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The Equivalence Principle and an Electric Charge in a Gravitational Field

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It is shown that there is no violation of the strong principle of equivalence in the case of an electric charge either falling freely or supported in a static uniform gravitational field. For a freely falling charge, the global electromagnetic field distribution at any instant is found to be the same as that of a charge which is moving uniformly with respect to an inertial frame with a velocity equal to the instantaneous velocity of the freely falling charge. In the case of a charge supported in the gravitational field, the total electromagnetic field energy, as measured by freely falling observers instantaneously at rest with respect to the charge, is shown to be equal to the Coulomb field energy of a charge permanently stationary in an inertial frame. The conclusion here, that in neither of the two cases does the charge emit electromagnetic radiation, is independent of our choice of the observer's frame of reference.

1. INTRODUCTION

The strong principle of equivalence appears to be violated in the case of a free fall of an electric charge in a static uniform gravitational field. As seen by an observer stationary in the gravitational field, a freely falling charge is accelerated "downwards" and should radiate at a rate proportional to the square of the acceleration due to gravity, according to Larmor's formula for radiation from the classical electromagnetic theory (see e.g. Ref. 1,

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p.658, Ref. 8, p.37). On the other hand such a charge is stationary with respect to an observer also falling freely in the same uniform gravitational field. From the strong principle of equivalence a freely falling observer is in an inertial frame of reference and according to the classical electromagnetic theory such an inertial observer should see no radiation emitted by a charge stationary in his/her frame. Since the electromagnetic radiation, as commonly understood, cannot be eliminated by a change of frame of reference (radiation could cause some physical effects which should be visible to all observers; photons may get Doppler-boosted or even red-shifted but not eliminated altogether, etc.), an inference of radiation from a charge should not depend upon our choice of the observer. Thus the strong principle of equivalence and the classical electromagnetic theory may appear to be incompatible in this case. An equally paradoxical case appears to be that of a charge stationary ("supported") in the static gravitational field. Such a charge is ever at rest with respect to a similarly supported observer. Everything is static and there are no temporal changes in the charge or in its fields, and consequently no electromagnetic radiation should be seen leaving the charge. But according to a freely falling inertial observer such a charge is accelerated upwards and should radiate continuously. Based on these paradoxical results, doubts have sometimes been raised about the *universal validity* of the strong principle of equivalence (see e.g. Refs. 2,3).

Both above problems, at their face values, may appear linked to the question of whether or not does a uniformly accelerated charge radiate. After Pauli first expressed doubts about the occurrence of radiation from a uniformly accelerated charge (Ref. 4, p.92), there have been many attempts in the literature to counter his arguments (see Refs. 2,3,5–7, Ref. 8, p.37; also see Ref. 9, p.367, for a review and for other references on the related works). Here it is important to note that in spite of the apparent similarity, the case of a freely falling charge is quite different in its nature from that of a charge uniformly accelerated with respect to an inertial frame. While in the former case there is an inertial frame available in which the charge remains at rest, no such inertial frame exists in the latter case. Therefore one should be able to resolve the two cases with rather independent arguments.

2. THE CASE OF A FREELY FALLING CHARGE

The results derived in classical electromagnetism are valid strictly only for observers stationed in inertial frames of reference. Therefore the conclusions of an observer in the freely falling frame (an inertial frame) should, in general, be correct and there should be no electromagnetic radiation from a freely falling charge. But the main problem is how to reconcile this result with the one expected by an observer stationary in the gravitational field. For that we need to look more carefully at the exact findings of such an observer. From the theory of relativity, all forms of energy (including that of the electromagnetic fields) have an associated inertial mass and which, by the principle of equivalence, will fall in a gravitational field in the same way as any other matter. This has been amply tested by the bending of light ("fall" of photons) in the gravitation field of the sun (see Ref. 10). Now even the electric field of a charge has a well-defined energy density and hence a mass density, and there is no reason why this field also should not be considered to "fall" along with the charge (after all from the principle of equivalence, *everything* falls in a gravitational field). This can be also seen from the fact that in the case of a uniform gravitational field, the space-time coordinate transformation between the freely falling inertial frame and the supported frame is identical for *all events* that are simultaneous in a horizontal plane (i.e., in a plane perpendicular to the direction of fall). Thus an observer stationary in a static, uniform gravitational field will find that, as a charged particle falls, so does the bundle of electric field lines alongside it, for *field points at all distances* in the horizontal plane containing the charge.

