligned} 0 = & - p\lambda^\mu_1 - r\lambda^\mu_2 - F^\nu G^\mu_{,w^\nu} + G^\nu F^\mu_{,w^\nu} \\ = & - p(G^\mu_{,z} + zF^\mu_{,z}) - rG^\mu_{,p} - (F^\nu \partial_{w^\nu})G^\mu + (G^\nu \partial_{w^\nu})F^\mu. \end{aligned} \tag{4.29}$$

The form of the last 2 terms suggest that things would, at least, be simpler to write if we would define vector fields over the pseudopotential variables, $\{w^\mu\}$, with coefficients determined by the F^μ and the G^ν. I will denote these particular vector fields, over pseudopotentials for generators of symmetries, either by using Latin capital letters, in boldface typefonts, **or** with Greek letters with over-tildes:

$$\underset{\sim}{\mathrm{F}} \equiv F^{\mu}\partial_{w^{\mu}} \text{ and } \underset{\sim}{\mathrm{G}} \equiv G^{\mu}\partial_{w^{\mu}} \quad . \tag{4.30}$$

Multiplying the entirety of Eq. (4.30) by $\partial_{w^{\mu}}$, summing, and also taking A and B as the obvious vector fields, our equation takes the form

$$[-\underset{\sim}{\mathrm{A}}, \underset{\sim}{\mathrm{A}}_{z}r + \underset{\sim}{\mathrm{B}}] = [\underset{\sim}{\mathrm{F}}\,, \underset{\sim}{\mathrm{G}}] = -p(\underset{\sim}{\mathrm{A}}_{,zz}r + \underset{\sim}{\mathrm{B}}_{,z}) - r\underset{\sim}{\mathrm{B}}_{,p} + pz\underset{\sim}{\mathrm{A}}_{,z} \quad . \tag{4.31}$$

This equation is a first order polynomial in r, so we may compare coefficients:

$$[\underset{\sim}{\mathrm{A}}_{z}, \underset{\sim}{\mathrm{A}}] = -p\,\underset{\sim}{\mathrm{A}}_{,zz} - \underset{\sim}{\mathrm{B}}_{,p}\,, \text{so application of } \partial_p \quad \Rightarrow \quad A_{,zz} + B_{,pp} = 0 \quad . \tag{4.32}$$

Since $\underset{\sim}{\mathrm{A}}_{,p} = 0$, we integrate this to give

$$\underset{\sim}{\mathrm{B}} = -\tfrac{1}{2}\underset{\sim}{\mathrm{A}}''p^2 + \underset{\sim}{\mathrm{C}}p + \underset{\sim}{\mathrm{E}} \quad , \tag{4.33}$$

where now all of $\{\underset{\sim}{\mathrm{A}}, \underset{\sim}{\mathrm{C}}, \underset{\sim}{\mathrm{E}}\}$ all depend only on z! We obtain then an equation for $\underset{\sim}{\mathrm{C}}$:

$$[\underset{\sim}{\mathrm{A}}', \underset{\sim}{\mathrm{A}}] = -\underset{\sim}{\mathrm{C}} \quad . \tag{4.34}$$

Picking up the 0-th order terms in the variable r, from Eq. (4.30), we obtain the requirement $[\underset{\sim}{\mathrm{B}}, \underset{\sim}{\mathrm{A}}] = -p\underset{\sim}{\mathrm{B}}_{,z} + pz\underset{\sim}{\mathrm{A}}_{,z}$. When Eq. (4.33) for $\underset{\sim}{\mathrm{B}}$ is inserted into this, we acquire a polynomial in p:

$$[-\tfrac{1}{2}\underset{\sim}{\mathrm{A}}''p^2 + \underset{\sim}{\mathrm{C}}p + \underset{\sim}{\mathrm{E}}, \underset{\sim}{\mathrm{A}}] = \tfrac{1}{2}\underset{\sim}{\mathrm{A}}'''p^3 - \underset{\sim}{\mathrm{C}}'p^2 - \underset{\sim}{\mathrm{E}}'p + pz\underset{\sim}{\mathrm{A}}' \quad , \tag{4.35}$$

