Physics 570

Tangent Vectors, Differential Forms, and Tensors

0. Symbols to describe various Vector and Tensor Spaces

1. We first note that Finley uses the (common) useful mathematical notations that $\mathbb{R}$ stands for the set of all real numbers, and then $\mathbb{R}^n$ is the set of all "n-tuples" of real numbers, while $\mathbb{C}$ stands for the set of all complex numbers. Then $\mathbb{Z}$ denotes the set of all integers, and also $\mathbb{Z}^+$ stands for the set of all non-negative integers.

We use the symbol $\mathcal{M}$ to denote a manifold. If the manifold is arbitrary, it may be, for instance, of dimension n; however, our spacetime is of course a manifold of dimension 4. [For more details on how a manifold is defined, see the notes on the geometrical views on manifolds, vectors, differential forms, and tensors.]

We are also interested in various functions, mapping manifolds into the real numbers, $\mathbb{R}$, and denote the set of all of them by the symbol $\mathcal{F}$, or $\mathcal{F}(\mathcal{M})$ if it is necessary to specify which manifold. We usually are only interested in functions which (at least almost everywhere) possess arbitrarily many continuous derivatives. We refer to such a mapping as being either "of class $C^{(\infty)}$, or simply as "smooth."

2. In principle there are two, physically-different kinds of vectors, each of which has their corresponding vector space. These are the space of **"tangent vectors,"** $\mathcal{T}^1$ (or sometimes just $\mathcal{T}$), and the space of **"differential forms,"** Λ^1, also called cotangent vectors. Tangent vectors are the usual sort of vectors that one regularly uses in, say, freshman physics; however, differential forms are really ways to label, and "add," etc. hypersurfaces (of dimension $n-1$ in an n-dimensional manifold, so that ours will usually be 3-surfaces). This is like the usual, freshman-physics notion of the "normal" to a surface, this time, of course, in a 3-dimensional space.

An equivalent, but very different sounding characterization of differential forms is as (continuous) linear maps of tangent vectors into scalars, e.g., $\mathbb{R}$ or $\mathbb{C}$. Common mathematical

language is to say that differential forms are *dual* to tangent vectors; i.e., Λ^1 is the dual vector space to $\mathcal{T}^1$.

Geometrically, one should think of tangent vectors as (locally) tangent to 1-dimensional curves on $\mathcal{M}$, while differential forms are (locally) tangent to hypersurfaces, i.e., $n-1$-dimensional surfaces. From that point of view, the "action" of a hypersurface on a tangent vector, i.e., what it does to map that vector into a scalar, is to determine how many times the one intersects the other, that being the resultant scalar.

Because the spaces we are considering usually have a scalar product, the distinction between these two kinds becomes blurred a bit since the scalar product can be used to map one kind into the other kind. That is to say, one may use the metric (tensor) to map tangent vectors into differential forms, or vice versa. We will discuss this in some detail in section III.4.b below. Geometrically this is the same as the usual approach where we characterize a hypersurface by the vector which is normal to it.

3.   Proceeding onward to notation for these various geometrical objects, we will generalize the more familiar use of "arrows" over symbols, long used to denote ordinary, 3-dimensional vectors, by using an "over-tilde" to indicate a tangent vector, and an "under-tilde" to indicate a differential form:

   i.) $\widetilde{v}$ denotes a tangent vector defined over (at least) some portion of the (spacetime) manifold; if we need to specify the particular point at which it is being considered, we may write $\widetilde{v}|_P$ to indicate that we are considering it at the point P on the manifold,

   ii.) $\underset{\sim}{\omega}$ is a differential form over the manifold, with the same notation if we need to specify it at some particular point, while

   iii.) $\vec{p}$ will continue to be used for the usual 3-dimensional vectors, while the notation above in (i) is intended for a 4-dimensional vector.

   iv.)   The action of a differential form on a tangent vector will then be written as $\underset{\sim}{\alpha}(\widetilde{v}) \in \mathbb{R}$.

4.   In any vector space, vectors are often described by giving their (scalar) components with

respect to some choice of basis. Our most common spaces of interest will be 4-dimensional; therefore, most examples will come from there.

a. Finley uses the standard symbols $\{\widetilde{\mathbf{e}}_1, \widetilde{\mathbf{e}}_2, \widetilde{\mathbf{e}}_3, \widetilde{\mathbf{e}}_4\} \in \mathcal{T}^1$ to denote an arbitrary basis for a space of tangent vectors; more compactly, we often write $\{\widetilde{\mathbf{e}}_\mu\}_{\mu=1}^4$, or just $\{\widetilde{\mathbf{e}}_\mu\}_1^4$, to mean the same thing.

Given any vector, $\widetilde{v}$, there always exist **unique**, *scalar* quantities, $\{v^\mu \mid \mu = 1, 2, 3, 4\}$, such that

$$\widetilde{v} = v^\mu \, \widetilde{\mathbf{e}}_\mu \ . \tag{2.1}$$

The index on the symbol v^μ is a **superscript**; this will always be true when the vector $\widetilde{v}$ is a "tangent vector." Remember that the Einstein summation convention acts here to cause the above equation to involve a sum over all the allowed values of μ. We also refer to these indices as *"contravariant."* The word comes from the fact that a set of contravariant quantities transforms—under change of basis, for instance—in the opposite way to the way that the basis vectors themselves transform.

b. We also need a choice for basis vectors for differential forms. Habitually, we will use the symbols $\{\boldsymbol{\omega}^\alpha\}_1^4 \in \Lambda^1$ for this basis. We will usually choose these basis elements so that the two basis sets, for tangent vectors and for differential forms, are *reciprocal bases*, which means that

$$\boldsymbol{\omega}^\alpha(\widetilde{\mathbf{e}}_\beta) = \delta_\beta^\alpha \ , \qquad \text{—the Kronecker delta: } \delta_\beta^\alpha = \begin{cases} 1, & \alpha = \beta, \\ 0, & \text{otherwise.} \end{cases} \tag{2.2}$$

c. For an arbitrary $\underset{\sim}{\mu} \in \Lambda^1$, its components are then the set of scalars $\{\mu_\alpha\}_1^4$ such that

$$\underset{\sim}{\mu} = \mu_\alpha \, \boldsymbol{\omega}^\alpha \ , \ \text{ and } \ \underset{\sim}{\mu}(\widetilde{v}) = \mu_\alpha v^\beta \, \boldsymbol{\omega}^\alpha(\widetilde{\mathbf{e}}_\beta) = \mu_\alpha v^\alpha \ . \tag{2.3}$$

The indices on the components of a 1-form are always lower indices, i.e., subscripts. We also refer to such indices as *"covariant".* We also have the following set of useful relations, rather analogous to the behavior of "dot products" which allows us to determine the

components of an arbitrary tangent vector or 1-form when we know explicitly the form of the basis elements: for $\widetilde{x} \in \mathcal{T}^1$ and $\underset{\sim}{\sigma} \in \Lambda^1$, we have

$$x^\alpha = \underset{\sim}{\omega}^\alpha(\widetilde{x}) \quad , \qquad \sigma_\beta = \underset{\sim}{\sigma}(\widetilde{\mathbf{e}}_\beta) \quad . \tag{2.4}$$