It still remains to be seen if some distortion in electric field lines may occur because of any differential motion between neighbouring freely-falling horizontal planes, as seen by observers supported in the gravitational field. For that purpose it seems necessary to be more specific about the meaning of a static uniform gravitational field. As is well known, an ideal homogeneous gravitational field, where the gravitational acceleration of objects just released from rest is the same everywhere, cannot have a static (time-independent) metric. The only non-trivial metric for a static gravitational field with a zero Riemann curvature (implying no geodesic deviation for freely falling objects) is given by [7,11,12],

$$ds^2 = -\frac{g_0^2 X^2}{c^2}\, dT^2 + dX^2 + dY^2 + dZ^2. \qquad (1)$$

Here the acceleration due to gravity is along the $-X$ axis. In these coordinates, a standard clock stationary at X, during a coordinate time interval dT, measures an interval of proper time ("local" time; Ref. 11) $g_0 X dT/c^2$. Moreover the acceleration due to gravity g for an object just released from rest at a point X is equal to c^2/X in such a field. Thus the arbitrary constant g_0 in eq. (1) represents the gravitational acceleration at a point $X_0 = c^2/g_0$, where the time intervals measured on a standard clock are equal to those of the coordinate time.

The motion of a freely falling object in the above gravitational field has been discussed in detail in the literature (see e.g. Ref. 11) and we should only highlight some interesting features of such a motion that are relevant for the discussion here. It turns out that the motion of a freely falling particle (assuming it to be momentarily at rest at X_i at time $T = 0$) in these coordinates is described by

$$\frac{1}{X}\frac{dX}{dT} = -\frac{g_0}{c}\tanh\left(\frac{g_0 T}{c}\right),$$

which gives

$$X = \frac{X_i}{\cosh(g_0 T/c)}. \tag{2}$$

It should be noted that $\gamma = \cosh(g_0 T/c)$ is the Lorentz factor corresponding to the "local" velocity, $\beta = -\tanh(g_0 T/c)$, of the falling particle (measured in terms of the local standard-clock rate at X). By applying eq. (2) to a set of such freely falling objects, which were momentarily at rest at different X_i's at $T = 0$, we see that the mutual separation between any pair of objects at some later time T would be less by a factor $\cosh(g_0 T/c)$ as compared to their separation at time $T = 0$. Therefore, as seen by observers supported in the above gravitational field, all spatial dimensions of a freely falling frame appear contracted along the direction of free fall by the Lorentz factor $\cosh(g_0 T/c)$ at time T. This can be visualized in another way. The above "uniform" gravitational field can be simulated (Ref. 11, Ref. 13, p.49) by a uniform (in time) proper acceleration of a set of observers, where the acceleration of an observer at X in an inertial frame that is a common instantaneous rest frame for all accelerated observers, is equal to c^2/X. For this particular set of observers, one such instantaneous, spatially, "coincident" inertial frame is always available. From the clock and length hypotheses [13], all momentarily spacetime measurements by the accelerated observers will exactly match with those made in their instantaneously coincident inertial rest frame. Thus not only will the accelerated observers measure the "local" velocity of the initial inertial frame (i.e. the freely–falling frame, which at $T = 0$ was the coincident inertial frame in the simulated gravitation field) to be the same everywhere in their frame [$\beta = -\tanh(g_0 T/c)$], all dimensions of the freely–falling frame (including those of the bundle of electric field lines around the charge) will also appear Lorentz contracted along the X–axis by a factor $\gamma = \cosh(g_0 T/c)$ at T.

