Spherically Symmetric, Static Space-Time

Metric Structure

We take the metric in the following form, along with this choice for (orthonormal) tetrad:

$$\begin{gathered} ds^2 = \mathbf{g} = J(r)\,dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\varphi^2) - H(r)\,dt^2 = (\underset{\sim}{\omega}^r)^2 + (\underset{\sim}{\omega}^\theta)^2 + (\underset{\sim}{\omega}^\varphi)^2 - (\underset{\sim}{\omega}^t)^2\ , \\ \underset{\sim}{\omega}^{\hat r} \equiv \sqrt{J}\,dr\ , \quad \underset{\sim}{\omega}^{\hat\theta} \equiv r\,d\theta\ , \quad \underset{\sim}{\omega}^{\hat\varphi} \equiv r\sin\theta\,d\varphi\ , \quad \underset{\sim}{\omega}^{\hat t} \equiv \sqrt{H}\,dt\ . \end{gathered} \tag{1}$$

The "guess" method of determining the connection 1-forms, $\underset{\sim}{\Gamma}^\mu{}_\nu$, works in a straight-forward way, giving the following results:

$$\textbf{Connections:}\quad \begin{cases} \underset{\sim}{\Gamma}_{\hat r\hat\theta} = -\frac{\underset{\sim}{\omega}^{\hat\theta}}{r\sqrt{J}}\ , \qquad \underset{\sim}{\Gamma}_{\hat r\hat\varphi} = -\frac{\underset{\sim}{\omega}^{\hat\varphi}}{r\sqrt{J}}\ , \quad \underset{\sim}{\Gamma}_{\hat r\hat t} = \frac{H'}{2\sqrt{JH}}\underset{\sim}{\omega}^{\hat t}\ , \\ \underset{\sim}{\Gamma}_{\hat\theta\hat\varphi} = -\frac{\cot\theta}{r}\underset{\sim}{\omega}^{\hat\varphi}\ , \qquad \underset{\sim}{\Gamma}_{\hat\theta\hat t} = 0\ , \qquad \underset{\sim}{\Gamma}_{\hat\varphi\hat t} = 0\ , \end{cases} \tag{2}$$

where the prime is used to indicate derivative with respect to r, i.e., $H' \equiv dH/dr$. The curvature 2-forms are then easily calculated, and give

$$\textbf{Curvatures:}\quad \begin{cases} \underset{\sim}{\Omega}_{\hat r\hat\theta} = \frac{J'}{2rJ^2}\,\underset{\sim}{\omega}^{\hat r}\wedge\underset{\sim}{\omega}^{\hat\theta}\ , & \underset{\sim}{\Omega}_{\hat r\hat\varphi} = \frac{J'}{2rJ^2}\underset{\sim}{\omega}^{\hat r}\wedge\underset{\sim}{\omega}^{\hat\varphi}\ , \\ \underset{\sim}{\Omega}_{\hat r\hat t} = \frac{1}{2\sqrt{JH}}\left(\frac{H'}{\sqrt{JH}}\right)'\underset{\sim}{\omega}^{\hat r}\wedge\underset{\sim}{\omega}^{\hat t}\ , & \underset{\sim}{\Omega}_{\hat\theta\hat\varphi} = \frac{1}{r^2}\left\{1 - J^{-1}\right\}\underset{\sim}{\omega}^{\hat\theta}\wedge\underset{\sim}{\omega}^{\hat\varphi}\ , \\ \underset{\sim}{\Omega}_{\hat\theta\hat t} = \frac{H'}{2rJH}\,\underset{\sim}{\omega}^{\hat\theta}\wedge\underset{\sim}{\omega}^{\hat t}\ , & \underset{\sim}{\Omega}_{\hat\varphi\hat t} = \frac{H'}{2rJH}\,\underset{\sim}{\omega}^{\hat\varphi}\wedge\underset{\sim}{\omega}^{\hat t}\ . \end{cases} \tag{3}$$

