Problem Session Questions

Physics 480/581

Problem Session No. 6 Monday, 8 October, 2018

1. Consider an electric field for a single, stationary charge, q, and write out the Faraday, using our usual orthonormal polar (Schwarzsch.) coordinates. Then, compute the associated stress-energy tensor,

$$T^{\mu\nu} = F^{\mu\alpha} g_{\alpha\beta} F^{\nu\beta} - \tfrac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \ .$$

2. Beginning with the usual Schwarzschild metric, change the time coordinate, t, to a new time coordinate, w:

$$w \equiv t - r - 2M \log(r/2M - 1) \ .$$

Determine the form of the metric in the these new coordinates, (r, θ, φ, w). Show that the metric is NOT singular in these coordinates at $r = 2M$, but it is no longer "diagonal." Check that the tangent vector $\widetilde{u} \equiv \pm\partial_r$ describes a null vector, i.e., a light ray, in these coordinates. What is the corresponding 1-form for this tangent vector, using the metric as usual to map tangent vectors into 1-forms (and vice versa)?

3. There is a way of re-defining the metric up to a (usually non-constant) scale factor, which is referred to as a *conformal transformation.* One of the values of such a transformation is that it allows the "points at infinity" to be brought in to some finite places, so that the entire (originally infinite, in several dimensions), can be viewed in a finite setting. I will note that the idea of "rolling up" the 2-dimensional plane of complex variables onto the (Riemann) sphere is an example of this, where the process of rolling up causes all paths—on that plane of complex variables—that go to infinity to arrive at the same point, which is the correct approach to use to study analytic functions of complex variables.

Let us study this approach for flat, 4-dimensional Minkowski space. We first define two new variables, p and q, by the following equations:

$$\tan p \equiv v \equiv t + r \ , \qquad \tan q \equiv w \equiv t - r \ .$$

Since we know that r is never negative, this allows us to see that

$$0 \leq 2r = v - w \quad \Longrightarrow \quad v \geq w \ .$$

a. First show that there is a single scalar factor, which we will Φ, such that the metric, in polar coordinates in 4-dimensional, flat Minkowski space may be written as

$$\mathbf{g} = \Phi^2 \,\overline{\mathbf{g}} = \Phi^2 \left(-4\, dp\, dq + \sin^2(p-q)\, d\Omega^2\right) \ , \quad d\Omega^2 \equiv d\theta^2 + \sin^2\theta\, d\varphi^2 \ .$$

b. Next, show that both p and q are allowed to vary only in the region $[-\frac{1}{2}\pi, +\frac{1}{2}\pi]$, but that, also, we must always have $p \geq q$, and then define yet another pair of variables, (R, T):

$$R \equiv p - q \in [0, \pi] \ , \qquad T \equiv p + q \in [-\pi, +\pi] \ .$$