To define the electromagnetic field at any event in this gravitational field, we can use the measurements of the electric and magnetic field components carried out in a local Lorentz frame, using the Lorentz force law

on a test charge, at that event (see e.g. Ref. 14, p.568). In our case this local Lorentz frame is the coincident inertial frame described above, which, as already mentioned, in this particular case happens to be a common instantaneous rest frame for all observers supported in the gravitational field. Therefore the electric and magnetic field measurements in the instantaneous rest frame will also describe the electromagnetic fields for all observers supported in the gravitational field. Now with respect to this coincident inertial frame our freely falling charge is moving with a uniform velocity β. Therefore, its electric and magnetic field components, in the $Z = 0$ cross–section plane, are given by (Ref. 1, p.552)

$$E_X = \frac{e\gamma(\Delta X)}{(\gamma^2(\Delta X)^2 + Y^2)^{3/2}}$$
$$E_Y = \frac{e\gamma Y}{(\gamma^2(\Delta X)^2 + Y^2)^{3/2}}$$
$$B_Z = \frac{e\beta\gamma Y}{(\gamma^2(\Delta X)^2 + Y^2)^{3/2}},$$

with all other field components being zero. Here (ΔX) represents the distance of the field point from the "present" position of the charge, along the X–axis. We have assumed that the charge remains on the X–axis, i.e. $Y = Z = 0$ for the charge motion throughout.

As discussed above, the field components of the freely falling charge in the supported frame at an instant T are also described by the above equation, however, now β and γ are to be functions of T, with $\beta = -\tanh(g_0 T/c)$ and $\gamma = \cosh(g_0 T/c)$. Therefore the nonvanishing components of the electromagnetic fields in the supported frame, in the $Z = 0$ cross-section plane, can be written as

$$E_X = \frac{e\cosh(g_0 T/c)(\Delta X)}{(\cosh^2(g_0 T/c)(\Delta X)^2 + Y^2)^{3/2}}$$
$$E_Y = \frac{e\cosh(g_0 T/c) Y}{(\cosh^2(g_0 T/c)(\Delta X)^2 + Y^2)^{3/2}} \tag{3}$$
$$B_Z = \frac{-e\sinh(g_0 T/c) Y}{(\cosh^2(g_0 T/c)(\Delta X)^2 + Y^2)^{3/2}}.$$

Here $(\Delta X) = X - X_i/\cosh(g_0 T/c)$ for a field point at X, where X_i is the initial position of the charge at $T = 0$. It should be noted that without any loss of generality we can choose $X_0 = X_i$, i.e., we could choose the arbitrary constant g_0 to be the value of the acceleration due to gravity at the point X_i where the freely falling charge is instantaneously at rest (at say, $T = 0$), and then T will be the proper time measured on a standard clock stationary at X_i in the gravitational field.

We can express the electric and magnetic field components at any point in the supported frame, in terms of a spherical coordinate system (r, θ, ϕ) with an origin at the "present" position of the charge, as

$$E_r = \frac{e\cosh(g_0T/c)}{r^2(\cosh^2(g_0T/c) - \sinh^2(g_0T/c)\sin^2\theta)^{3/2}}$$
$$B_\phi = \frac{-e\sinh(g_0T/c)\sin\theta}{r^2(\cosh^2(g_0T/c) - \sinh^2(g_0T/c)\sin^2\theta)^{3/2}}, \quad (4)$$

with all other field components being zero. Here $r^2 = (\Delta X)^2 + Y^2 + Z^2$, and angle θ is measured with respect to the X–axis.

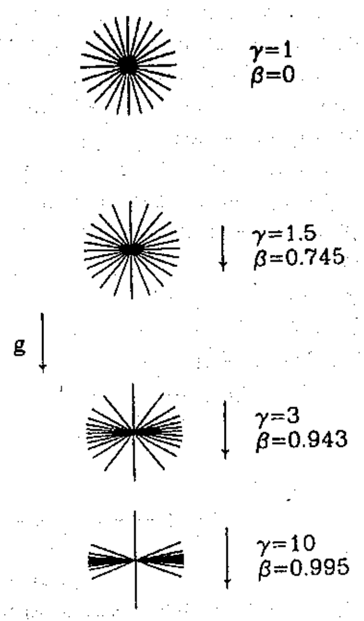

Figure 1. The electric field lines distribution of a charge as seen at different instants during its free fall in a "uniform" gravitational field.

