Problem Session Questions

Physics 480/581

Problem Session No. 4 Monday, 24 September, 2018

1. On a vector space of 1-forms over a flat manifold with coordinates $\{x, y, z, t\}$, choose as a basis the following:

$$\underset{\sim}{\theta}^1 \equiv dx + i\,dy\,, \quad \underset{\sim}{\theta}^2 \equiv dx - i\,dy\,, \quad \underset{\sim}{\theta}^3 \equiv dz - dt\,, \quad \underset{\sim}{\theta}^4 \equiv dz + dt\,.$$

For 2 timelike-separated events, A and B on the manifold, with coordinates $\{x^\mu_A\}$ and $\{x^\nu_B\}$ the (Lorentz invariant) interval between them would be given by

$$\mathbf{g} \equiv \boldsymbol{\eta}_{\mu\nu}\,(x^\mu_B - x^\mu_A)\,(x^\nu_B - x^\nu_A) \equiv -(\Delta\tau_{AB})^2\,,$$

assuming that τ is the parameter along a worldline of some observer who passed between A and B. An infinitesimal version of the interval, using $\{dx^\mu\}$ as a choice for a basis of 1-forms or the choice given above $\{\underset{\sim}{\theta}^\alpha\}$, would just be

$$\boldsymbol{\eta}_{\mu\nu}\,dx^\mu dx^\nu \equiv Q_{\alpha\beta}\underset{\sim}{\theta}^\alpha\,\underset{\sim}{\theta}^\beta\,,$$

where $Q_{\alpha\beta}$ is the form of what is usually referred to as *the metric tensor* relative to the first choice of basis 1-forms, as given in the beginning of this problem. Please determine the matrix which has the $Q_{\alpha\beta}$ as its elements.

2. Using the connection 1-forms for the Schwarzschild metric, in the orthonormal basis

$$\underset{\approx}{\omega}^{\hat r} \equiv \frac{1}{\mathcal{H}}\,dr\,, \quad \underset{\approx}{\omega}^{\hat\theta} \equiv r\,d\theta\,, \quad \underset{\approx}{\omega}^{\hat\varphi} \equiv r\sin\theta\,d\varphi\,, \quad \underset{\approx}{\omega}^{\hat t} \equiv \mathcal{H}\,dt$$

the connection 1-forms are the following:

$$\textbf{Connections:} \quad \begin{cases} \underset{\sim}{\Gamma}_{\hat r\hat\theta} = -\mathcal{H}\frac{\underset{\approx}{\omega}^{\hat\theta}}{r}\,, & \underset{\sim}{\Gamma}_{\hat r\hat\varphi} = -\mathcal{H}\frac{\underset{\approx}{\omega}^{\hat\varphi}}{r}\,, \quad \underset{\sim}{\Gamma}_{\hat r\hat t} = \mathcal{H}'\underset{\approx}{\omega}^{\hat t}\,, \\ \underset{\sim}{\Gamma}_{\hat\theta\hat\varphi} = -\frac{\cot\theta}{r}\underset{\approx}{\omega}^{\hat\varphi}\,, & \underset{\sim}{\Gamma}_{\hat\theta\hat t} = 0\,, \quad \underset{\sim}{\Gamma}_{\hat\varphi\hat t} = 0\,, \end{cases}$$

write out the 4 differential equations that define a geodesic in this spacetime, remembering that the connection 1-forms are skew-symmetric in their indices.

3. The 4-velocity can be written as

$$\frac{d}{d\tau} = \widetilde{u} = \frac{dx^\mu}{d\tau}\frac{\partial}{\partial x^\mu}\,.$$

If the coordinates $\{x^\mu\}$ are the usual spherical coordinates, $\{r, \theta, \varphi, t\}$, what is a matrix representation of the components of the 4-velocity?

4. In the orthonormal basis, show that the components of the 4-velocity are such that

$$\gamma = \mathcal{H}\frac{dt}{d\tau}\,, \quad v^{\hat r} = \frac{dr/dt}{\mathcal{H}}\,, \quad v^{\hat\theta} = \frac{r}{\mathcal{H}}\frac{d\theta}{dt}\,, \quad v^{\hat\varphi} = \frac{r\sin\theta}{\mathcal{H}}\frac{d\varphi}{dt}\,.$$

Remember that $(\widetilde{u})^2 = -1$.