5. At any given point, $p \in \mathcal{M}$, there are also other interesting vector spaces. Tensor spaces, in general, are linear, continuous maps of some number, s, of tangent vectors and some number, r, of 1-forms into $\mathbb{R}$. One may also say that they are contravariant of type r and covariant of type s, or simply that they are "of type $[r, s]$." Since a tensor of type $[r, s]$ is a member of the tensor product of r copies of $\mathcal{T}^1$ and s copies of Λ^1, the natural choice of basis is

$$\{\widetilde{\mathbf{e}}_{\mu_1} \otimes \widetilde{\mathbf{e}}_{\mu_2} \otimes \ldots \otimes \widetilde{\mathbf{e}}_{\mu_r} \otimes \underset{\sim}{\omega}^{\lambda_1} \otimes \underset{\sim}{\omega}^{\lambda_2} \otimes \ldots \otimes \underset{\sim}{\omega}^{\lambda_s} \mid \mu_1, \ldots, \mu_r, \lambda_1, \ldots \lambda_s = 1, \ldots, n\} . \tag{2.5}$$

Here the symbol $\otimes$ is referred to as *the tensor product* of the two geometric objects on either side of it. It is a method of considering both objects at the same time, with all possible components of each, but that product is required to preserve linearity, as the most important operation in any sort of vector space. Therefore we require for the tensor product relations like

$$(a\widetilde{e}_1) \otimes (b\widetilde{e}_2 + c\widetilde{e}_3) = ab(\widetilde{e}_1) \otimes (\widetilde{e}_2) + ac(\widetilde{e}_2) \otimes (\widetilde{e}_3) .$$

A good analogy is the so-called outer product (or Kronecker product) of the matrix, or column vector, representing the components of a vector and the transposed matrix, or row vector. The result of the multiplication of these two matrices, in that order, is a larger matrix, with more than one row and column.

Some physically-interesting examples of tensors are given by

a. an ordinary tangent vector, which is of type [1,0],

b. a differential form, or 1-form, which is of type [0,1],

c. the metric tensor, of type [0,2], and symmetric

d. the electromagnetic field tensor, also of type [0,2] but skew-symmetric,

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e.   the stress-energy tensor, of type [1,1], and

f.   the curvature of the manifold, of type [1,3].

g.   Of special interest are the tensor spaces made up of combinations of skew-symmetric tensor products of a number, p, of 1-forms. These objects are often called *p-forms*. We use a special skew-symmetric version of the tensor product in these spaces, referred to as a *Grassmann product*, or, more commonly, just a *"wedge" product*, since it is denoted with the symbol $\wedge$ between the two objects for which this is the product. If we define the wedge product of two basis 1-forms as

$$\underset{\sim}{\omega}^{\alpha} \wedge \underset{\sim}{\omega}^{\beta} \equiv \underset{\sim}{\omega}^{\alpha} \otimes \underset{\sim}{\omega}^{\beta} - \underset{\sim}{\omega}^{\beta} \otimes \underset{\sim}{\omega}^{\alpha} \, , \tag{2.6}$$

then a basis for the vector space of 2-forms, Λ^2, is just

$$\{\underset{\sim}{\omega}^{\alpha} \wedge \underset{\sim}{\omega}^{\beta} \mid \alpha, \beta = 1, \ldots, n \,;\, \alpha < \beta\} \, . \tag{2.7}$$

We then use the associativity of the tensor product to extend this definition of the wedge product to Λ^p, and obtain sets of basis vectors accordingly; for instance, for Λ^3 we have

$$\underset{\sim}{\omega}^{\alpha} \wedge \underset{\sim}{\omega}^{\beta} \wedge \underset{\sim}{\omega}^{\gamma} \equiv \underset{\sim}{\omega}^{\alpha} \otimes \underset{\sim}{\omega}^{\beta} \otimes \underset{\sim}{\omega}^{\gamma} - \underset{\sim}{\omega}^{\beta} \otimes \underset{\sim}{\omega}^{\alpha} \otimes \underset{\sim}{\omega}^{\gamma} + \underset{\sim}{\omega}^{\beta} \otimes \underset{\sim}{\omega}^{\gamma} \otimes \underset{\sim}{\omega}^{\alpha}$$

$$- \underset{\sim}{\omega}^{\alpha} \otimes \underset{\sim}{\omega}^{\gamma} \otimes \underset{\sim}{\omega}^{\beta} + \underset{\sim}{\omega}^{\gamma} \otimes \underset{\sim}{\omega}^{\alpha} \otimes \underset{\sim}{\omega}^{\beta} - \underset{\sim}{\omega}^{\gamma} \otimes \underset{\sim}{\omega}^{\beta} \otimes \underset{\sim}{\omega}^{\alpha} \, , \tag{2.8}$$

$$\left\{\underset{\sim}{\omega}^{\alpha} \wedge \underset{\sim}{\omega}^{\beta} \wedge \underset{\sim}{\omega}^{\gamma} \mid \alpha, \beta, \gamma = 1, \ldots, n \,;\, \alpha < \beta < \gamma\right\} \, .$$

As one can infer from the comments above, the vector space of all p-forms is denoted by Λ^p, and has dimension $\binom{n}{p} = n(n-1)\ldots(n-p+1)/p!$. Therefore, over a manifold of dimension n, the variety of p-forms extends only from $p = 1$ to $p = n$, although often one also takes $\mathcal{F}$, the space of all smooth functions, as though they were 0-forms, Λ^0. It will also allow us to introduce a very important mapping, called the *exterior derivative*, $d : \Lambda^p \to \Lambda^{p+1}$, which is a generalization of the ordinary 3-dimensional notions of "gradient," "curl," and "divergence."

For **p-forms** of rank higher than 1, we will also use an "under-tilde" so that such a symbol does not automatically tell us "the value of p." It is very unlikely that this will cause much

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confusion, however, since we will only truly discuss relatively few distinct p-forms; each one of interest will generally have its own particular symbol, never used for anything else. More discussion is given elsewhere.

h.   Area and volume forms, and the totally anti-symmetric tensor density, related to Levi-Civita's symbol, $\epsilon^{\mu\nu\lambda\eta}$. (Hodge) duality is a very useful map from $\Lambda^p \to \Lambda^{n-p}$, which is created by the Levi-Civita symbol and the metric. [More discussion about the Levi-Civita symbol, and the volume n-form, will be given later.]

For higher-rank tensors that are not p-forms, we will put their special symbol in boldface letters when not explicitly indicating their indices; an example will be the metric tensor, of type [0,2], denoted $\boldsymbol{\eta}$.

III.   Matrix presentations for the components of geometrical objects

1.   Matrices are not, from first principles, true geometrical objects. Rather, they are just arrays of quantities along with (standard) rules concerning their display, and their manipulation to create new matrices. One must therefore have conventions concerning the rules that are used to relate the matrix arrays of quantities with the arrays of objects that form the set of components of some geometrical object. We will discuss below Finley's conventions concerning such things, which are commonly agreed on, but not universally.

2.   To begin with, it is very common to use matrices to present, or display, the components of a vector relative to some specific choice of the set of basis vectors. Of course the components would change were one to change the choice of basis vectors; therefore, a very important geometrical question is how this happens. That will be discussed later, as we are now concerned—more so than with anything else—with just conventions concerning notation. It is convenient, and very conventional, to display the set of components of a vector by means of a matrix with only one column, usually referred to as a column-vector. However, since we have two sorts of vectors, we generalize this convention—not done by all authors—so that we represent our geometrical vectors so that

a.   contravariant components are represented via **column-vectors**, i.e., **matrices with only one column**, and

b.   covariant components are represented via row-vectors,

where **row-vectors** are actually **matrices with only one row**.