Figure 1 shows a plot of the electric field lines in the supported frame, for different instants of time. At any instant the field lines are everywhere radial with respect to the instantaneous position of the charge, though,

depending upon the "present" velocity of the freely falling charge, are bunched towards the plane perpendicular to the direction of motion. It is of course well known that for a uniformly moving charge the electric field lines appear as if all scales along the direction of motion have been squashed by the Lorentz contraction factor (see e.g. Ref. 15, p.163). The global electromagnetic field configuration for a freely falling charge, as seen at any instant in a static uniform gravitational field, thus appears to be that of a charge moving uniformly with respect to an inertial frame with a velocity equal to the instantaneous velocity of the freely falling charge. It should be noted here that in the case of a simulated gravitational field though the possible measurements by uniformly accelerated observers remain confined to only a part of the total space-time observable in an inertial frame [13], the electromagnetic field in the space-time even outside that part, as observed from the coincident inertial rest frame of the accelerated observers, will be that of a charge moving with a uniform velocity. Thus even if the accelerated observers cannot observe beyond their horizon at $X = 0$, the inertial observers do see the field lines to be continuing in radial directions with respect to the instantaneous charge position, even beyond $X = 0$ (i.e. for all $X < 0$ as well).

It has been proposed in the literature [6] that the nonvanishing of the Poynting vector at an observer's position should be taken as the criterion for the existence of radiation. The fallacy inherent in such a proposition is easily seen from the case of a charge moving uniformly with respect to an inertial frame. It is a well accepted fact that no radiation will be seen by inertial observers from a uniformly moving charge. However, even in this case there is a nonvanishing Poynting vector at every location of the inertial observer. The reason of course being that as the uniformly moving charge approaches (or recedes from) an observer's position, its field strength (both in magnitude and direction) undergoes a change at the observer's location, resulting in a finite Poynting vector there. The Poynting vector at any location, in the case of a freely falling charge, can be readily calculated from our eqs. (3) or (4) and the formulae appear to be the same as those given by Kovetz and Tauber [6] (apart from a small difference in one of their Poynting vector components (S_A), which appears to be due to the presence of a small mistake in their given formulae, as could be also verified from a dimensional comparison of their expression for S_A with that of the other Poynting vector component). The Poynting flux, cited by Kovetz and Tauber [6] as evidence for the presence of radiation in this case, represents actually the "convective" flow of the self-field energy of the charge due to its "present" motion at any instant, like it happens in our quoted example of a uniformly moving charge. The absence of radiation in the case of a freely falling charge can be also demonstarated in the following manner.

In the standard picture for radiation from a charge accelerated with respect to an inertial frame, because of the finite value of the wave propagation speed c, the electric field values at a distance R respond to any change in the motion of the charge, only at a time R/c later. Thus while the field values in regions "near" to the charge ($R < ct$) will have adjusted to a change in motion (acceleration) of the charge, the fields in the far away regions would still correspond to a previous unchanged motion of the charge. Electromagnetic radiation (in the famous J. J. Thomson picture; see e.g. Ref. 15, p.163, Ref. 16, p.193) is supposed to represent the transverse fields in the transition zone between the nearby regions (where electric field vectors point in radially outward directions from an actual present position of the charge) and the far-off regions (fields in radial directions from a would-have-been charge position, assuming a uniform velocity for the charge). But as we discussed above, in the case of a freely falling charge in a uniform gravitational field, the electric field around the charge *everywhere* keeps "in step" with the charge motion (i.e. the field lines will everywhere be radial with respect to the instantaneous position of the charge; eq. (4), Fig. 1). At any field point this happens due to a "local cause" (i.e., because of the acceleration due to gravity at the location). No extra "information" from the charge position, in the form of a transverse field (radiation!), is required to travel towards the field points (to adjust the fields there to the changing position and velocity of the charge). Therefore even an observer stationary in the gravitational field will not see any electromagnetic radiation emanating from a freely falling charge.