the coefficients of which we then separately set equal to 0:

$$p^3: \quad \underset{\sim}{\mathrm{A}}''' = 0\,, \;\Rightarrow \quad \underset{\sim}{\mathrm{A}} = \tfrac{1}{2}\tilde{\alpha}z^2 + \tilde{\beta}z + \tilde{\gamma} \quad , \tag{4.35a}$$

$$p^2: \quad \tfrac{1}{2}[\underset{\sim}{\mathrm{A}}\,, \underset{\sim}{\mathrm{A}}''] = -\underset{\sim}{\mathrm{C}}' = -[\underset{\sim}{\mathrm{A}}, \underset{\sim}{\mathrm{A}}'']\,, \;\Rightarrow \quad [\underset{\sim}{\mathrm{A}}, \underset{\sim}{\mathrm{A}}''] = [\tfrac{1}{2}\tilde{\alpha}z^2 + \tilde{\beta}z + \tilde{\gamma}, \tilde{\alpha}] = 0 \quad ,$$
$$\Rightarrow \quad [\tilde{\alpha}\,, \tilde{\beta}] = 0 = [\tilde{\alpha}, \tilde{\gamma}] \quad , \text{ and } C \text{ is a constant!} \tag{4.35b}$$

$$p^1: \quad [\underset{\sim}{\mathrm{C}}, \underset{\sim}{\mathrm{A}}] = -\underset{\sim}{\mathrm{E}}' + z\underset{\sim}{\mathrm{A}}' = [\underset{\sim}{\mathrm{C}}, \tfrac{1}{2}\tilde{\alpha}z^2 + \tilde{\beta}z + \tilde{\gamma}] = \tilde{\alpha}z^2 + \tilde{\beta}z - \underset{\sim}{\mathrm{E}}' \quad , \tag{4.35c}$$

$$p^0 = 1: \quad [E, A] = 0 \quad . \tag{4.35d}$$

Inserting this information into our equation for $\underset{\sim}{C}$ gives a new expression for $\underset{\sim}{C}$:

$$\underset{\sim}{C} = [\underset{\sim}{A}, \underset{\sim}{A}'] = [\tfrac{1}{2}\tilde{\alpha}z^2 + \tilde{\beta}z + \tilde{\gamma}, \tilde{\alpha}z + \tilde{\beta}] = -[\tilde{\beta}, \tilde{\gamma}] \quad , \quad \Rightarrow \quad \underset{\sim}{C} = [\gamma, \beta] \quad . \tag{4.36}$$

The p^1 constraint above, Eq. (4.35c), wants us to have the value for $[\underset{\sim}{C}, \tilde{\alpha}]$, which we now determine to vanish:

$$[\underset{\sim}{C}\,, \tilde{\alpha}] = [[\tilde{\gamma}, \tilde{\beta}], \tilde{\alpha}] = [\tilde{\beta}, [\tilde{\gamma}, \tilde{\alpha}]] + [\tilde{\gamma}, [\tilde{\alpha}, \tilde{\beta}]] = 0 \quad . \tag{4.37}$$

Solving Eq. (4.35c) for $\underset{\sim}{E}'$ now allows us to integrate it to determine a form for $\underset{\sim}{E}$:

$$\underset{\sim}{E} = \tfrac{1}{3}\tilde{\alpha}z^3 + \tfrac{1}{2}\{\tilde{\beta} + [\tilde{\beta}, \underset{\sim}{C}]\}z^2 + [\tilde{\gamma}, \underset{\sim}{C}]z + \tilde{\delta} \quad . \tag{4.38}$$