An additional problem, however, is that different choices of basis would generate different sets of scalar quantities for the very same vector; therefore $\widetilde{x}$ is not **equal** to the set of scalars, x^μ, so that, instead of writing an equality, I use the symbol $\Longrightarrow$, which is read as "is represented by." When this symbol is used, it should remind us that we must know which particular choice of basis has been made, and what the choice of ordering is, before we may understand whatever comes next. Finally, then, examples might be

$$V^a\,\widetilde{\mathbf{e}}_a = \widetilde{V} \;\Longrightarrow\; V^a \;\Longrightarrow\; \begin{pmatrix} V^1 \\ V^2 \\ \vdots \end{pmatrix} ,$$

$$\Upsilon_b\,\omega^b = \underset{\sim}{\Upsilon} \;\Longrightarrow\; \Upsilon_b \;\Longrightarrow\; \begin{pmatrix} \Upsilon_1 & \Upsilon_2 & \ldots \end{pmatrix} . \tag{3.1}$$

3.   We also need conventions to describe the generic elements of a matrix. My standard convention for matrices is that the entries within matrices are labelled by their **row** and their **column**, with the row index coming first. We use this convention independent of whether the indices for the components are upper or lower, i.e., whether they are being used to form a presentation of tensorial objects that transform contravariantly or covariantly. Therefore, for example, we could easily have (different) matrices with elements denoted in the following ways: $F^a{}_b$, G_{ab}, $H_a{}^b$, J^{ab}. For instance $F^3{}_2$ is a notation labeling the element of the matrix F that is found in the 3rd row and the 2nd column.

In each case the row index is the one that comes first.

4.   We also use matrices to display the components of other geometrical quantities, especially those sorts of tensors that relate two vectors at once—called second-rank tensors. Here we give appropriate **conventions** for two, very common examples, the metric tensor and the electromagnetic field tensor.

a. The metric tensor is a bilinear mapping that takes two tangent vectors and gives a scalar—their "scalar product." This makes it an element of $\Lambda^1 \otimes \Lambda^1$, a basis for which is the set of all $\widetilde{\omega}^\alpha \otimes \widetilde{\omega}^\beta$. Therefore we can define the components in either of two equivalent ways:

$$\boldsymbol{\eta} = \eta_{\alpha\beta}\, \widetilde{\omega}^\alpha \otimes \widetilde{\omega}^\beta \quad, \qquad \text{or} \quad \eta_{\alpha\beta} = \boldsymbol{\eta}\left(\widetilde{\mathbf{e}}_\alpha, \widetilde{\mathbf{e}}_\beta\right) \quad, \tag{3.2}$$

and then we can use a matrix representation to display these components:

$$\boldsymbol{\eta} \implies \eta_{\mu\nu} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \,, \quad \mu, \nu = 1, 2, 3, 4 \quad, \tag{3.3}$$

Notice that $\boldsymbol{\eta}$ is that particular metric tensor that is commonly used in special relativity, for an "orthonormal" metric. In a more general context, I would use the symbol $\mathbf{g} = g_{\alpha\beta}\, \widetilde{\omega}^\alpha \otimes \widetilde{\omega}^\beta$ to refer to the metric tensor, that determines the interval.

b. As already suggested, the existence on our manifold of a metric "blurs" the (calculational) distinction between 1-forms and tangent vectors; more precisely, it allows a mapping between them: since $\boldsymbol{\eta} : \mathcal{T}^1 \otimes \mathcal{T}^1 \to \mathbb{R}$, the result of giving it one vector but not the other would be some tensor which, when given the second vector, would give you a scalar; however, that is exactly what a 1-form is, so that we see that there is a different behavior for the metric, namely $\boldsymbol{\eta} : \mathcal{T}^1 \to \Lambda^1$. We use the notation $\boldsymbol{\eta}(\widetilde{v}, \cdot)$ to indicate this mapping, where the centered dot is a "placeholder" awaiting the arrival of a second tangent vector so that the metric may compute the scalar product of the two; therefore, $\boldsymbol{\eta}(\widetilde{v}, \cdot)$ is actually a 1-form, able and willing to give a scalar for any vector given it, and in a continuous, linear fashion. We may "explain" all this in index notation as follows:

$$\forall\, \widetilde{v} = v^\alpha\, \widetilde{\mathbf{e}}_\alpha \in \mathcal{T}^1 \,, \quad \exists!\; \underset{\sim}{v} = v_\beta\, \widetilde{\omega}^\beta \in \Lambda^1 \;\; \text{where} \;\; v_\beta \equiv \eta_{\beta\alpha}\, v^\alpha \,,$$

$$\text{so that } \underset{\sim}{v}(\widetilde{w}) = v_\beta\, w^\beta = \eta_{\beta\alpha}\, v^\alpha w^\beta = \eta_{\alpha\beta}\, v^\alpha w^\beta = \boldsymbol{\eta}(\widetilde{v}, \widetilde{w}) \; \equiv \; \widetilde{v} \cdot \widetilde{w} \,. \tag{3.4}$$

This process is usually just referred to as "lowering an index," from contravariant to covariant. As well, at the end of the equation one sees that it is also common practice to simply use a "centered *dot*" to indicate the scalar product.

c. Clearly the process should have a reverse as well, since we may "undo" the process of lowering the index by using the inverse matrix for the matrix presenting $\boldsymbol{\eta}$, which we can refer to by the symbol $\boldsymbol{\eta}^{-1}$, although we will usually use exactly the same symbol for the inverse, but with the indices raised:

$$\boldsymbol{\eta}^{-1} \implies \eta^{\mu\nu} \quad \text{where} \quad \eta^{\mu\nu}\eta_{\nu\lambda} \equiv \delta^{\mu}_{\lambda} \impliedby \mathbf{I}_4 \ . \tag{3.5}$$

Having this inverse, we immediately have two different functions that it can perform:

i.) The first is simply that it maps 1-forms to tangent vectors:

$$\boldsymbol{\eta}^{-1} : \Lambda^1 \to \mathcal{T}^1 \ , \text{where} \ \forall \underset{\sim}{\alpha} = \alpha_\mu \underset{\sim}{\boldsymbol{\omega}}^\mu \in \Lambda^1 \ , \quad \exists! \ \widetilde{\alpha} = \alpha^\mu \widetilde{\mathbf{e}}_\mu \in \mathcal{T}^1$$

$$\text{with } \alpha^\mu \equiv \eta^{\mu\nu}\alpha_\nu \text{ and } \forall \underset{\sim}{\beta} \in \Lambda^1 \ , \quad \underset{\sim}{\beta}(\widetilde{\alpha}) = \beta_\mu \, \alpha^\mu = \beta_\mu \eta^{\mu\nu}\alpha_\nu \ . \tag{3.6}$$

This process, of course, is referred to as "raising an index," from covariant to contravariant.

ii.) The second function is already put into evidence in the last phrases of the equation above, namely we may use $\boldsymbol{\eta}^{-1}$ as a scalar product in the vector space of differential forms, Λ^1:

$$\boldsymbol{\eta}^{-1} : \Lambda^1 \times \Lambda^1 \to \mathbb{R} \ , \quad \text{via} \quad \boldsymbol{\eta}^{-1}(\underset{\sim}{\alpha}, \underset{\sim}{\beta}) \equiv \eta^{\mu\nu}\alpha_\mu\beta_\nu \ \equiv \ \underset{\sim}{\alpha} \cdot \underset{\sim}{\beta} \ \in \mathbb{R} \ . \tag{3.7}$$