A question that may arise is whether one can still apply the standard formulae to calculate radiation from a freely falling charge in the more realistic case of a static but non-uniform gravitational field by using the value of g (at the charge location) as the acceleration parameter in Larmor's formula. It is obvious that the answer cannot be a generic yes. As we have already seen above, at least in one case (i.e. in a "uniform" gravitational field case) radiation is not determined by the value of g. It is true that in the case of a non-uniform gravitational field, the electric field lines may not "fall" everywhere in step with the charge. But then any distortions in the field lines in such a case would depend only upon the departure of g from the uniform gravitational field case (tidal-effects of gravity!) implying that these distortions arise primarily not because of g itself but rather depend upon its spatial differentials and are thus only of a second order in nature. Any consequential transverse bending

in field lines will vary from case to case and could be totally different for different tidal fields, even for similar local values of g (as measured say, by locally supported observers). This picture is qualitatively different from the standard picture of radiation where the transverse bends in the field lines are determined by the actual acceleration of the charge, and where the radiated power at any instant is calculated from the value of the charge acceleration at that moment.

3. A CHARGE "SUPPORTED" IN THE GRAVITATIONAL FIELD

In contrast with the case of a freely falling charge, in the case of a supported charge *no inertial frame* exists in which the charge remains at rest. In fact it is only in this case (a supported charge seen by a freely falling observer) that the question of electromagnetic radiation from a uniformly accelerated charge arises. The strong principle of equivalence demands that the conclusions about the absence of radiation from a supported charge, as inferred by a co-supported observer, should also be equally valid for an inertial observer that may, during his/her free fall, be instantaneously at rest alongside of the supported observer. We show here that in the case of a uniformly accelerated charge, the R^{-1} dependent acceleration fields are cancelled exactly by the transverse component of the R^{-2} velocity fields, *at all distances* in the inertial frame in which the charge comes to rest instantaneously, when the effects of the retarded time are properly taken into account.

The electromagnetic fields, as derived from the Liénard-Wiechert potentials for a moving charge, (in the notations of Ref. 1) are given by,

$$\mathbf{B} = \mathbf{n} \times \mathbf{E}\,,$$

$$\mathbf{E} = e\frac{\mathbf{n}-\boldsymbol{\beta}}{\gamma^2 R^2(1-\boldsymbol{\beta}.\mathbf{n})^3} + \frac{e}{c}\frac{\mathbf{n}\times\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\}}{R(1-\boldsymbol{\beta}.\mathbf{n})^3}\,.$$

All quantities on the right hand side are to be evaluated at the retarded time. It is usually assumed that the acceleration fields (second term on the right hand side), which fall with distance as $1/R$ and are transverse in nature (perpendicular to $\mathbf{n}$), solely represent the radiation from a charge, since the contribution of the velocity fields ($\propto 1/R^2$) appears to be negligible for a large enough value of R.

Now we are interested in a one-dimensional motion ($\boldsymbol{\beta}\|\dot{\boldsymbol{\beta}}$), for which the electric field vector reduces to

$$\mathbf{E} = e\frac{\mathbf{n}-\boldsymbol{\beta}}{\gamma^2 R^2(1-\boldsymbol{\beta}.\mathbf{n})^3} + \frac{e}{c}\frac{\mathbf{n}\times\{\mathbf{n}\times\dot{\boldsymbol{\beta}}\}}{R(1-\boldsymbol{\beta}.\mathbf{n})^3}\,.$$

Using the vector identity $\boldsymbol{\beta} = \mathbf{n}(\boldsymbol{\beta}.\mathbf{n}) - \mathbf{n}\times\{\mathbf{n}\times\boldsymbol{\beta}\}$, we can rewrite the electric field in terms of the radial (along $\mathbf{n}$) and transverse components as,

$$\mathbf{E} = e\frac{\mathbf{n}}{\gamma^2 R^2(1-\boldsymbol{\beta}.\mathbf{n})^2} + e\frac{\mathbf{n}\times\{\mathbf{n}\times(\gamma\boldsymbol{\beta}+\gamma^3\dot{\boldsymbol{\beta}}R/c)\}}{\gamma^3 R^2(1-\boldsymbol{\beta}.\mathbf{n})^3}. \qquad (5)$$

The second term on the right hand side includes transverse terms both from the velocity and acceleration fields together. It should be noted that while we might separate the velocity fields and acceleration fields for the purpose of simplification in our calculations, but as such there is no fundamental difference in the nature of fields calculated from the two terms and that the net electric field at any point is given by the vector sum of all terms.