All this information, inserted into Eq. (4.35d), the coefficient of p^0, gives us

$$\begin{aligned} &[\tfrac{1}{3}\tilde{\alpha}z^3 + \tfrac{1}{2}\{\tilde{\beta} + [\tilde{\beta}\,, \underset{\sim}{C}]\}z^2 + [\tilde{\gamma}\,, \underset{\sim}{C}]z + \tilde{\delta}, \tfrac{1}{2}\tilde{\alpha}z^2 + \tilde{\beta}z + \tilde{\gamma}] \\ &\quad = z^4\{\tfrac{1}{4}[\tilde{\beta} + [\tilde{\beta}\,, \underset{\sim}{C}], \tilde{\alpha}] + \tfrac{1}{3}[\tilde{\alpha}, \tilde{\beta}]\} \\ &\qquad + z^3\{\tfrac{1}{3}[\tilde{\alpha}, \tilde{\gamma}] + \tfrac{1}{2}[[\tilde{\beta}, \underset{\sim}{C}], \tilde{\beta}] + \tfrac{1}{2}[[\tilde{\gamma}, \underset{\sim}{C}], \tilde{\alpha}]\} \\ &\qquad + z^2\{\tfrac{1}{2}[\tilde{\beta} + [\tilde{\beta}, \underset{\sim}{C}], \tilde{\gamma}] + [[\tilde{\gamma}, \underset{\sim}{C}], \tilde{\beta}] + \tfrac{1}{2}[\tilde{\delta}, \tilde{\alpha}]\} \\ &\qquad + z\{[[\tilde{\gamma}, \underset{\sim}{C}], \tilde{\gamma}] + [\tilde{\delta}, \tilde{\beta}]\} + [\tilde{\delta}, \tilde{\gamma}] = 0 \quad , \end{aligned} \tag{4.39}$$

from which we infer the following:

$$[\tilde{\delta}, \tilde{\gamma}] = 0 \quad , \tag{4.40a}$$

$$[\tilde{\delta}, \tilde{\beta}] + [\tilde{\gamma}, [\tilde{\gamma}, \underset{\sim}{C}]] = 0 \quad , \tag{4.40b}$$

$$-\tfrac{1}{2}\underset{\sim}{C} - \tfrac{1}{2}[\tilde{\gamma}, [\tilde{\beta}, \underset{\sim}{C}]] - [\tilde{\beta}, [\tilde{\gamma}, \underset{\sim}{C}]] - \tfrac{1}{2}[\tilde{\alpha}, \tilde{\delta}] = 0 \quad , \tag{4.40c}$$

$$[\tilde{\alpha}, [\tilde{\gamma}, \underset{\sim}{C}]] + [\tilde{\beta}, [\tilde{\beta}, \underset{\sim}{C}]] = 0 \quad , \tag{4.40d}$$

$$[\tilde{\alpha}, [\tilde{\beta}, \underset{\sim}{C}]] = 0 \quad . \tag{4.40e}$$

Having now satisfied **all** the implied parts of Eq. (4.30), we may write down the complete answers for $\underset{\sim}{F}$ and $\underset{\sim}{G}$, in terms of vector fields lying only over the (fiber) space of allow, new pseudopotential variables, where we replace the symbol $\underset{\sim}{C}$ by its value as given above, namely $[\tilde{\gamma}, \tilde{\beta}]$:

$$\begin{aligned} \underset{\sim}{F} &= -\tfrac{1}{2}\tilde{\alpha}z^2 - \tilde{\beta}z - \tilde{\gamma} \quad , \\ \underset{\sim}{G} &= (\tilde{\alpha}z + \tilde{\beta})r - \tfrac{1}{2}\tilde{\alpha}p^2 - [\tilde{\beta}, \tilde{\gamma}]p + \tfrac{1}{3}\tilde{\alpha}z^3 \\ &\quad + \tfrac{1}{2}\{\tilde{\beta} - [\tilde{\beta}, [\tilde{\beta}, \tilde{\gamma}]]\}z^2 + [\tilde{\gamma}, [\tilde{\gamma}, \tilde{\beta}]]z + \tilde{\delta} \quad . \end{aligned} \tag{4.41}$$