Do note that **all of the above concerning raising and lowering indices** applies equally well when using some (vector) basis where the metric might have quite a different form than does the simple, Minkowski presentation of $\eta_{\mu\nu}$. Under those circumstances, we might well refer to the metric as $\mathbf{g}$, although in spacetime this would still be a matrix with the same signature and significance relative to the interval.

d. The electromagnetic tensor is a skew-symmetric, second-rank tensor, and therefore is actually a 2-form, i.e., an element of Λ^2. Therefore we use the (skew-symmetric) wedge products that provide a basis for Λ^2:

$$\underset{\sim}{F} \equiv \tfrac{1}{2}F_{\alpha\beta} \, \underset{\sim}{\boldsymbol{\omega}}^\alpha \wedge \underset{\sim}{\boldsymbol{\omega}}^\beta \quad , \tag{3.8}$$

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where the $\frac{1}{2}$ is a convention making it consistent with a different possible definition, namely $\underset{\sim}{F} = F_{\alpha\beta}\underset{\sim}{\omega}^{\alpha} \otimes \underset{\sim}{\omega}^{\beta}$, and reminding us that the components are skew-symmetric, i.e., $F_{\alpha\beta} = -F_{\beta\alpha}$.

In 4 dimensions, any skew-symmetric, second-rank tensor is specified by exactly 6 degrees of freedom; therefore, it is customary to label those 6 degrees of freedom via a pair of 3-dimensional vectors. In the case of the electromagnetic tensor, we refer to those two 3-vectors as "the electric field," $\vec{E}$, and "the magnetic field," $\vec{B}$, justifying the following representations, with respect to the bases vectors noted in Eq. (3.8) above:

$$\underset{\sim}{F} \implies F_{\mu\nu} = \begin{pmatrix} 0 & B^z & -B^y & E_x \\ -B^z & 0 & B^x & E_y \\ B^y & -B^x & 0 & E_z \\ -E_x & -E_y & -E_z & 0 \end{pmatrix} \;, \quad \mu, \nu = 1, 2, 3, 4 \quad, \quad \text{(Finley's order)}$$

$$(3.9a)$$

The slightly peculiar arrangement of the components of the magnetic field, $\vec{B}$, in the 4×4 matrix comes from the fact that we usually find that the magnetic field finds itself in cross products—in 3-dimensional space. Therefore we note that if we take the (3-dimensional) portion of the matrix above that contains the magnetic field, and let it operate on a 3-dimensional vector, we have

$$\begin{pmatrix} 0 & B^z & -B^y \\ -B^z & 0 & B^x \\ B^y & -B^x & 0 \end{pmatrix} \begin{pmatrix} A^x \\ A^y \\ A^z \end{pmatrix} = \begin{pmatrix} A^y B^z - A^z B^y \\ A^z B^x - A^x B^z \\ A^x B^y - A^y B^x \end{pmatrix} \impliedby \vec{A} \times \vec{B} = -\vec{B} \times \vec{A}. \quad (3.9b)$$

Therefore it is often the case that we specify that portion of the matrix with the symbol $-\vec{B}\times$ as in the following

$$\underset{\sim}{F} \implies F_{\mu\nu} = \begin{pmatrix} -\vec{B}\times & \vec{E} \\ -\vec{E}^T & 0 \end{pmatrix} \;. \tag{3.9c}$$

Just as an additional help, I note that when one has these matrices the choice for the ordering of the indices creates a considerable difference in the appearance of the displayed matrix. Therefore, the electromagnetic tensor, expressed as a matrix (as above), but with the indices ordered according to the order used in the textbook by Schutz, i.e., $(0, 1, 2, 3)$, gives

the following appearance:

$$F_{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B^z & -B^y \\ E_y & -B^z & 0 & B^x \\ E_z & B^y & -B^x & 0 \end{pmatrix} \quad , \quad \alpha, \beta = 0, 1, 2, 3 \quad , \quad \text{(Schutz's order)} \qquad (3.9')$$

Following our discussion of raising and lowering indices, we may now consider various different "index locations" for the components of the electromagnetic tensor, all presented, as usual, in Finley's index ordering:

$$F^\mu{}_\nu \equiv \eta^{\mu\lambda} F_{\lambda\nu} \implies \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & B^z & -B^y & E_x \\ -B^z & 0 & B^x & E_y \\ B^y & -B^x & 0 & E_z \\ -E_x & -E_y & -E_z & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & B^z & -B^y & E_x \\ -B^z & 0 & B^x & E_y \\ B^y & -B^x & 0 & E_z \\ E_x & E_y & E_z & 0 \end{pmatrix} , \qquad (3.10)$$

$$F_\mu{}^\nu \equiv F_{\mu\lambda} \eta^{\lambda\nu} \implies \begin{pmatrix} 0 & B^z & -B^y & E_x \\ -B^z & 0 & B^x & E_y \\ B^y & -B^x & 0 & E_z \\ -E_x & -E_y & -E_z & 0 \end{pmatrix} \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & B^z & -B^y & -E_x \\ -B^z & 0 & B^x & -E_y \\ B^y & -B^x & 0 & -E_z \\ -E_x & -E_y & -E_z & 0 \end{pmatrix} . \qquad (3.11)$$

It should be noted that these matrix presentations of the electromagnetic field tensor when it has one index contravariant and one covariant do not give skew-symmetric matrices. Skew symmetry on the indices, or symmetry, requires that the indices be of the same type!

IV. Display of Matrix Arithmetic without explicit indices

1. Basic Notions:

We sometimes want to conserve ink, and simplify the life of the reader, by not explicitly writing out the indices on symbols representing matrices. In fact, this justifies the fact that there are very standard conventions concerning matrix multiplication. To proceed, we first give notation to relate the statement that some matrix is named A, and the statement that the elements of that matrix are labelled by the symbols $A^\alpha{}_\mu$ through the following convention:

$$A = ((A^\alpha{}_\nu)) \qquad \text{means } A \text{ is the matrix with components } A^\alpha{}_\nu. \tag{4.1}$$

Note that the *transpose* of a matrix requires switching its rows and columns; therefore, we could have the following examples, where we use an upper ^{T} to denote the transpose of a given matrix:

$$A = ((A^\alpha{}_\nu)) \quad \Longleftrightarrow \quad A^T = ((A_\nu{}^\alpha)) \quad \text{and} \quad B = ((B_{ab})) \quad \Longleftrightarrow \quad B^T = ((B_{ba})) . \tag{4.2}$$

We may then state succinctly the rules concerning matrix multiplication, of two matrices, A and B, say:

$$C \equiv AB , \quad \Longleftrightarrow \quad C^a{}_b = A^a{}_e B^e{}_b , \tag{4.3}$$

where the symbol $\Longleftrightarrow$ is to be read as meaning "if and only if."