Now for a uniformly accelerated particle, the expression $\gamma\boldsymbol{\beta}+\gamma^3\dot{\boldsymbol{\beta}}R/c$ represents the "present" velocity of the charge (for all values of R) and is zero in the instantaneous rest-frame of the charge. Actually as we go to a larger value of R, in order to calculate the retarded position and velocity of the charge, we also have to go further back in time. For a uniformly accelerated charge, in its instantaneous rest-frame, the retarded value of velocity is directly proportional to R. The net effect being that

$$\gamma\boldsymbol{\beta}+\gamma^3\dot{\boldsymbol{\beta}}\frac{R}{c} = (\gamma\boldsymbol{\beta})_{\rm present} = 0\,, \qquad (6)$$

for all R. Therefore

$$\mathbf{E} = e\frac{\mathbf{n}}{\gamma^2 R^2(1-\boldsymbol{\beta}.\mathbf{n})^2}\,, \qquad (7)$$

in the instantaneous rest-frame of a uniformly accelerated charge. Moreover $\mathbf{B} = 0$ everywhere. Thus we see that the acceleration fields throughout are cancelled neatly by the transverse component of the velocity fields, everywhere in the instantaneous rest-frame, implying no radiation fields for a charge supported in a gravitational field, in conformity with the strong principle of equivalence.

Using Born's solution [17] for the fields of a charge undergoing a hyperbolic motion, Pauli [4] first drew attention to the fact that in the instantaneous rest-frame of a uniformly accelerated charge $\mathbf{B} = 0$ throughout and from this he further construed that there is no radiation for such a motion. Subsequently it has been argued [3] that while $\mathbf{B} = 0$ may be something unusual for accelerated motion and of some interest, it has nothing to do with the presence or absence of radiation. However, as we show below, there is something more to it than just a matter of mere curiosity. From the expressions for field strengths as given in Ref. 3, eq. (2.6), we see that the electric field vector is independent of the sign of the time parameter

t, while the magnetic field changes sign with t, but for any of the field components the magnitude at t is exactly the same as that at $-t$. Also the spatial location of the charge is the same at t as it was at $-t$. Let us consider a spherical surface centered at the charge position at t or $-t$ (we choose the radius of the sphere to be small enough so that its surface lies well within the region where eq. (2.6) of Ref. 3 is applicable at both t and $-t$). Now the Poynting vector, at any point on the spherical surface, at time t is exactly equal but in opposite direction to its value at time $-t$. Thus if there is an outflow of electromagnetic energy through the surface at t, it immediately follows that there was an equal inflow of energy through that surface at $-t$ (the energy flow being null at $t = 0$ as $\mathbf{B} = 0$). Also the velocity of the charge at t is equal but in opposite direction to that at $-t$. Therefore any change in the kinetic energy of the charge or in its self-field energy contained in the volume enclosed within the spherical surface during the time intervals between t and $t+dt$ is equal and opposite to the change taken place between $-(t+dt)$ and $-t$. Applying Poynting's theorem to this case we see that the rate of energy being "fed" into fields by the charge during its acceleration phase is exactly equal to that of the energy being "retrieved" from the fields during the deceleration phase. It is important to note here that Poynting's theorem is strictly defined only for *fixed* instants of time in any inertial frame (see e.g. Ref. 1, p.236). That means, it allows us to relate the instantaneous rate of energy loss of the charges or of electromagnetic fields enclosed within a volume to the Poynting flux through the enclosing surface, all to be calculated for the *same* instant of time. It is here immaterial that the fields both within and at the surface were caused by the motion of the charges at some retarded times. It appears that the Poynting flux in the case of a uniformly accelerated charge merely accounts for the rate of increase in the self-field of the charge during the acceleration phase and an equal but opposite rate of decrease during the deceleration phase. There does not appear to be any radiated power, which should be positive both during the acceleration and the deceleration phases (since, from Larmor's formula, it is proportional to the square of acceleration), and which when algebraically added to the equal but opposite Poynting flux required for the rate of self-field energy changes at t and $-t$, should have made the magnitude of the net Poynting flux through the spherical surface unequal at these two times. We may note that a positive Poynting flux at $R \to \infty$ (corresponding to $t \to \infty$), taken in [3] as proof of radiation, merely points out to the fact that the self-field energy of the uniformly accelerated charge is still increasing (as the velocity of the charge increases indefinitely).