where $\{\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}\}$ are vector fields over the space of variables, $\{w^\mu\}$, but are subject, however, to many commutator constaints as we have defined above. It is useful to list them in the form of a table, as we have done before. Many of the commutators have already been determned, above, as required to have certain values. Some of the commutators may be determined from those that we already know, by the use of the Jacobi identity. Lastly, some of the commutators are simply not determined by the calculations above. Typically, these are the commutators of the elements that are involved only in $\underset{\sim}{F}$, among themselves, and also those involved only in $\underset{\sim}{G}$, among themselves. Recall that for any 3 vector fields, the Jacobi identity is simply an identity concerning the way commutators work:

$$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0 \quad . \tag{4.42}$$

Although one could use a table as I have done previously, a simpler approach in this case is probably to just list the requirements:

$$\begin{aligned} [\tilde{\alpha}\,,\tilde{\beta}] = 0 = [\tilde{\alpha}\,,\tilde{\gamma}] &= [\tilde{\beta},[\tilde{\beta},[\tilde{\beta},\tilde{\gamma}]]] = [\tilde{\gamma},\tilde{\delta}] \quad , \\ [\tilde{\alpha},\tilde{\delta}] - [\tilde{\beta},\tilde{\gamma}] - 3[\tilde{\beta},[\tilde{\gamma},[\tilde{\beta},\tilde{\gamma}]]] &= 0 = [\tilde{\beta},\tilde{\delta}] + [\tilde{\gamma},[\tilde{\gamma},[\tilde{\beta},\tilde{\gamma}]]] \quad . \end{aligned} \tag{4.43}$$

The idea is, of course, that $F^\mu = w^\mu_{,x}$, $G^\mu = w^\mu_{,y}$. The integrability conditions on this set are already satisfied—by virtue of the KdV equation—so such w^i's will exist provided I can find vector fields $\{\alpha, \beta, \gamma, \delta\}$ satisfying the relations given, Eq. (4.43). There are various ways one can attempt to do this; however, they can basically be characterized in two different ways: the first is to "guess" functions of some number of w^μ's satisfying these relations. The second is to create an algebra satisfying only these relations and use that knowledge to determine what functions of w^μ the F^μ and G^μ are. In either case, having determined how many w^μ one will consider, and what functions they are, we then have found our "potentials." Since they were found by this method whereby they depend on themselves, they are actually called *pseudopotentials.*

For a first example, I make a relatively simple choice for these functions, which nonetheless will demonstrate several interesting things one may do with them! We first decide to look only for one such pseudopotential, so that the number of w^μ's is simply 1. A particular realization for the commutation relations—the simplest one that exists—is then given by the following homomorphism:

$$\begin{aligned} \tilde{\alpha} &\rightarrow 0 \quad , \\ \tilde{\beta} &\rightarrow \tfrac{1}{2}\partial_w \Longleftarrow \quad \tfrac{1}{2}\widetilde{\mathbf{f}} \,, \\ \tilde{\gamma} &\rightarrow \tfrac{1}{3}w^2\partial_w \Longleftarrow \quad -\tfrac{1}{3}\widetilde{\mathbf{e}} \,, \\ \tilde{\delta} &\rightarrow 0 \quad , \end{aligned} \tag{4.44}$$

which implies $\widetilde{\mathbf{C}} = [\tilde{\gamma},\tilde{\beta}] \rightarrow -\tfrac{1}{3}w\partial_w \Longleftarrow \quad -\tfrac{1}{6}\widetilde{\mathbf{h}}\,,$

where $\{\widetilde{\mathbf{h}}, \widetilde{\mathbf{e}}, \widetilde{\mathbf{f}}\}$ are (a 1-dimensional) vector-field realization of the standard Chevalley generators for $\boldsymbol{sl}(2)$.