Since there are only 4 independent functional quantities involved, it may be useful to define

$$\begin{aligned} \mathbf{A} &\equiv \frac{J'}{2rJ^2} = R_{\hat r\hat\theta\hat r\hat\theta} = R_{\hat r\hat\varphi\hat r\hat\varphi}\ , \qquad & \mathbf{B} &\equiv \frac{1}{r^2}\left\{1 - J^{-1}\right\} = R_{\hat\theta\hat\varphi\hat\theta\hat\varphi}\ , \\ \mathbf{C} &\equiv \frac{H'}{2rJH} = R_{\hat\theta\hat t\hat\theta\hat t} = R_{\hat\varphi\hat t\hat\varphi\hat t}\ , \qquad & \mathbf{D} &\equiv \frac{1}{2\sqrt{JH}}\left(\frac{H'}{\sqrt{JH}}\right)' = R_{\hat r\hat t\hat r\hat t}\ , \end{aligned} \tag{4}$$

Notice that all the components of $R_{\mu\nu\lambda\eta}$ are diagonal in the sense that they are zero unless $\mu\nu$ is the same as $\lambda\eta$, or of course the reverse, $\eta\lambda$.

Riemann Curvature for our spherically-symmetric, static metrics

Returning to our particular case, where we are insisting that we want spherical symmetry, and also no dependence on time, returning to the values already presented in Eqs. (3) and (4), we find that $\mathbf{R}$ is diagonal, along with $\mathbf{N} \equiv 0$:

$$\mathbf{R} \Longrightarrow \begin{pmatrix} \mathbf{M} & \mathbf{N} \\ \mathbf{N}^T & \mathbf{Q} \end{pmatrix}\ , \quad \mathbf{M} = \begin{pmatrix} \mathbf{B} & 0 & 0 \\ 0 & \mathbf{A} & 0 \\ 0 & 0 & \mathbf{A} \end{pmatrix}\ , \qquad \mathbf{Q} = \begin{pmatrix} \mathbf{D} & 0 & 0 \\ 0 & \mathbf{C} & 0 \\ 0 & 0 & \mathbf{C} \end{pmatrix}\ . \tag{6}$$

Notice that the equality of the second and third diagonal elements of each of $\mathbf{M}$ and $\mathbf{Q}$ comes from the spherical symmetry of the problem, i.e., the fact that θ and φ are being treated equally. Therefore, we may now determine the Ricci tensor, $\mathcal{R}_{\mu\nu} \equiv R^{\lambda}{}_{\mu\lambda\nu}$, or the Einstein tensor, $\mathcal{G}_{\mu\nu} \equiv \mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R}^{\lambda}{}_{\lambda}$, and also the conformal tensor. We have

$$\mathcal{R}_{\hat{r}\hat{r}} = 2\mathbf{A} - \mathbf{D} \ , \quad \mathcal{R}_{\hat{\theta}\hat{\theta}} = \mathbf{A} + \mathbf{B} - \mathbf{C} = \mathcal{R}_{\hat{\varphi}\hat{\varphi}} \ , \quad \mathcal{R}_{\hat{t}\hat{t}} = \mathbf{D} + 2\mathbf{C} \ , \tag{7}$$

$$G_{\hat{t}\hat{t}} = \text{trace}(\mathbf{M}) = \mathbf{B} + 2\mathbf{A} \ , \quad G_{\hat{r}\hat{r}} = 2\mathbf{C} - \mathbf{B} \ , \quad G_{\hat{\theta}\hat{\theta}} = \mathbf{D} + \mathbf{C} - \mathbf{A} = G_{\hat{\varphi}\hat{\varphi}} \ , \tag{8}$$

$$\mathcal{R} = \text{trace}(\mathit{Ricci}) = -\text{trace}(\mathit{Einstein}) = 2\,\text{trace}(\mathbf{Q} - \mathbf{M}) = 2(2\mathbf{A} + \mathbf{B} - \mathbf{D} - 2\mathbf{C}) \tag{9}$$