To verify it further, we can calculate the total energy in the fields of a charge at time $t = 0$, the fields arising from its uniformly accelerated motion at retarded times. The field energy is given by the volume integral

$$\mathcal{E} = \int \frac{E^2}{8\pi}\, dv.$$

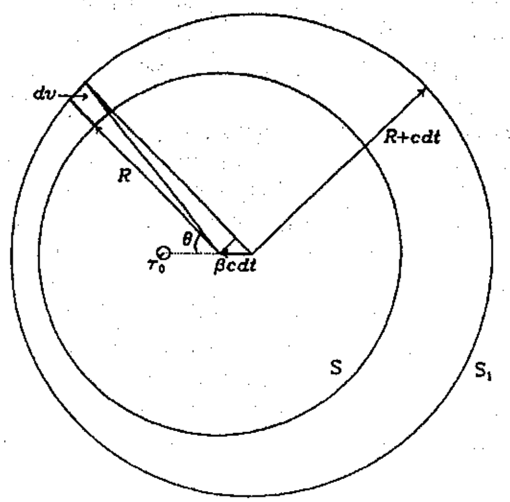

Figure 2. Volume element for calculating the field energy.

It is possible to calculate this volume integral in terms of the retarded quantities in the following manner. First we calculate the field energy in the region between two spherical surfaces S and S_1 (Fig. 2) at the retarded distances R and $R + dR$ where $dR = cdt$. This field energy also includes the contribution of the energy radiated, if any, by the charge between time t and $t + dt$, where $t = -R/c$. The two spheres are centered on two different positions of the charge, separated by a distance βcdt. Therefore the radial distance between the two spherical surfaces is not dR but is instead $dR(1 - \beta\cos\theta)$ (see also e.g. Ref. 18, p.359) with the volume element $dv = 2\pi R^2(1 - \beta\cos\theta)\sin\theta\, dR\, d\theta$. From the integral

$$\int_0^\pi \frac{\sin\theta}{(1 - \beta\cos\theta)^3}\, d\theta = 2\gamma^4,$$

the total field energy in the volume enclosed within S and S_1 is found to be

$$d\mathcal{E} = \frac{e^2}{2R^2}\, dR. \qquad (8)$$

To calculate the total field energy, we can sum over volume elements enclosed between all such spherical surfaces. This in fact implies that we integrate the above expression over all values of R. As may be expected, the integral diverges for $R \to 0$, but we can restrict the lower limit of R at a small but finite value r_0, which may indeed represent the radius of the charged particle. In that case

$$\mathcal{E} = \int_{r_0}^{\infty} \frac{e^2}{2R^2}\, dR = \frac{e^2}{2r_0}. \qquad (9)$$

Now this is exactly the expression for the field energy $U = e^2/2r_0$, of a charge that is permanently at rest in an inertial frame. But in our calculations we included the contribution of the acceleration fields also, for all R. Now $\mathcal{E}$ could not have been equal to U *if there were radiation emitted at a constant rate* (as given by Larmor's radiation formula) from such a charge *all along* its accelerated motion. Thus it follows naturally that there is no electromagnetic radiation from a uniformly accelerated charge.