Inserting these values into Eq. (4.41), we obtain the following explicit Bäcklund transformation, between the KdV and another pde:

$$\begin{aligned} \Rightarrow\quad w_{,x} &= F^1 = -\tfrac{1}{3}w^2 - \tfrac{1}{2}z \Longleftarrow\quad -\tfrac{1}{2}\widetilde{\mathbf{f}}z + \tfrac{1}{3}\widetilde{\mathbf{e}}\ , \\ w_{,t} &= G^1 = \tfrac{1}{2}z_{xx} - \tfrac{1}{3}z_x w + \tfrac{1}{6}z^2 + \tfrac{1}{9}w^2 z \Longleftarrow\quad \tfrac{1}{2}\widetilde{\mathbf{f}}z_{xx} - \tfrac{1}{6}\widetilde{\mathbf{h}}z_x + \tfrac{1}{6}\widetilde{\mathbf{f}}z^2 - \tfrac{1}{9}\widetilde{\mathbf{e}}z\ . \end{aligned} \tag{4.45}$$

The integrability condition for the existence of a simultaneous solution to both of these equations is of course

$$w_{,xt} - w_{,tx} = \tfrac{1}{2}\left\{z_{xxx} + zz_x + z_t\right\} \quad , \tag{4.46}$$

as was expected. However, one may also eliminate the z's from Eq. (4.45) which gives us the (new) pde:

$$w_{xxx} - \tfrac{2}{3}w^2 w_x + w_t = 0 \quad , \tag{4.47}$$

often referred to as the "potential KdV" equation since one can acquire it by first setting the u in the KdV equation to w_x, integrating once, and then changing the constants. On the other hand, just doing this integration to acquire the other equation would not, also, provide us with the transformation Eq. (4.45), which actually provides a mapping between solutions of these two equations.

A more complicated, but more interesting and useful, realization is found by modifying this only slightly, with the insertion of a constant parameter, λ:

$$\begin{aligned} \tilde{\alpha} &\to 0 \quad , \\ \tilde{\beta} &\to \tfrac{1}{2}\partial_w \Longleftarrow\quad \tfrac{1}{2}\widetilde{\mathbf{f}};\, , \\ \tilde{\gamma} &\to \tfrac{1}{3}(w^2 - \lambda)\partial_w \Longleftarrow\quad -\tfrac{1}{3}(\widetilde{\mathbf{e}} + \lambda\widetilde{\mathbf{f}})\ , \\ \tilde{\delta} &\to \tfrac{4}{9}\lambda\tilde{\gamma} \Longleftarrow\quad -\tfrac{4}{27}(\lambda\widetilde{\mathbf{e}} + \lambda^2\widetilde{\mathbf{f}})\ , \\ \text{which implies}\quad \widetilde{\mathbf{C}} = [\tilde{\gamma}, \tilde{\beta}] &\to -\tfrac{1}{3}w\partial_w \Longleftarrow\quad -\tfrac{1}{6}\widetilde{\mathbf{h}}\ , \end{aligned} \tag{4.48}$$

which then transforms a solution into a family of solutions, parametrized by λ. In this case the new formulation for the generators $\widetilde{\mathbf{F}}$ and $\widetilde{\mathbf{G}}$ is

$$\begin{aligned} \widetilde{\mathbf{F}} &= -\tfrac{1}{2}\widetilde{\mathbf{f}}z + \tfrac{1}{3}(\widetilde{\mathbf{e}} + \lambda\widetilde{\mathbf{f}})\ , \\ \widetilde{\mathbf{G}} &= +\tfrac{1}{2}\widetilde{\mathbf{f}}z_{xx} - \tfrac{1}{6}\widetilde{\mathbf{h}}z_x + \tfrac{1}{6}\widetilde{\mathbf{f}}z^2 + \tfrac{1}{9}(-\widetilde{\mathbf{e}} + \lambda\widetilde{\mathbf{f}})z - \tfrac{4}{27}\lambda(\widetilde{\mathbf{e}} + \lambda\widetilde{\mathbf{f}})\ . \end{aligned}$$

The integrability equations for this more complicated realization are

$$\begin{aligned} & w_t + w_{xxx} - \tfrac{2}{3}(w^2 - \lambda)w_x = 0 \\ &\quad \Longrightarrow h_t + h_{xxx} - \tfrac{2}{3}h^2 h_x = 0 \text{ with } h(x,t) \equiv w(x, t + \tfrac{2}{3}\lambda x)\ . \end{aligned} \tag{4.49}$$

This of course includes the earlier ones as a special case, when $\lambda = 0$.

