Physics 570

The Weak-Field Limiting Behavior: Linearized Gravity

I. Introduction

We want to study gravity as a weak "perturbation" away from the Minkowski spacetime (of special relatiity). We therefore suppose that this weak gravitational field can be described by a tensor of the same tensorial rank as the metric, which we call $h_{\mu\nu}$, **but that, in addition there exists some choice of coordinates such that** one can write the true metric, $g_{\mu\nu}$ in the following form:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \ , \tag{1.1}$$

where it is actually the components of the perturbing metric that are each small, and small enough that we will only need to keep the lowest orders of powers of $h_{\mu\nu}$. We also want to introduce the additional constraint, generated by the sort of physics for which this approximation is intended—such as the weak fields generated by the earth, that we have already been discussing withut quite the amount of formalism we are about to introduce, or for gravitational waves received here on earth, which we certainly hope are small when received—that $h_{\mu\nu}$ should vanish as some sort of radial coordinate—like the usual spherical coordinate, r—goes to infinity.

It should, however, be explicitly noted that there could be a coordinate transformation that would cause some of the components to become large; therefore,**this equation also amounts to an assumption about the existence of coordinates that allow this, and a restriction on further coordinates, so as to continue to allow this.** Do remember that this is a perfectly reasonable first approximation to the behavior of gravity near the surface of the earth!

Continuing with the relevant tensorial structures, we can do the following:

$$\begin{aligned} & h^{\mu\nu} \equiv g^{\mu\lambda} g^{\nu\sigma} h_{\lambda\sigma} = \eta^{\mu\lambda}\eta^{\nu\sigma} h_{\lambda\sigma} + O^2 \ , \quad \text{and define} \ \ h \equiv h^{\mu}{}_{\mu} \equiv \eta^{\mu\nu} h_{\mu\nu} \ ; \\ \Longrightarrow \qquad & g^{\mu\nu} = \eta^{\mu\nu} - \epsilon\, h^{\mu\nu} \ ; \qquad g \equiv \det(g_{\mu\nu}) = -1 - \eta^{\mu\nu} h_{\mu\nu} \equiv -1 - h \ , \end{aligned} \tag{1.2}$$

where we have given a symbol for the trace of our small grav. field.

To see this last line of these equalities, concerning the determinant of the metric, g, remember that

$$\begin{aligned} g \equiv \det(g_{\mu\nu}) &= g_{m1}g_{n2}g_{b3}g_{c4}\epsilon^{mnbc} = (\eta_{m1}+h_{m1})(\eta_{n2}+h_{n2})(\eta_{b3}+h_{b3})(\eta_{c4}+h_{c4})\epsilon^{mnbc} \\ &= -1 + h_{m1}\epsilon^{m234} + h_{n2}\epsilon^{1m34} + h_{a3}\epsilon^{12a4} + h_{b4}\epsilon^{123b} \\ &= -1 - h_{11} - h_{22} - h_{33} - h_{44} = -1 - h^{\alpha}{}_{\alpha} + O^2 \ , \end{aligned} \tag{1.3}$$

where of course the determiant of $\eta_{\mu\nu}$ is -1, and it also helps to notice that $h_{m4}\epsilon^{m123} = -h_{m1}\epsilon^{m234}$, and similar relations.

The Christöffel symbols are then of the same order of smallness as is this perturbing metric:

$$\left\{ \begin{matrix} \mu \\ \lambda\eta \end{matrix} \right\} = \tfrac{1}{2}\eta^{\mu\nu}\left[-h_{\lambda\eta,\nu} + h_{\eta\nu,\lambda} + h_{\nu\lambda,\eta}\right] + O^2 \ . \tag{1.4}$$

We can then go ahead and determine the Riemann curvature tensor associated with this field, remembering that we can ignore products of the Christöffel symbols:

$$\begin{aligned} R^{\mu}{}_{\nu\lambda\eta} &= \left\{ \begin{matrix} \mu \\ \nu[\eta \end{matrix} \right\}_{,\lambda]} + O^2 \ , \\ R_{\mu\nu\lambda\eta} &= -\tfrac{1}{2}\, h_{\mu[\lambda,\nu\eta]} - \tfrac{1}{2}\, h_{\nu[\eta,\mu\lambda]} + O^2 \ . \end{aligned} \tag{1.5}$$

This now puts us on our way to the Einstein equations:

$$\begin{aligned} \mathcal{R}_{\mu\lambda} &\equiv \eta^{\nu\eta}R_{\mu\nu\lambda\eta} \\ &= -\tfrac{1}{2}h_{\mu\lambda,\nu}{}^{,\nu} + \tfrac{1}{2}h_{\mu\eta,\lambda}{}^{,\eta} - \tfrac{1}{2}h^{\nu}{}_{\nu,\mu\lambda} + \tfrac{1}{2}h^{\nu}{}_{\lambda,\mu\nu} = \tfrac{1}{2}h^{\nu}{}_{(\mu,\lambda),\nu} - \tfrac{1}{2}h_{\mu\lambda,\nu}{}^{,\nu} - \tfrac{1}{2}h_{,\mu\lambda} \ , \\ \text{and} \quad \mathcal{R} &= h^{\mu\nu}{}_{,\mu\nu} - h^{,\nu}{}_{,\nu} \ . \end{aligned} \tag{1.6}$$