It may be pointed out that in the case of a charge that is being uniformly accelerated beginning from an infinite past, as seen in an inertial frame, there appears to be a plane of discontinuity (corresponding to $R = \infty$) for the field lines. From the divergence of the electric field vectors Leibovitz and Peres [19] derived a surface charge density at the plane of discontinuity, amounting to a total charge $-e$, and which led them to the conclusion that the Maxwell's equations are incompatible with the existence of a *single* charge uniformly accelerated at all times. This curious result, however, may not be so strange as it appears at a first glance. We can get a similar picture in the case of a charge (say, e) permanently stationary in an inertial frame, if we consider it to be surrounded by a charge $-e$ distributed uniformly on a spherical surface of radius R, and then let $R \to \infty$. In fact, this will be the more appropriate picture of the Coulomb fields of a charge if we strictly hold to the view point that the electric field lines should always terminate on a charge. The discontinuous fields in the case of a uniformly accelerated charge actually correspond to the original (Lorentz transformed) Coulomb fields of the charge "before" it began its acceleration at a time $t \to -\infty$ [7]. It is important to note here that any radiation energy supposed to be emitted by the charge during any finite interval in the past, should of course lie only within a sphere of a finite retarded radius R, without having anything to do with the field energy in the plane of discontinuity at $R = \infty$, which could as such have causal relation with events belonging only to an infinite past (see also the discussion in Ref. 3). To see it more explicitly, we consider the situation at time $t_0 = 0$, when the charge momentarily comes to rest ($\gamma_0\beta_0 = 0$). Any radiation emitted by the charge during a time interval between $-t_1$ and t_0 could lie only within a spherical volume of a retarded radius $R_1 = ct_1$. But as is easily seen from eqs. (8) or (9), the *total* field energy in the region within R_1 for the uniformly accelerated charge at time $t_0 = 0$, is only $e^2/2r_0 - e^2/R_1$, exactly the amount expected in the self-Coulomb field energy of a "presently" stationary charge. Thus it is clear in this case that there has been no field energy "radiated" by the charge during the time interval between $-t_1$ and t_0 as there is none whatsoever excess (radiated!) field energy within $R = ct_1$. Further, this argument is true for any value of t_1. If anything, the field energy in the plane of discontinuity could be said to have been radiated away from the charge due to a *rate of change of acceleration*, in accordance with the radiation reaction equation (see e.g. Ref. 8, p.27), at the instant of the "start" of acceleration (the event at $t = -\infty$ to which the fields in the plane of discontinuity are causally connected), without implying any radiation losses during the uniform acceleration phase.

We should point out that even though the field energy of a uniformly accelerated charge, as measured in its instantaneous rest frame, is equal to that in Coulomb fields of a charge permanently stationary in an inertial frame, the electric field vectors at various points are not identical in both cases. In the accelerated case the field vectors are in radial directions with respect to the retarded positions of the charge and not with respect to the "present" position of the charge. It has an interesting consequence in the case of an accelerated sphere with a uniform distribution of charge over the spherical surface. The electric field inside the sphere is not zero. It can be shown that a uniformly accelerated sphere, which has a uniform surface charge density, has a finite electric field inside it, whose value to a first order is a constant and is equal to $-2e\mathbf{g}/3r_0c^2$, where $\mathbf{g} = \gamma^3\dot{\beta}$ is the acceleration vector (see e.g. Ref. 20). It follows from the strong principle of equivalence that a sphere with a uniform surface charge density, but supported in a gravitational field (say, on the surface of earth), also has a finite electric field inside it, in the same direction as the acceleration due to gravity. The energy in these "inside-fields" ($\propto r_0$) is extremely small for small r_0, and could be ignored for most purposes. But in principle a detection of such a field inside a finite-sized sphere with a uniform charge distribution could be a test of the strong principle of equivalence, though the practical difficulties might be immense because of the weak nature of this field ($\propto \mathbf{g}/c^2$) as compared to the effects of any non-uniformity in the spherical distribution (whose effects perhaps could partially be eliminated by say, a 180° rotation of the sphere, without disturbing the charge distribution). For one thing the sphere will necessarily have to be made of a highly non-conducting material to avoid cancellation of the acceleration dependent inside-electric fields by a redistribution of the conduction electrons on the surface of the sphere.

4. CONCLUSIONS

We have shown that from the strong principle of equivalence the electric field of a freely falling charge in a static, uniform gravitational field would appear to fall along with the charge, remaining everywhere in a radial direction from the instantaneous position of the charge. Accordingly there will be no transverse fields (radiation!) from a freely falling charge in such a gravitational field. It is further shown that in the case of a charge supported in such a gravitational frame, the electric field energy, as measured by freely falling observers instantaneously at rest with respect to the charge, is equal to the Coulomb field energy of a charge permanently stationary in an inertial frame. It follows that in neither of the two cases will there be any electromagnetic radiation.

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