Returning, for the moment, to that realization, I want to note that if we make the following new names,

$$\begin{aligned} \underset{\sim}{\mathrm{H}} &\equiv 6\underset{\sim}{\mathrm{C}} \to -2w\partial_w \quad , \\ \underset{\sim}{\mathrm{E}}_+ &\equiv -2\tilde{\beta} \to -\partial_w \quad , \\ \underset{\sim}{\mathrm{E}}_- &\equiv +3\tilde{\gamma} \to w^2\partial_w , \end{aligned} \tag{4.50}$$

then the resulting objects have rather simple commutation realtions:

$$\begin{aligned} [\underset{\sim}{\mathrm{H}}\, , \underset{\sim}{\mathrm{E}}_\pm] &= \pm 2\, \underset{\sim}{\mathrm{E}}_\pm \quad , \\ [\underset{\sim}{\mathrm{E}}_+\, , \underset{\sim}{\mathrm{E}}_-] &= \underset{\sim}{\mathrm{H}} \quad , \end{aligned} \tag{4.51}$$

analogous to the commutation relations for i times the $\hat{z}$- and $\pm$-components of ordinary angular momentum. This allows us to see yet another (interesting) role that the rotation group plays. You might try to see what you can do with creating a similar way of looking at the commutation relations with λ, in Eq. (4.48), taking perhaps the hint that $\frac{9}{4}\tilde{\delta}$ and $\frac{9}{4}[\tilde{\gamma}\, , [\tilde{\gamma}\, , \underset{\sim}{\mathrm{C}}]]$, being linear in λ, could be used to create infinite series of powers of λ with coefficients that, otherwise, look much like the angular-momentum-like ones in Eq. (4.50).

Appendix I. Calculation of the pull-back of $\bar{D}_x$ for Burgers' equation:

Following on the discussion near Eq. (3.9), I insert here the details of the proof that $\bar{D}_x$ for Burgers' equation has the structure that is claimed. We begin by looking at the details of the mapping that defines the subspace where the solutions live. On J^∞ the set $\{x, t, z, z_x, z_t, z_{xx}, z_{xt}, z_{tt}, \ldots\}$ constitute our choice of coordinates, giving us $D_x = \partial_x + z_x\partial_z + z_{xx}\partial_{z_x} + z_{xt}\partial_{z_t} + z_{xxx}\partial_{z_{xx}} + z_{xxt}\partial_{z_{xt}} + \ldots$ while on Y^∞ the coordinates above are functions of a set of intrinsic coodinates, which I take to be $\{x, t, \bar{z}, \bar{z}_x, \bar{z}_{xx}, \bar{z}_{xxx}, \ldots\}$ and I claim that

$$\bar{D}_x = \partial_x + \bar{z}_x\partial_{\bar{z}} + \bar{z}_{xx}\partial_{\bar{z}_x} + \bar{z}_{xxx}\partial_{\bar{z}_{xx}} + \ldots \quad . \qquad (A1.1)$$

To prove this belief we note the explicit correspondence mentioned above:

$$\begin{aligned} z &= \bar{z} \quad , \\ z_x &= \bar{z}_x \quad , \\ z_t &= -(\bar{z}_{xx} + \bar{z}\bar{z}_x) \quad , \\ z_{xx} &= \bar{z}_{xx} \quad , \\ z_{xt} &= -(\bar{z}_{xxx} + \bar{z}\bar{z}_{xx} + \bar{z}_x^2) \quad , \\ z_{tt} &= \bar{z}_{(4)} + 2\bar{z}\bar{z}_{(3)} + 4\bar{z}_1\bar{z}_2 + \bar{z}^2\bar{z}_{(2)} + 2\bar{z}\bar{z}_{(1)}^2 \quad . \end{aligned} \qquad (A1.2)$$