It is perhaps also worth noting that a matrix with a contravariant row index and a covariant column index—the most usual form we see—is one that is presenting an operator that maps tangent vectors into tangent vectors; i.e., we have the following relationship between a linear operator and its matrix presentation:

$$A : \mathcal{T}^1 \to \mathcal{T}^1 \quad \Longleftrightarrow A = ((A^\mu{}_\nu)) \text{ or, equivalently } A \; \Longrightarrow \; A^\mu{}_\nu . \tag{4.4}$$

This is consistent with our various conventions: we begin with a tangent vector, which has contravariant indices, and the final result of the action of our operator on that tangent vector

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still has just one contravariant (free) index, so it should still be a representation of a tangent vector.

On the other hand an operator, B, that, for instance, is presented by a matrix with two covariant indices maps tangent vectors into 1-forms:

$$B : \mathcal{T}^1 \to \Lambda^1 \quad \Longleftrightarrow B \implies B_{\mu\nu} \ . \tag{4.5}$$

A slightly different example begins with the tensor which has as components the (usual) Kronecker delta symbol, $\delta^a{}_b$, which has either the value 1 or 0, depending on whether a equals b, or not; it should surely be conceived of as denoting the components of the identity matrix, I. Therefore we can have the parallel statements, as above, one using the conventions of matrix multiplication, and the second using the rules for indices:

$$AB = I = BA \quad \Longleftrightarrow \quad B = A^{-1} \ ,$$
$$A^\beta{}_\nu \, B^\nu{}_\alpha = \delta^\beta{}_\alpha \ , \qquad B^\mu{}_\alpha \, A^\alpha{}_\nu = \delta^\mu{}_\nu \ . \tag{4.6}$$

Yet more structure enters when we work out products of three matrices:

$$W_{\mu\nu} = A^\alpha{}_\mu \, A^\beta{}_\nu \, M_{\alpha\beta} \quad \Longleftrightarrow \quad W = A^T M A \ ,$$
$$J^\mu{}_\nu = (A^{-1})^\mu{}_\alpha \, A^\beta{}_\nu \, F^\alpha{}_\beta \quad \Longleftrightarrow \quad J = A^{-1} F A \quad . \tag{4.7}$$

In the first line of Eqs. (4.7) the transpose of the matrix A is necessary because of the ordering of the indices. Matrix multiplication always sums the column indices of the matrix on the left with the row indices of the matrix on the right. Of the two summations indicated on that line, via the Einstein summation convention, the second sum has the correct ordering for matrix multiplication, and so is indicated simply by the matrix product GA. However, the first sum has the opposite ordering, necessitating the use of the transpose on the matrix A, so that the matrix multiplication rules represent correctly the desired summation, namely $A^T G$.

2. Examples generated by **change of basis vectors**:

It is probably important to preface these remarks that the entire "scheme" of how geometric objects transform when their components are considered with respect to different choices of

a basis set is the place where more classical (à la 1930's to, perhaps, 1960's) treatments, and definitions, of tensor analysis begin. Therefore, this material is really more important than its location here might immediately suggest!

We begin by considering two distinct choices for basis elements of the cotangent space (of differential forms), namely $\{\boldsymbol{\omega}^{\alpha}\}$ and $\{\boldsymbol{\sigma}^{\mu}\}$. Since either set forms a basis, we may immediately write down each member of the one choice of basis in terms of the members of the other; i.e., there exists uniquely a (square array) of scalar quantities $A^{\alpha}{}_{\mu}$ and, alternatively, the quantities $B^{\mu}{}_{\alpha}$ such that

$$\boldsymbol{\omega}^{\beta} = A^{\beta}{}_{\nu}\,\boldsymbol{\sigma}^{\nu}\,, \qquad \boldsymbol{\sigma}^{\mu} = B^{\mu}{}_{\alpha}\,\boldsymbol{\omega}^{\alpha} \quad . \tag{4.8}$$

Using the elements of these arrays, we may phrase the fact that if one goes "backward," she should surely arrive at the place she started; i.e., we must have the relationships

$$A^{\beta}{}_{\nu}\,B^{\nu}{}_{\alpha} = \delta^{\beta}{}_{\alpha}\,, \qquad B^{\mu}{}_{\alpha}\,A^{\alpha}{}_{\nu} = \delta^{\mu}{}_{\nu}\,. \tag{4.9}$$

Continuing, we can easily see that it would be possible, and probably advisable, to consider these square arrays as square matrices, where we give the (square) array involving the quantities $A^{\alpha}{}_{\mu}$ the name A, and do the similar thing of giving the name B to the square array with element $B^{\nu}{}_{\alpha}$.

It is also a very useful notational tool to now also create new sorts of matrices, which contain elements which are 1-forms, via the following presentations, we may rewrite Eqs. (4.8) and (4.9) in pure matrix form, where Ω and Σ are column vectors with the respective 1-form basis elements as their elements:

$$((\boldsymbol{\omega}^{\mu})) = \begin{pmatrix} \boldsymbol{\omega}^{1} \\ \boldsymbol{\omega}^{2} \\ \boldsymbol{\omega}^{3} \\ \boldsymbol{\omega}^{4} \end{pmatrix} \Longleftarrow \Omega\,, \quad ((\boldsymbol{\sigma}^{\nu})) = \begin{pmatrix} \boldsymbol{\sigma}^{1} \\ \boldsymbol{\sigma}^{2} \\ \boldsymbol{\sigma}^{3} \\ \boldsymbol{\sigma}^{4} \end{pmatrix} \Longleftarrow \Sigma\,, \tag{4.10}$$

$$\Longrightarrow \quad \Omega = A\,\Sigma\,, \qquad \Sigma = B\,\Omega\,, \qquad AB = I = BA\,, \text{ or } B = A^{-1}\,.$$

Knowing that the geometrical objects themselves, i.e., the local tangent planes to physical hypersurfaces (3-surfaces in our case), are independent of any choice of basis, we can induce

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transformations of various other associated objects. We begin with the components of an arbitrary 1-form $\underset{\sim}{\tau}$:

$$\underset{\sim}{\tau} = \tau_\alpha \, \underset{\sim}{\omega}^\alpha \qquad \text{or} \quad \underset{\sim}{\tau} \implies T \equiv ((\tau_\alpha)) \text{ , relative to the basis } \{\underset{\sim}{\omega}^\alpha\}$$

$$\underset{\sim}{\tau} = \tau'_\mu \, \underset{\sim}{\sigma}^\mu \qquad \text{or} \quad \underset{\sim}{\tau} \implies T' \equiv ((\tau'_\mu)) \text{ , relative to the basis } \{\underset{\sim}{\sigma}^\mu\} \qquad (4.11)$$

$$\underset{\sim}{\sigma}^\mu = (A^{-1})^\mu{}_\alpha \, \underset{\sim}{\omega}^\alpha \qquad \implies \qquad \tau'_\mu = A^\alpha{}_\mu \, \tau_\alpha \ \text{ or } \quad T' = TA \ ,$$

where we have agreed that the matrix presentation of the components of a 1-form, such as $\underset{\sim}{\tau}$, being covariant, would be taken as a row-vector, which insures that the order indicated above for the matrix multiplication is indeed the correct order.