Displayed, also as a 6×6 matrix, the Weyl tensor is also diagonal, and traceless:

$$\begin{aligned} C_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}} = -\rho = C_{\hat{r}\hat{\varphi}\hat{r}\hat{\varphi}} \ , \quad C_{\hat{\theta}\hat{\varphi}\hat{\theta}\hat{\varphi}} = +2\rho \ , \\ C_{\hat{\theta}\hat{t}\hat{\theta}\hat{t}} = +\rho = C_{\hat{\varphi}\hat{t}\hat{\varphi}\hat{t}} \ , \quad C_{\hat{r}\hat{t}\hat{r}\hat{t}} = -2\rho \ , \\ \rho \equiv \frac{1}{6}\left(\mathbf{C} - \mathbf{A} + \mathbf{B} - \mathbf{D}\right) \ . \end{aligned} \tag{10}$$

The conformal tensor may also be described in terms of what are usually called the Petrov scalars, which turn out to be in this case:

$$\Psi_{++} = \Psi_{+} = 0 = \Psi_{-} = \Psi_{--} \ , \quad \Psi_0 = -2\rho \ . \tag{11}$$

which corresponds, generically, to Petrov Type D.

Geodesic Structure

The equations for parallel transport of an arbitrary vector $\widetilde{w}$, along a curve with tangent vector $\widetilde{u}$, may be written in the following form, where prime means the action of $\widetilde{u}$ on the scalar in question, i.e., it is the derivative with respect to the parameter along the curve:

$$\nabla_{\widetilde{u}} \widetilde{w} = 0 = \begin{cases} (w^{\hat{r}})' - \dfrac{1}{r\sqrt{J}}\left(w^{\hat{\theta}}u^{\hat{\theta}} + w^{\hat{\varphi}}u^{\hat{\varphi}}\right) + \dfrac{H'}{2\sqrt{J}H} w^{\hat{t}}u^{\hat{t}} = 0 \ , \\ (w^{\hat{\theta}})' + \dfrac{1}{r\sqrt{J}} w^{\hat{r}}u^{\hat{\theta}} - \dfrac{\cot\theta}{r} w^{\hat{\varphi}}u^{\hat{\varphi}} = 0 \ , \\ (w^{\hat{\varphi}})' + \dfrac{1}{r\sqrt{J}} w^{\hat{r}}u^{\hat{\varphi}} + \dfrac{\cot\theta}{r} w^{\hat{\theta}}u^{\hat{\varphi}} = 0 \ , \\ (w^{\hat{t}})' + \dfrac{H'}{2\sqrt{J}H} w^{\hat{r}}u^{\hat{t}} = 0 \ , \end{cases} \ . \tag{12}$$

When $\widetilde{u}$ is timelike, i.e., is the tangent vector to a curve describing a possible motion for a physical creature, and also when we use that creature's proper time, τ, as the parameter along that curve, then we refer to it as the *4-velocity* for that creature, and because we are using the proper time as the parameter, it has the property that $\widetilde{u}^2 = -1$. If we use an orthonormal basis for our vectors, then we may divide it further, and easily relate it to the more ordinary 3-velocity, $\vec{v}$. In such an orthonormal basis, we may write

$$\widetilde{u}^2 = (\vec{u})^2 - (u^{\hat{t}})^2 = -1 \; , \qquad \begin{cases} u^{\hat{t}} = \gamma \; , \\ u^{\hat{i}}/u^{\hat{t}} = v^i \; , \end{cases} \; , \tag{13a}$$

so that the content of the statement that $\widetilde{u}^2 = -1$ is now the same as the familiar statement that $\gamma^{-2} = 1 - (\vec{v})^2$. In our problem this means

$$\begin{gathered} u^{\hat{\mu}} \Longrightarrow \quad (\sqrt{J}\, r' , \quad r\,\theta' , \quad r \sin\theta\, \varphi' , \quad \sqrt{H}\, t') \; , \\ \gamma = \sqrt{H}\, t' = \sqrt{H} \frac{dt}{d\tau} \; , \\ v^{\hat{r}} = u^{\hat{r}}/u^{\hat{t}} = \sqrt{\frac{J}{H}} \frac{dr}{dt} \; , \quad v^{\hat{\theta}} = u^{\hat{\theta}}/u^{\hat{t}} = \frac{r}{\sqrt{H}} \frac{d\theta}{dt} \; , \quad v^{\hat{\varphi}} = u^{\hat{\varphi}}/u^{\hat{t}} = \frac{r \sin\theta}{\sqrt{H}} \frac{d\varphi}{dt} \; . \end{gathered} \tag{13b}$$