However, although $z = \bar{z}$, it is **certainly not true** that $\partial_z = \partial_{\bar{z}}$ because they must hold different things constant! We recall the chain rule, which says that if

$$z_\sigma = Z_\sigma(\bar{z}_\tau),$$

then

$$\partial_{\bar{z}_\tau} = \left(\frac{\partial Z_\sigma}{\partial_{\bar{z}\tau}}\right)\partial_{z_\sigma}.$$

Using the coordinate restricitions in Eq. (3.11), we then find that

$$\begin{aligned} \partial_{\bar{z}} &= 1\partial_z - z_x\partial_{z_t} - z_{xx}\partial_{z_{xt}} + 2(z_{xxx} + zz_{xx} + z_x^2)\partial_{z_{tt}} + \ldots \quad , \\ \partial_{\bar{z}_x} &= \partial_{z_x} - z\partial_{z_t} - 2z_x\partial_{z_{xt}} + 4(z_{xx} + zz_x)\partial_{z_{tt}} + \ldots \quad , \\ \partial_{\bar{z}_{xx}} &= \partial_{z_{xx}} - \partial_{z_t} - z\partial_{z_{xt}} + (4z_x + z^2)\partial_{z_{tt}} + \ldots \quad , \\ \partial_{\bar{z}_{xxx}} &= \partial_{z_{xxx}} - \partial_{z_{tx}} + 2z\partial_{z_{tt}} + \ldots \quad , \\ \partial_{\bar{z}_{xxxx}} &= \partial_{z_{xxxx}} + \partial_{z_{tt}} + \ldots \quad . \end{aligned} \qquad (A1.4)$$

Then, inserting all this we have

$$\begin{aligned}
\bar{D}_x =& \partial_x + \bar{z}_x \partial_{\bar{z}} + \bar{z}_{xx} \partial_{\bar{z}_x} + \bar{z}_{xxx} \partial_{\bar{z}_{xx}} + \bar{z}_{xxxx} \partial_{\bar{z}_{xxx}} + \dots \\
=& \partial_x + z_x \{\partial_z - z_x \partial_{z_t} - z_{xx} \partial_{z_{xt}} - 2 z_{xt} \partial_{z_{tt}} + ...\} \\
&+ z_{xx} \{\partial_{z_x} - z \partial_{z_t} - 2 z_x \partial_{z_{xt}} - 4 z_t \partial_{z_{tt}} + \dots\} \\
&+ z_{xxx} \{\partial_{z_{xx}} - \partial_{z_t} - z \partial_{z_{xt}} + (4 z_x + z^2) \partial_{z_{tt}} + \dots\} \\
&+ z_{xxxx} \{\partial_{z_{xxx}} - \partial_{z_{tx}} + 2 z \partial_{z_{tt}} + \dots\} + z_{(5)} \{\partial_{z_{(4)}} + \partial_{z_{tt}} + \dots\} + \dots \\
=& \partial_x + z_x \partial_z + z_{xx} \partial_{z_x} - (z_{xxx} + z z_{xx} + z_x^2) \partial_{z_t} + z_{xxx} \partial_{z_{xx}} \\
&- (z_{xxxx} + z z_{xxx} - 2 z_x z_{xx} - z_x z_{xx}) \partial_{z_{xt}} \\
&+ (z_{(5)} + 2 z z_{(4)} + 4 z_x z_{xxx} + z^2 z_{xxx} - 4 z_{xx} z_t - 2 z_x z_{xt}) \partial_{z_{tt}} + \dots \\
=& \partial_x + z_x \partial_z + z_{xx} \partial_{z_x} + z_{xt} \partial_{z_t} + z_{xxx} \partial_{z_{xx}} + z_{xxt} \partial_{z_{xt}} + z_{xtt} \partial_{z_{tt}} + \dots \ = \ D_x,
\end{aligned}$$

as was hoped!