We do not yet write out the Einstein tensor explicitly since the formalism makes it very long and ugly. It is simplified a fair amount by introducing the so-called "trace-reversed" perturbing field:

$$\overline{h}_{\mu\nu} \equiv h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}h \ , \ \text{with}\ \overline{h} \equiv \eta^{\mu\nu}\overline{h}_{\mu\nu} = -h \ . \tag{1.7}$$

Considerable cancellation then gives us

$$2\mathcal{G}_{\mu\nu} = \overline{h}^{\lambda}{}_{(\mu,\lambda\nu)} - \overline{h}_{\mu\nu,\lambda}{}^{,\lambda} - \eta_{\mu\nu}\overline{h}^{\lambda\eta}{}_{,\lambda\eta} = 16\pi T_{\mu\nu} \ , \tag{1.8}$$

where the last equals sign is of course Einstein's equations, ignoring Λ.

One can easily notice the term which might be desired in such an expression, namely the term with the d'Alembert differential operator—the 4-dimensional wave operator—acting on the field in question, since we are looking for gravitational waves: this is the middle term in the expression on the left-hand side, namely

$$\overline{h}_{\mu\nu,\lambda}{}^{,\lambda} \equiv \Box \overline{h}_{\mu\nu} \ . \tag{1.8b}$$

However, there are also several other terms, which are not as desirable.

The plan is then to consider a transformation of the coordinates—which in an ordinary field theory would be called a gauge transformation. Labelling the new coordinates with primes, we can write

$$x^\mu \ \longrightarrow \ x'^\mu \equiv x^\mu + \zeta^\mu \quad \text{or} \quad x^\mu = \delta^\mu_\alpha(x'^\alpha - \zeta^\alpha) \ \Longrightarrow \quad X^\mu{}_\alpha \equiv \frac{\partial x^\mu}{\partial x'^\alpha} = \delta^\mu_\alpha - \zeta^\mu_{,\alpha} \ , \tag{1.9}$$

where we insist that all derivatives of ζ^α should be of the same order, or smaller, than our small perturbing field, $h_{\mu\nu}$.

We can then determine the change in the metric that this causes:

$$\begin{aligned} g'_{\alpha\beta} =& X^\mu{}_\alpha X^\nu{}_\beta g_{\mu\nu} = g_{\alpha\beta} - \zeta^\mu_{,(\alpha} g_{\mu\beta)} \\ \Longrightarrow& h'_{\alpha\beta} = h_{\alpha\beta} - \zeta_{(\alpha,\beta)} \ . \end{aligned} \tag{1.10}$$

To agree that these are actually transformations of the sort that should be called gauge transformations, we now determine their effect on the curvature itself. We begin with the connections:

$$\begin{aligned} \left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\}' = \left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} &- \tfrac{1}{2}\eta^{\alpha\delta}\left(-\zeta_{\beta,\gamma\delta} - \zeta_{\gamma,\beta\delta} + \zeta_{\gamma,\delta\beta} + \zeta_{\delta,\gamma\beta} + \zeta_{\delta,\beta\gamma} + \zeta_{\beta,\delta\gamma}\right) \\ = \left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} &- \zeta^\alpha_{,\beta\gamma} \ , \end{aligned} \tag{1.11}$$

and then pass forward to the curvature:

$$R'^{\alpha}{}_{\beta\gamma\delta} = R^{\alpha}{}_{\alpha\beta\gamma} - \zeta^{\alpha}{}_{,\beta[\gamma\delta]} = R^{\alpha}{}_{\beta\gamma\delta} \ , \tag{1.12}$$

so that the curvature tensor is unchanged by these gauge transformations, i.e., the essential physics is unchanged.

It is common—actually following Landau and Lifshitz, to adopt a so-called *harmonic gauge condition*, where we ask of the gauge functions that they cause the trace—on the two lower indices—of the transformed Christöffel symbols to vanish, which can be written out as follows, using Eqs. (111) above:

$$\Box \zeta^{\alpha} \equiv \zeta^{\alpha,\beta}{}_{,\beta} = \eta^{\beta\gamma} \left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} = \ \dots \ = \overline{h}^{\alpha\beta}{}_{,\beta} \ . \tag{1.13}$$

We now suppose that we are now using the re-gauged coordinates, but no longer write the primes, so that we know that

$$\eta^{\beta\gamma} \left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} = 0 \ , \quad \text{and} \ \ \overline{h}^{\alpha\beta}{}_{,\beta} = 0 \ ,$$

which reduces Einstein's equations to the very simple form:

$$\begin{aligned} \Box \overline{h}_{\mu\nu} &= -\ 16\pi T_{\mu\nu} \ ; \\ \text{or} \ \ \Box h_{\mu\nu} &= -\ 16\pi (T_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu} T) \equiv -16\pi S_{\mu\nu} \ , \end{aligned} \tag{1.14}$$

where T is the trace of $T_{\mu\nu}$, so that

$$\Box \overline{h} = -\Box h = -16\pi T \equiv -16\pi \eta^{\mu\nu} T_{\mu\nu} \ . \tag{1.15}$$

However, we do still have some gauge freedom remaining, namely any further choice of some additional such ζ^{α}—still with derivatives small of the order of $h_{\mu\nu}$, of course—which we will denote by ϵ^{α}, and which satisfy

$$\Box \epsilon^{\alpha} = 0 \ . \tag{1.15}$$