In the case that we choose $\widetilde{u}$ itself as the vector $\widetilde{w}$ that was being parallel transported back in Eqs. (13), i.e., when we are determining equations to insist that $\widetilde{u}$ defines a geodesic path, then the symmetries of the metric immediately allow all of these equations to be integrated, where we put directly into evidence those equations which contain the 3 constants of integration:

$$
\left\{
\begin{array}{l}
\qquad \sqrt{H}\, u^{\hat{t}} = H\, t' = A \\[1ex]
\qquad\qquad\qquad\qquad = -p_t/\mu \text{ —the energy per unit (test particle) mass, dimensionless,} \\[1ex]
r \sin\theta\, u^{\hat{\varphi}} = r^2 \sin^2\theta\, \varphi' = B \equiv p_\varphi/\text{mass} \ , \\[1ex]
\qquad\qquad\qquad \text{—the z-component of angular momentum per unit (test particle) mass ,} \\[1ex]
\qquad\qquad\qquad\qquad \text{which has the dimension of length,} \\[1ex]
\qquad r u^{\hat{\theta}} = r^2 \theta' = \pm\sqrt{\ell^2 - B^2/\sin^2\theta} \ , \\[1ex]
\qquad\qquad\qquad \text{or } \ (r^2\Omega')^2 = r^2\{(u^{\hat{\theta}})^2 + (u^{\hat{\varphi}})^2\} = (r^2\theta')^2 + (r^2 \sin\theta\, \varphi')^2 = \ell^2 \ , \\[1ex]
\qquad\qquad\qquad \text{with } \ell \text{ the total angular momentum per unit (test particle) mass,} \\[1ex]
\qquad\qquad\qquad\qquad \text{of the dimension of length,} \\[1ex]
\qquad\qquad \text{where } \ A, B, \text{ and } \ell \text{ are constants.}
\end{array}
\right.
\tag{14}
$$

We do not attempt to solve the ode for $u^{\hat{r}}$—however, see the last line of Eqs. (16)—since it is much simpler to write down the standard normalization condition for the affine parameter τ and its associated 4-velocity, and insert those conserved quantities we already have:

$$
-\mu = (u^{\hat{r}})^2 + (u^{\hat{\theta}})^2 + (u^{\hat{\varphi}})^2 - (u^{\hat{t}})^2 = (u^{\hat{r}})^2 + \frac{\ell^2}{r^2} - \frac{A^2}{H} \ , \tag{15}
$$

where μ is either $+1$ for timelike motions or 0 for lightlike motions, which gives us a moderately-simple 1st-order ode for $u^{\hat{r}}$, or a nonlinear, 2nd-order ode for $dr/d\tau$.

However, we won't even do that because we can also immediately notice that

(1) if one takes $\theta = \pi/2$, and $\theta' = 0$, then the geodesic remains, always, within the equatorial plane, with $\ell = B$ so that θ' remains zero, and

(2) the remainder of the equations may be written in a form appropriate for motion in that equatorial plane:

$$
\begin{aligned}
\theta'' &= 0 \ , \\
\varphi' &= B/r^2 \ , \\
t' &= A/H \ , \\
(u^{\hat r})' = (\sqrt{J}\, r')' &= \frac{B^2}{r^3\sqrt{J}} - \frac{A^2\, H'}{2\sqrt{J}\, H^2} \ ,
\end{aligned}
\tag{16}
$$

along with the normalization equation

$$
J\,(r')^2 + \left(\frac{B}{r}\right)^2 - \frac{A^2}{H} = -\mu \quad . \tag{17a}
$$

where μ takes on the values $+1$ or 0, depending on whether the geodesic is timelike or null.