We may then consider the reciprocal bases for tangent vectors, relative to the two bases $\{\underset{\sim}{\omega}^\alpha\}$ and $\{\underset{\sim}{\sigma}^\mu\}$, respectively, as follows:

$$\underset{\sim}{\omega}^\alpha(\widetilde{\mathbf{e}}_\beta) = \delta^\alpha{}_\beta \qquad \text{and} \ \ \underset{\sim}{\sigma}^\mu(\widetilde{\mathbf{f}}_\nu) = \delta^\mu{}_\nu \ ,$$

$$\underset{\sim}{\sigma}^\mu = (A^{-1})^\mu{}_\alpha \, \underset{\sim}{\omega}^\alpha \qquad \implies \qquad \widetilde{\mathbf{f}}_\mu = A^\alpha{}_\mu \, \widetilde{\mathbf{e}}_\alpha \ , \ \text{ or } \ F = E \, A \ , \qquad (4.12)$$

where we have named the row vectors (with vector entries) called $F = ((\widetilde{\mathbf{f}}_\mu))$ and $E = ((\widetilde{\mathbf{e}}_\mu))$. This gives us the capability to look at the components of an arbitrary tangent vector, $\widetilde{v}$:

$$\widetilde{v} = v^\alpha \, \widetilde{\mathbf{e}}_\alpha \qquad \text{or} \quad \widetilde{v} \implies V \equiv ((v^\alpha)) \text{ , relative to the basis } \{\widetilde{\mathbf{e}}_\alpha\}$$

$$\widetilde{v} = v'^\mu \, \widetilde{\mathbf{f}}_\mu \qquad \text{or} \quad \widetilde{v} \implies V' \equiv ((v'^\mu)) \text{ , relative to the basis } \{\widetilde{\mathbf{f}}_\mu\} \qquad (4.13)$$

$$\underset{\sim}{\sigma}^\mu = (A^{-1})^\mu{}_\alpha \, \underset{\sim}{\omega}^\alpha \qquad \implies \qquad v'^\mu = (A^{-1})^\mu{}_\alpha \, v^\alpha \ \text{ or } \quad V' = A^{-1} \, V \ ,$$

where we have agreed that the matrix representation of the components of a tangent vector, such as $\widetilde{v}$, would be taken as a column-vector, so that the order indicated above for the matrix multiplication is again the correct order.

Notice that the components of a 1-form, conventionally taken as lower indices, are such that they transform according to the same matrix, $A = B^{-1}$, as do the tangent vector basis elements, $\{\widetilde{\mathbf{e}}_\alpha\}$; it is for this reason that they are referred to as "covariant," in the sense that they transform in the same manner. On the other hand, we see that the components of a tangent vector transform in the inverse manner to the basis elements for tangent vectors, for which reason they are referred to as "contravariant."

Following the reasoning above, one may now generalize and discover the transformation laws of more complicated objects, i.e., higher-rank tensors. This time, again, I give two examples. Firstly, we consider the metric tensor, $\mathbf{g} \in \Lambda^1 \otimes \Lambda^1$:

$$\mathbf{g} = g_{\alpha\beta}\, \underset{\sim}{\boldsymbol{\omega}}^\alpha \otimes \underset{\sim}{\boldsymbol{\omega}}^\beta \quad \text{or} \quad \mathbf{g} \implies G \equiv ((g_{\alpha\beta})) \text{ , relative to the basis } \{\underset{\sim}{\boldsymbol{\omega}}^\alpha\}$$

$$\mathbf{g} = g'_{\mu\nu}\, \underset{\sim}{\boldsymbol{\sigma}}^\mu \otimes \underset{\sim}{\boldsymbol{\sigma}}^\nu \quad \text{or} \quad \mathbf{g} \implies G' \equiv ((g'_{\mu\nu})) \text{ , relative to the basis } \{\underset{\sim}{\boldsymbol{\sigma}}^\mu\} \qquad (4.14)$$

$$\underset{\sim}{\boldsymbol{\sigma}}^\mu = (A^{-1})^\mu{}_\alpha\, \underset{\sim}{\boldsymbol{\omega}}^\alpha \quad \implies \quad g'_{\mu\nu} = A^\alpha{}_\mu\, A^\beta{}_\nu\, g_{\alpha\beta} \text{ or } \quad G' = A^T G A \,,$$

The transpose of the matrix in the last line is necessary because of the ordering of the indices.

As a second (important) example, consider the type [1,1] tensor created in Eqs. (3.8-9), but in the [1,1] format described in Eq. (3.10). As given there it has components $F^\alpha{}_\beta$, which gives us

$$F'^\mu{}_\nu = (A^{-1})^\mu{}_\alpha\, A^\beta{}_\nu\, F^\alpha{}_\beta \quad \text{or} \quad F' = A^{-1}\, F\, A \quad . \qquad (4.15)$$

We are not surprised that this sort of transformation looks "familiar." It is a similarity transformation, as one might have encountered in classical or quantum mechanics. Therefore we consider for a moment the following.

3. **The study of matrix transformations has a long history.**

Transformations of the type appropriate for F, as given in Eqs. (4.15), are called *similarity transformations*, $F' = A^{-1}\, F\, A$; they preserve all the eigenvalues of the matrix, and, therefore, also its determinant. For *normal matrices*, this sort of transformation can be used to bring the matrix into diagonal form. (Normal matrices are defined as those which commute with their transpose; both symmetric and skew-symmetric matrices are examples, although there are many more less common ones.)

On the other hand, transformations of the type appropriate for the metric matrix G, as given in Eqs. (4.14) above, are called *congruency transformations*: $G' = A^T G A$. They do not preserve the determinant; instead they preserve the *signature of the matrix*, which is a particular set of +1's, -1's, and 0's for that matrix. Sylvester's theorem says every symmetric matrix has a congruency transformation which will not only diagonalize it, but in fact bring

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it to have only +1, -1, or 0 in all the places on the diagonal. This set, independent of order, is called the signature. (Sometimes, only the sum of these quantities is referred to, also, as the signature; this is completely understandable only when the dimension of the manifold is a priori known.)

Therefore, in a spacetime, the equivalence principle of Einstein assures us that there is always a change of basis for $\mathcal{T}$, effected by an invertible matrix, M, such that the components of the metric can be converted from $g_{\alpha\beta}$ to just those of an arbitrary inertial frame of special relativity, $\eta_{\mu\nu}$:

$$G = M^T H M , \quad \textbf{or} \quad g_{\alpha\beta} = M^\mu{}_\alpha \, \eta_{\mu\nu} \, M^\nu{}_\beta \quad , \tag{4.16}$$

where the symbol H is a capital Greek η.

This fact assures us that we may always, in some neighborhood of a given point in spacetime, find a special basis set for which the components are what they would have been were the spacetime flat and special relativity valid. It is only to be noted that this basis set may not be defined everywhere when the spacetime is not flat.

4. Comments on Determinants of Matrices.

In addition to rules for matrix multiplication, matrices also come equipped with a definition of various scalars created from them. The only important ones we are likely to use are *the trace*, tr A, and *the determinant*, det A.

a. The definition of trace must be given so that it will be invariant under tensor transformation rules, i.e., it should be a scalar; therefore, we can write

$$\text{tr } A \equiv A^\alpha{}_\alpha \quad = g^{\alpha\beta} A_{\beta\alpha} \quad , \tag{4.17}$$

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where $g^{\alpha\beta}$ are the components of the inverse metric tensor, if the manifold admits one to exist there.

If there is no metric tensor, or if it has no inverse, then the trace will not be an invariant quantity. In addition we should explicitly note that $\sum_{\alpha=1}^{4} A_{\alpha\alpha}$ is **not** the trace, since it would have a different value in every distinct coordinate basis.

b. The definition of the determinant of an $n \times n$ matrix is independent of the properties of the underlying manifold. It involves taking products of n elements, one from each row and from each column, in certain orders, with certain signs.