The solutions of these equations, then, have trajectories, i.e., r versus φ in the equatorial plane—where $\theta = \pi/2,\ \theta' = 0$—of the form

$$
\left(\frac{dr}{d\varphi}\right)^2 = \frac{1}{J}\left[-\mu\frac{r^4}{B^2} - r^2 + \frac{A^2}{B^2}\frac{r^4}{H}\right] \ , \tag{17b}
$$

Applications to Vacuum

The general, relevant solution to the Einstein vacuum, field equations is given by $J^{-1} = H = 1 - 2\frac{M}{r}$, where M is a constant of integration, interpreted as the central mass, that causes the gravitational field at large distances consonant with Newtonian gravity. The particular value of that constant is determined, of course, by knowing that $H \approx (1+\phi)^2$, for small values of ϕ, the gravitational potential, and that the gravitational potential for a central mass is such that $\phi \underset{r\to\infty}{\longrightarrow} 0$.

Under these circumstances, i.e., in vacuum, it is useful to rewrite some things taking account of these values of H and J. We especially now rewrite the particle-motion equations, **for the vacuum case**, with prime denoting the derivative with respect to proper time, τ:

take the following definition: $\mathcal{H} \equiv \sqrt{H} = 1/\sqrt{J}$;

$$
\begin{gathered}
(u^{\hat t})' + \mathcal{H}_{,r} u^{\hat r}\, u^{\hat t} = 0 \ , \\
(u^{\hat\theta})' + \frac{\mathcal{H}}{r} u^{\hat r}\, u^{\hat\theta} - \frac{\cot\theta}{r}(u^{\hat\varphi})^2 = 0\,,\ (u^{\hat\varphi})' + \frac{\mathcal{H}}{r} u^{\hat r}\, u^{\hat\varphi} + \frac{\cot\theta}{r} u^{\hat\theta}\, u^{\hat\varphi} = 0\,, \\
(u^{\hat r})' - \ell^2 \frac{\mathcal{H}}{r^3} + A^2 \frac{\mathcal{H}_{,r}}{\mathcal{H}^2} = 0 \ ,
\end{gathered}
\tag{18a}
$$

and the normalization $\left(\dfrac{r'}{\mathcal{H}}\right)^2 + \dfrac{\ell^2}{r^2} - \dfrac{A^2}{\mathcal{H}^2} = -\mu = -1 \text{ or } 0$, while the orbit equation—the specialization to vacuum of Eq. (17b)—has the form

$$\left(\frac{du}{d\varphi}\right)^2 = \frac{1}{r^4}\left(\frac{dr}{d\varphi}\right)^2 = \frac{A^2}{B^2} - (1 - \frac{2m}{r})\left(\frac{1}{r^2} + \frac{\mu}{B^2}\right) , \tag{18b}$$

where $u \equiv 1/r$ and $\mu = 0$ or 1, as usual, for null or timelike geodesics. The general solution of this equation may be written in the form

$$\frac{2m}{r} = \mathcal{P}(\tfrac{1}{2}(\varphi + \delta)) + \frac{1}{3} , \tag{18c}$$

where $\mathcal{P}(z)$ is the Weierstrass elliptic function. The Weierstrass function is an even function of complex z, with a double pole at $z = 0$, and has two independent periods, whose ratio is always complex, and satisfies the first-order, nonlinear ode:

$$(\mathcal{P}')^2 = 4\mathcal{P}^3 - a\mathcal{P} - b ,$$

where a and b are constants determined by the periods.

We will, however, discuss in more detail some interesting special cases, where the orbits are considerably simpler. For example, radial or circular, or ellipses with precessing perihelia (and aphelia).

Applications to Spherically-Symmetric, Ideal Fluids at Rest

If we agree to model a non-rotating star by an ideal fluid, then, at rest, it is characterized totally by its density, ρ, and its pressure P, both of which must depend only upon the radial variable r. Then the Einstein equations read

$$G_{\mu\nu} = 8\pi\, T_{\mu\nu} - \Lambda\, g_{\mu\nu} = 8\pi\left\{P\, g_{\mu\nu} + (\rho + P)u^\mu u^\nu\right\} - \Lambda\, g_{\mu\nu}\ . \tag{19}$$

We can see that, were one to care, the cosmological constant acts something like a *negative pressure* in this situation.