The Levi-Civita symbol, $\epsilon^{b_1 b_2 \ldots b_n} \equiv \epsilon_{b_1 b_2 \ldots b_n}$, has n indices, is skew-symmetric under the interchange of any two of those indices, and is such that $\epsilon^{1234} = +1$. **Since this statement of numerical value is required to be true in whatsoever coordinate system, the Levi-Civita symbol is not a tensor quantity!**

It has been defined precisely so that it creates determinants via summations, as expressed by the **Fundamental Theorem of Determinants** as written below. In fact this theorem is simply a mathematical re-phrasing of the language one learned long ago about how to take determinants, involving "signed minors," etc.:

$$\epsilon^{b_1 b_2 \ldots b_n} A^{a_1}{}_{b_1} A^{a_2}{}_{b_2} \ldots A^{a_n}{}_{b_n} = \epsilon^{a_1 a_2 \ldots a_n} \det(A) . \tag{4.18}$$

The following rather complicated (numerical) relations involving the values of a product of Levi-Civita symbols, are worth writing down, since they are occasionally of use, especially in those cases where one or more pair of the indices are being summed. Their proof is very straightforward but quite lengthy, and will be omitted here:

$$\epsilon^{a_1 \ldots a_n} \epsilon_{b_1 \ldots b_n} = \begin{vmatrix} \delta^{a_1}_{b_1} & \cdots & \delta^{a_1}_{b_n} \\ \vdots & \ddots & \vdots \\ \delta^{a_n}_{b_1} & \cdots & \delta^{a_n}_{b_n} \end{vmatrix} \equiv \delta^{a_1 \ldots a_n}_{b_1 \ldots b_n} \ , \tag{4.19}$$

where the vertical bars imply the determinant of a matrix. For instance, in the simple case of 2 dimensions, the above equation simply says

$$\epsilon^{a_1 a_2} \, \epsilon_{b_1 b_2} = \begin{vmatrix} \delta_{b_1}^{a_1} & \delta_{b_2}^{a_1} \\ \delta_{b_1}^{a_2} & \delta_{b_2}^{a_2} \end{vmatrix} = \delta_{b_1}^{a_1} \delta_{b_2}^{a_2} - \delta_{b_2}^{a_1} \delta_{b_1}^{a_2} \equiv \delta_{b_1 b_2}^{a_1 a_2}$$

$$\implies \quad \epsilon^{a_1 a_2} \, \epsilon_{a_1 b_2} = \delta_{a_1}^{a_1} \delta_{b_2}^{a_2} - \delta_{b_2}^{a_1} \delta_{a_1}^{a_2} = 2 \, \delta_{b_2}^{a_2} - \delta_{b_2}^{a_2} = \delta_{b_2}^{a_2} \; .$$

As well, we will regularly consider some $q \times q$ submatrix of the general matrix given above, and the $q \times q$ determinant made with Kronecker delta entries, as above. We will denote such a determinant with the *generalized Kronecker delta symbol*, that has q upper and q lower indices. Using that notation, we may then write out an expression, easily provable from the general one above, for partial sums of products of Levi Civita symbols. Writing $q + q' \equiv n$, it follows that

$$\epsilon^{a_1 \ldots a_q c_1 \ldots c_{q'}} \, \epsilon_{b_1 \ldots b_q c_1 \ldots c_{q'}} \;\; = \;\; \delta_{b_1 \ldots b_q c_1 \ldots c_{q'}}^{a_1 \ldots a_q c_1 \ldots c_{q'}} \equiv \;\; (q')! \, \delta_{b_1 \ldots b_q}^{a_1 \ldots a_q} \quad . \tag{4.20}$$

In 4 dimensions we have the following explicit values:

$$\epsilon^{1234} = +1 = \epsilon^{2143} = \epsilon^{4321} = \epsilon^{1423} = \; \ldots$$

$$\epsilon^{2134} = -1 = \epsilon^{2341} = \epsilon^{1243} = \epsilon^{4312} = \epsilon^{4123} = \; \ldots \tag{4.21}$$

$$\epsilon^{\alpha\beta\gamma\delta} = 0 \text{ whenever 2 indices are equal.}$$

One should notice that $\epsilon^{\alpha\beta\gamma\delta}$ is **not a tensor**; instead, it is something referred to as a *weighted tensor*. For such an object, in addition to the usual transformation rules, one must multiply it by some power of the square root of the negative of the determinant of the metric. The power in question is called the weight of the tensor. We now begin a round-about discussion as to how to obtain a tensor that is indeed proportional to the Levi-Civita symbol:

V. Hodge Duality for p-forms

    a.) Over 4-dimensional manifolds, there are 5 distinct spaces of p-forms, Λ^p:

        i. Λ^0 is just the space of continuous (C^∞) functions, also denoted by $\mathcal{F}$. We say that it has dimension 1, since no true "directions" are involved.

ii. Λ^1 is the space of 1-forms, already considered; it has as many dimensions as the manifold, so for 4-dimensional spacetime, it has dimension 4.

iii. Λ^2 is the space of 2-forms, i.e., skew-symmetric tensors, or linear combinations of wedge products of 1-forms; therefore in general it has dimension $\frac{1}{2}n(n-1)$, which becomes 6 for 4-dimensional spacetime. A basis can be created by taking all wedge products of the basis set for 1-forms: $\{\omega^\alpha \wedge \omega^\beta \mid \alpha, \beta = 1, \ldots 4; \alpha < \beta\}$.

iv. Λ^3 is the space of 3-forms, i.e., linear combinations of wedge products of 1-forms, three at a time; in general it has dimension $\binom{n}{3} = \frac{1}{6}n(n-1)(n-2)$, which becomes 4 for 4-dimensional spacetime.

v. Λ^4 is the space of 4-forms; in general it has dimension $\binom{n}{4}$. For 4-dimensional spacetime, this is a 1-dimensional space; i.e., every 4-form is proportional to every other; we refer to some particular choice of basis for this space as *the volume form*. (In general n dimensions, the volume form is always an n-form.)

vi. Over n-dimensional spacetime, it is impossible to have more than n things skew all at once; therefore, the volume form is always the last in the sequence of basis sets for p-forms. So, in 4 dimensions, there is no Λ^p for $p \geq 5$.

b.) Working in the usual (local) Minkowski coordinates, where it is reasonable to choose $\{dx, dy, dz, dt\}$ as a basis for 1-forms, we choose the particular 4-form

$$\underset{\sim}{V} \equiv dx \wedge dy \wedge dz \wedge dt \qquad \text{the (standard) volume form} \qquad (5.1)$$

as our choice of a volume form.