Using these equations, and setting $\Lambda \equiv 0$, we get

$$\begin{aligned} J^{-1} =& 1 - \frac{8\pi}{r}\int_0^r r^2\, dr\, \rho(r) \equiv 1 - \frac{2\,\mathcal{M}(r)}{r}\ , \\ \mathcal{M}(r) \equiv& 4\pi\int_0^r r^2\, dr\, \rho(r) = \int_0^r r^2\, dr \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\varphi\, \rho(r)\ . \end{aligned} \tag{20}$$

Progressing onward to the other equations, we find the Tolman-Oppenheimer-Volkov equation, which gives a relation between the pressure, P, and the density ρ, namely

$$\frac{dP}{dr} = -(\rho + P)\frac{\mathcal{M}(r) + 4\pi r^3 P}{r(r - 2\mathcal{M}(r))}\ , \tag{21}$$

and the equation which, in principle, can be used to determine $H = H(r)$, namely

$$\frac{d}{dr}(\log H) = \frac{H'}{H} = 8\pi\, rPJ(r) + \frac{J-1}{r} = \frac{2\mathcal{M}(r) + 8\pi\, r^3 P}{r(r - 2\mathcal{M}(r))}\ . \tag{22}$$

In the simple case, where we assume $\rho = \rho_0$, i.e., a constant, we can integrate these equations and find that

$$\begin{aligned} \frac{P(r)}{\rho_0} &= \frac{\sqrt{1 - \frac{2M}{r}} - \sqrt{1 - \frac{2M}{R}\left(\frac{r}{R}\right)^2}}{\sqrt{1 - \frac{2M}{R}\left(\frac{r}{R}\right)^2} - 3\sqrt{1 - \frac{2M}{r}}}\ , \\ H(r) &= \left\{\tfrac{3}{2}\sqrt{1 - \frac{2M}{R}} - \tfrac{1}{2}\sqrt{1 - \frac{2M}{R}\left(\frac{r}{R}\right)^2}\right\}^2\ , \\ J^{-1}(r) &= 1 - \frac{2\mathcal{M}(r)}{r} = 1 - \frac{2M}{R}\left(\frac{r}{R}\right)^2 . \end{aligned} \tag{23}$$

Since we want to insist that the pressure at the center not be infinite, this puts a bound on R, namely that it must be greater than $\frac{9}{4}M$, which is already greater than $2M$, thus keeping the horizon of the exterior solution inside itself—for this case of constant density. We note that in fact this bound on R/M can be shown without such a stringent assumption; it is sufficient to assume that

i.) there exists a quantity R such that $\rho(r) = 0$ for all $r > R$,

ii.) that the density is monotone decreasing, i.e., that $d\rho/dr \leq 0$,

iii.) and that $2\mathcal{M}(r) < r$, i.e., that J is non-singular within the fluid.

Applications to Time-Dependence, as well

In the event where J and H and both allowed to depend on time, as well as the radial coordinate, then there are slight changes in the connections and curvature. We find that only one connection 1-form is changed, namely

$$\underset{\sim}{\Gamma}_{\hat{r}\hat{t}} \text{ acquires an additional term } \frac{1}{2\sqrt{JH}} \frac{\dot{J}}{\sqrt{J}} \underset{\approx}{\omega}^{r} , \tag{24}$$

where the overdot indicates a time derivative. In the same way, 3 of the Cartan curvature 2-forms acquire extra terms:

$$\begin{aligned} R_{\hat{r}\hat{t}\hat{r}\hat{t}} \text{ has the additional term } &\frac{H(\dot{J})^2 - 2JH\ddot{J} + J\dot{J}\dot{H}}{4(JH)^2} , \\ R_{\hat{r}\hat{\theta}\hat{\theta}\hat{t}} = -\,1/2r &\frac{\dot{J}/J}{\sqrt{JH}} = R_{\hat{r}\hat{\varphi}\hat{\varphi}\hat{t}} . \end{aligned} \tag{25}$$

This generates a single non-diagonal term in the Ricci tensor, as well as some additional terms in $\mathcal{G}_{\hat{\theta}\hat{\theta}} = \mathcal{G}_{\hat{\varphi}\hat{\varphi}}$:

$$\mathcal{R}_{\hat{r}\hat{t}} = \frac{\dot{J}/J}{r\sqrt{JH}} . \tag{26}$$