If $\{\omega^\alpha\}_1^4$ is an arbitrary basis for 1-forms, we may define the very important quantity $\eta^{\alpha\beta\gamma\delta}$, which gives the "components" of the volume form relative to this choice of basis:

$$\omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma \wedge \omega^\delta \equiv \eta^{\alpha\beta\gamma\delta} \underset{\sim}{V} \quad , \qquad (5.2)$$

$$\underset{\sim}{V} = \frac{1}{4!}\eta_{\alpha\beta\gamma\delta}\, \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma \wedge \omega^\delta \equiv \frac{1}{4!}\{g_{\alpha\rho}g_{\beta\sigma}g_{\gamma\tau}g_{\delta\varphi}\eta^{\rho\sigma\tau\varphi}\}\, \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma \wedge \omega^\delta \; . \qquad (5.3)$$

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This tensor is completely skew-symmetric, i.e., it changes sign when any two indices are interchanged, and so must be proportional to the *Levi-Civita symbol*, $\epsilon^{\alpha\beta\gamma\delta}$, used for determinants. One verifies that the following defines tensors of type [4,0] and [0,4], respectively, related as usual by raising/lowering of indices via the metric tensor:

$$\eta^{\alpha\beta\gamma\delta} = \frac{1}{m}\,\epsilon^{\alpha\beta\gamma\delta} \ , \qquad \eta_{\alpha\beta\gamma\delta} = (-1)^s\,m\,\epsilon_{\alpha\beta\gamma\delta} \ ,$$

$$\text{where} \qquad m \equiv \det(M)\ , \qquad G = M^T\,H\,M\ , \tag{5.4}$$

$$g \equiv detG = m^2\,\det(H) = (-1)^s\,m^2\ , \quad s = 0 \text{ or } 1\ .$$

Of course the values of $\eta^{\alpha\beta\gamma\delta}$ depend on the basis chosen; however, for an orthonormal tetrad, where the metric components are just $\eta_{\mu\nu}$, we have that $\eta^{1234} = +1 = -\eta_{1234}$. On the other hand, in ordinary, Cartesian 3-dimensional space, in an orthonormal frame, we simply have $\eta^{123} = +1 = \eta_{123}$, as expected.

The following rather complicated relations are worth writing down, since they are occasionally of considerable use:

$$\eta^{a_1 \ldots a_n}\,\eta_{b_1 \ldots b_n} \;=\; (-1)^s \begin{vmatrix} \delta^{a_1}_{b_1} & \ldots & \delta^{a_1}_{b_n} \\ \vdots & \ddots & \vdots \\ \delta^{a_n}_{b_1} & \ldots & \delta^{a_n}_{b_n} \end{vmatrix} \equiv (-1)^s\,\delta^{a_1 \ldots a_n}_{b_1 \ldots b_n} \ , \tag{5.5}$$

where the vertical bars imply a determinant, and we regularly intend to denote a $q \times q$ determinant made with Kronecker delta entries, as above, with the *generalized Kronecker delta symbol* indicated above, that has q upper and q lower indices.

Writing $p + p' \equiv n$, it then follows that

$$\eta^{a_1 \ldots a_p c_1 \ldots c_{p'}}\,\eta_{b_1 \ldots b_p c_1 \ldots c_{p'}} \;=\; (-1)^s\,\delta^{a_1 \ldots a_p c_1 \ldots c_{p'}}_{b_1 \ldots b_p c_1 \ldots c_{p'}} \;=\; (p')!\,(-1)^s\,\delta^{a_1 \ldots a_p}_{b_1 \ldots b_p} \ . \tag{5.6}$$

c.) **the (Hodge) dual**, $* : \ \Lambda^p \ \longrightarrow \ \Lambda^{n-p}$

Let $\underset{\sim}{\alpha}$ be an arbitrary p-form; then we denote the (Hodge) dual by $^*\underset{\sim}{\alpha}$, an $(n-p)$-form. They are related as follows:

$$\underset{\sim}{\alpha} = \frac{1}{p!}\, \alpha_{\mu_1\ldots\mu_p}\, \underset{\sim}{\omega}^{\mu_1} \wedge \ldots \wedge \underset{\sim}{\omega}^{\mu_p} \quad ,$$

$$^*\underset{\sim}{\alpha} \equiv \frac{i^{pp'+s}}{p!(p')!}\, \alpha^{b_1\ldots b_p}\, \eta_{b_1\ldots b_p c_1\ldots c_{p'}}\, \underset{\sim}{\omega}^{c_1} \wedge \ldots \wedge \underset{\sim}{\omega}^{c_{p'}} \equiv \frac{1}{(p')!}\,(^*\alpha)_{c_1\ldots c_{p'}}\, \underset{\sim}{\omega}^{c_1} \wedge \ldots \wedge \underset{\sim}{\omega}^{c_{p'}} \quad .$$

(5.7)

The factors of $i \equiv \sqrt{-1}$ have been inserted in **just such a way that** the dual of the dual brings one back to where she started:

$$^*\{^*\underset{\sim}{\alpha}\} = \underset{\sim}{\alpha} \quad .$$

(5.8)

There are various conventions concerning the i's in the definition. My convention, using the factors of i, allows for eigen-2-forms of the * operator, since Eq. (5.8) obviously tells us that the eigenvalues of * are ± 1.

d.) Since the definition of (Hodge) duality appears quite complicated, it is worthwhile writing it down for **all** plausible exemplars that may occur, in our 4-dimensional spacetime. We do this for the standard Minkowski tetrad, $\{dx, dy, dz, dt\}$, and the bases of each Λ^p:

$$\Lambda^1 \leftrightarrow \Lambda^3 : \; * \begin{pmatrix} dx \\ dy \\ dz \\ dt \end{pmatrix} = - \begin{pmatrix} dy \wedge dz \wedge dt \\ dz \wedge dx \wedge dt \\ dx \wedge dy \wedge dt \\ dx \wedge dy \wedge dz \end{pmatrix} , \quad \Lambda^0 \leftrightarrow \Lambda^4 : \; *1 = -i\, dx \wedge dy \wedge dz \wedge dt ,$$

$$\Lambda^2 \leftrightarrow \Lambda^2 : \; * \begin{pmatrix} dx \wedge dy \\ dy \wedge dz \\ dz \wedge dx \end{pmatrix} = -i \begin{pmatrix} dz \wedge dt \\ dx \wedge dt \\ dy \wedge dt \end{pmatrix} , \quad .$$

(5.9)

As an example, consider the electromagnetic 2-form, in Eqs. (2.8), from which we have:

$$\underset{\sim}{F} \implies F_{\mu\nu} = \begin{pmatrix} 0 & B^z & -B^y & E_x \\ -B^z & 0 & B^x & E_y \\ B^y & -B^x & 0 & E_z \\ -E_x & -E_y & -E_z & 0 \end{pmatrix} ,$$

(5.10)

$$^*\underset{\sim}{F} \implies (^*F)_{\mu\nu} = -i \begin{pmatrix} 0 & -E^z & E^y & B_x \\ E^z & 0 & -E^x & B_y \\ -E^y & E^x & 0 & B_z \\ -B_x & 0 - B_y & -B_z & 0 \end{pmatrix} .$$

Note that the map from $\underset{\sim}{F}$ to $i^*\underset{\sim}{F}$ is accomplished by sending $\vec{B} \to -\vec{E}$ and $\vec{E} \to +\vec{B}$.

e. To complete the picture, we also give details for **3-dimensional, Euclidean space**, with Cartesian basis, $\{dx, dy, dz\}$:

$$\Lambda^1 \leftrightarrow \Lambda^2 : \ * \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = - \begin{pmatrix} dy \wedge dz \\ dz \wedge dx \\ dx \wedge dy \end{pmatrix} \quad , \qquad \Lambda^0 \leftrightarrow \Lambda^3 : \ *1 = dx \wedge dy \wedge dz \qquad .$$

Valid only in 3-dimensional space, without time as a coordinate.